Related Rates

1. An airplane is flying towards a radar station at a constant height of 6 km above the ground. If the distance s between the airplane and the radar station is decreasing at a rate of 400 km per hour when s = 10 km., what is the horizontal speed of the plane?

2. A light is on the ground 20 m from a building. A man 2 m tall walks from the light directly toward the building at 1 m/s. How fast is the length of his shadow on the building changing when he is 14 m from the building?

3. A conical cup is 4 cm across and 6 cm deep. Water leaks out of the bottom at the rate of 2 cm^3/sec. How fast is the water level dropping when the height of the water is 3 cm?

4. A person 2 m tall walks towards a lamppost on level ground at a rate of 0.5 m/sec. The lamp on the post is 5 m high. How fast is the length of the person's shadow decreasing when the person is 3 m from the post?

5. Air is escaping from a spherical balloon at the rate of 2 cm^3 per minute. How fast is the surface area shrinking when the radius is 1 cm? $V = 4/3\ \pi r^3$ and $S = 4\pi r^2$ where V is the volume and S is the surface area, r is the radius.

6. A funnel in the shape of an inverted cone is 30 cm deep and has a diameter across the top of 20 cm. Liquid is flowing out of the funnel at the rate of 12 cm^3/sec. At what rate is the height of the liquid decreasing at the instant when the liquid in the funnel is 20 cm deep?

7. Find the rate of change of the area A, of a circle with respect to its circumference C.

8. A boat is being pulled into a dock by attached to it and passing through a pulley on the dock, positioned 6 meters higher than the boat. If the rope is being pulled in at a rate of 3 meters/sec, how fast is the boat approaching the dock when it is 8 meters from the dock?

9. A man 6 feet tall walks at the rate of 5 ft/sec toward a street light that is 16 ft above the ground.
   a) At what rate is the tip of his shadow moving?
   b) At what rate is the length of his shadow changing when he is 10 feet from the base of the light?

10. A water tank has the shape of an inverted right-circular cone, with radius at the top 15 meters and depth 12 meters. Water is flowing into the tank at the rate of 2 cubic meters per minute. How fast is the depth of water in the tank increasing at the instant when the depth is 8 meters?

11. A ladder 10 meters long is leaning against a vertical wall with its other end on the ground. The top end of the ladder is sliding down the wall. When the top end is 6 meters from the ground it is sliding down at 2 m/sec. How fast is the bottom moving away from the wall at this instant?

12. Gas is escaping a spherical balloon at the rate of 4 cm^3 per minute. How fast is the surface area shrinking when the radius is 24 cm? For a sphere, $V = 4/3\pi r^3$ and $S = 4\pi r^2$ where V is volume, S is surface area and r is the radius of the balloon.

13. The radius of a right circular cylinder is increasing at the rate of 4 cm/sec but its total surface area remains constant at 600π cm^2. At what rate is the height changing when the radius is 10 cm?

14. A block of ice, in the shape of a right circular cone, is melting in such a way that both its height and its radius r are decreasing at the rate of 1 cm/hr. how fast is the volume decreasing when r = h = 10 cm?

15. In a rig ht triangle, leg x is increasing at the rate of 2 m/s while leg y is decreasing so that the area of the triangle is always equal to 6 m^2. How fast is the hypotenuse z changing when x = 3 m?

16. A girl is flying a kite on a string. The kite is 120 ft. above the ground and the wind is blowing the kite horizontally away from her at 6 ft/sec. At what rate must she let out the string when 130 ft. of string has been let out?

17. A thin circular metal disk changes size (but not shape) when heated. The disk is being heated so that its radius is increasing at a rate of 0.03 mm/sec. How fast is the area of the disk changing when the radius is 200 mm?

18. A right circular cylinder of constant volume is being flattened. At the moment when its radius is 3 cm, the height is 4 cm and the height is decreased at the rate of 0.2 cm/sec. At that moment, what is the rate of change of the radius?

19. Assume that sand allowed to pour onto a level surface will form a pile in the shape of a cone, with height equal to diameter of the base. If sand is poured at 2 cubic meters per second, how fast is the height of the pile increasing when the base is 8 meters in diameters?

20. A boat is pulled into a dock by rope attached to it and passing through a pulley on the dock positioned 5 meters higher then the boat. If the rope is being pulled in at a rate of 2 m/sec, how fast is the boat approaching the dock when it is 12 meters away from the dock?

21. Jim, who is 180 cm tall, is walking towards a lamp-post which is 3 meters high. The lamp casts a shadow behind him. He notices that his shadow gets shorter as he moves closer to the lamp. He is walking at 2.4 meters per second.
    a) When he is 2 meters from the lamp-post, how fast is the *length* of his shadow decreasing?
    b) How fast is the *tip* of his shadow moving?

Answers: 1) - 500 k/ hr 2) - 10/9 m/ s 3) - 2/π cm/ s 4) - 1/3 m/ s 5) - 4 cm^2 /min

6) 27/(100π) cm/ s 7) c / (2 π) 8) – 30/ 8 m/ s 9a) tip ~ 8 ft/ s b) shadow -3 ft/ s must do (b) first

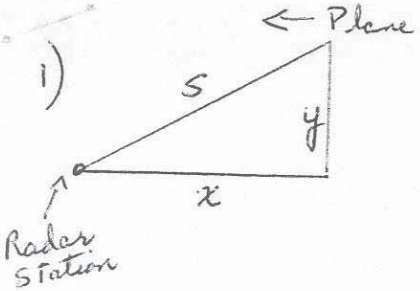
10) 1 /(50π) m/ s 11) 3/ 2 m/ s 12) - 1/3 cm^2/ s 13) – 16 cm/ s 14) – 100π

15) -14/ 15 m/ s 16) 30/13 ft /s 17) 12π m/ s 18) 3/40 cm/ s 19) 1/ (8π m/ s

20)-13/6 m/ s 21a) Shadow decreasing 3.6 m/ s b) Tip decreasing 6 m/ s

Related Rates

1)



$\frac{dy}{dt} = 0 \quad \frac{ds}{dt} = -400 km/hr \quad \frac{dx}{dt} = ?$

$x^2 + y^2 = s^2$

$x^2 + 36 = 100$

$x = 8$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2s\frac{ds}{dt}$

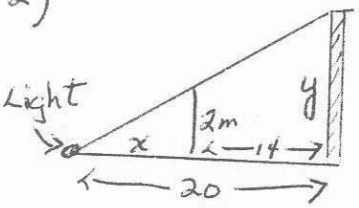
$8\frac{dx}{dt} + 0 = (10)(-400)$

$\frac{dx}{dt} = -500 km/hr$

The horizontal speed of the plane toward the station is 500 km/hr

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2)



$\frac{x}{20} = \frac{2}{y} \quad \frac{dx}{dt} = 1 m/s \quad \frac{dy}{dt} = ? \Big|_{x=6}$

$xy = 40$

$\frac{dx}{dt}y + x\frac{dy}{dt} = 0$

$(1)(y) + (6)\frac{dy}{dt} = 0$

$-\frac{20}{3} = 6\frac{dy}{dt}$

$\frac{dy}{dt} = -\frac{10}{9} m/s$

$xy = 40$

$6y = 40$

$y = \frac{20}{3}$

The length of the shadow is changing by $-\frac{10}{9} m/s$

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3)

$V = \frac{1}{3}\pi r^2 h$ diameter = 4 r = radius = 2 h = height = 6 $\frac{r}{h} = \frac{1}{3}$ $r = \frac{h}{3}$

$V = \frac{1}{3}\pi\left(\frac{h}{3}\right)^2 h$

$V = \frac{1}{27}\pi h^3$

$\frac{dV}{dt} = \frac{\pi}{27} \cdot 3h^2\frac{dh}{dt}$

$-2 = \pi\frac{dh}{dt}$

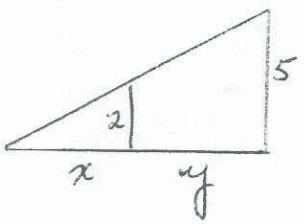
$\frac{dV}{dt} = -2 cm^3/s$

$\frac{dh}{dt} = ? \Big|_{h=3}$

$\frac{dh}{dt} = -\frac{2}{\pi} cm/s$

Height dropping at $\frac{2}{\pi} cm/s$

4.



$\frac{dx}{dt} = \quad m/s \quad \frac{dy}{dt} = -.5 m/s$

$\frac{2}{5} = \frac{x}{x+y}$

$2x + 2y = 5x$

$2y = 3x$

$2\frac{dy}{dt} = 3\frac{dx}{dt}$

$2(-.5) = 3\frac{dx}{dt}$

$-\frac{1}{3} m/s = \frac{dx}{dt}$

The shadow is decreasing at a rate of $\frac{1}{3} m/s$

5. $V = \frac{4}{3}\pi r^3$ $\quad \frac{dV}{dt} = -2\,cm^3/min$ $\quad \frac{dS}{dt} = ?$ $\quad r = 1$

$\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \frac{dr}{dt}$

$-2 = 4\pi(1)^2 \frac{dr}{dt}$

$\frac{-1}{2\pi} = \frac{dr}{dt}$ cm/min

$S = 4\pi r^2$

$\frac{dS}{dt} = 4\pi \cdot 2r \frac{dr}{dt}$

$\frac{dS}{dt} = 4\pi \cdot 2(1)\left(\frac{-1}{2\pi}\right)$

$= -4\,cm^2/min$

Surface area is shrinking by $4\,cm^2/min$

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6. $V = \frac{1}{3}\pi r^2 h$

diameter = 20 cm

radius = 10 cm

$h = 30$ cm

$\frac{r}{h} = \frac{1}{3}$

$3r = h$

$r = \frac{h}{3}$

$V = \frac{1}{3}\pi\left(\frac{h}{3}\right)^2 h$

$V = \frac{\pi}{3} \cdot \frac{h^3}{9}$

$\frac{dV}{dt} = -12\,cm^3/s$ $\quad \frac{dh}{dt} = ?\Big|_{h=20}$

$\frac{dV}{dt} = \pi \cdot \frac{h^2}{9} \frac{dh}{dt}$

$-12 = \pi \frac{(20)^2}{9} \frac{dh}{dt}$

$\frac{(-12)(9)}{400\pi} = \frac{dh}{dt}$

$\frac{dh}{dt} = -\frac{27}{100\pi}$ cm/s

Liquid decreasing at a rate of $\frac{27}{100\pi}$ cm/s

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7) $A = \pi r^2$ $\quad C = 2\pi r$ $\quad \frac{C}{2\pi} = r$

$A = \pi\left(\frac{C}{2\pi}\right)^2$

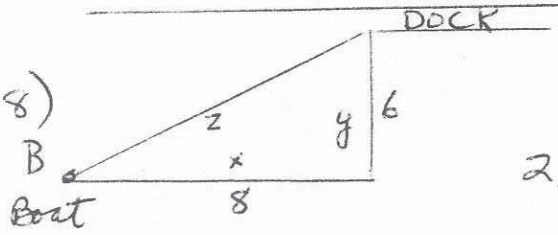
$A = \frac{C^2}{4\pi}$

$\frac{dA}{dC} = \frac{2C}{4\pi} \cdot \frac{dC}{dC}$ ← with respect to circumference

$\frac{dA}{dC} = \frac{C}{2\pi}$ units

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8)



$x^2 + y^2 = z^2$

$z = 10$

$\frac{dx}{dt} = ?$ $\quad \frac{dy}{dt} = 0$ $\quad \frac{dz}{dt} = -3\,m/s$

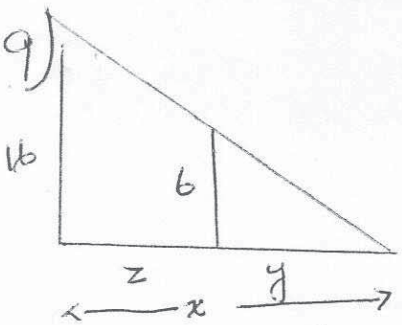
$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$

$8\frac{dx}{dt} = 10(-3)$

$\frac{dx}{dt} = -\frac{30}{8}$ m/s

Boat being pulled in at a rate of $\frac{30}{8}$ m/s

9)



@ Tip of shadow

$\frac{dx}{dt} = ?$ $\frac{dy}{dt} =$ ft/s

$\frac{6}{y} = \frac{16}{x}$

$6x = 16y$

$6\frac{dx}{dt} = 16\frac{dy}{dt}$

$\frac{dz}{dt} = -5$ ft/s $\frac{dy}{dt} = ?$

$\frac{dx}{dt} = \frac{8}{3}(-3) = -8$ ft/s

Tip of shadow decreasing at $\underline{-8 \text{ ft/s}}$

(b) Length of shadow $\frac{dy}{dt}$ must do (b) first

$\frac{6}{y} = \frac{16}{y+z}$

$6y + 6z = 16y$

$6z = 10y$

$6\frac{dz}{dt} = 10\frac{dy}{dt}$

$-\frac{30}{10} = \frac{dy}{dt}$

-3 ft/s

Length of shadow decreasing by 3 ft/s

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10) $V = \frac{\pi}{3}r^2h$

$V = \frac{\pi}{3}\left(\frac{5h}{4}\right)^2 h$

radius $r = 15$

$h = 12$

$\frac{dV}{dt} = 2\text{ m}^3/\text{min}$

$\frac{r}{h} = \frac{5}{4}$

$4r = 5h$ $r = \frac{5}{4}h$

$\frac{dV}{dt} = \frac{\pi}{3}\cdot\frac{25}{16}\cdot 3h^2\frac{dh}{dt}$

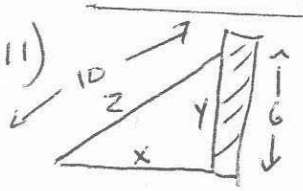
$\frac{dh}{dt} = ?\Big|_{h=8}$

$2 = \frac{\pi}{16}\cdot 25(8)^2\frac{dh}{dt}$

$\frac{dh}{dt} = \frac{1}{50\pi}$ m/s

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11)



$\frac{dz}{dt} = 0$ $\frac{dx}{dt} = ?$ $\frac{dy}{dt} = -2$ m/s $y = 8$ m

$x^2 + y^2 = z^2$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$

$(8)\frac{dx}{dt} + 6(-2) = 0$

$\frac{dx}{dt} = \frac{3}{2}$ m/s

Bottom moving away from wall at rate of $3/2$ m/s

12. $\frac{dV}{dt} = -4\,cm^3/m$ $\quad \frac{ds}{dt} = ?$ $\quad r = 24$

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \frac{dr}{dt}$

$-4 = 4\pi(24)^2 \frac{dr}{dt}$

$\frac{-1}{24^2\pi} = \frac{dr}{dt}$

$S = 4\pi r^2$

$\frac{ds}{dt} = 4\pi \cdot 2r \frac{dr}{dt}$

$= 8\pi(24)\left(\frac{-1}{24^2\pi}\right)$

$= -\frac{1}{3}\,cm^2/min$

Surface area decreasing at rate of $\frac{1}{3}\,cm^2/min$

13. 

$\frac{dr}{dt} = 4\,cm/s$ $\quad \frac{dh}{dt} = ?$ $\quad \frac{ds}{dt} = 0$ $\quad r = 10$

Surface Area $= S = 2\pi r^2 + 2\pi r h$

$\frac{ds}{dt} = 4\pi r \frac{dr}{dt} + 2\pi r \frac{dh}{dt} + 2\pi h \frac{dr}{dt}$

$0 = 4\pi(10)(4) + 2\pi(10)\frac{dh}{dt} + 2\pi(20)(4)$

$= 160\pi + 20\pi \frac{dh}{dt} + 160\pi$

$-320\pi = 20\pi \frac{dh}{dt}$

$-16\,cm/s = \frac{dh}{dt}$

$S = 600\pi =$

$2\pi(100) + 20\pi h$

$400\pi = 20\pi h$

$20 = h$

The height is changing at a rate of $-16\,cm/s$

14. $V = \frac{\pi}{3} r^2 h$ $\quad \frac{dh}{dt} = -1\,cm/hr$ $\quad \frac{dr}{dt} = -1\,cm/hr$

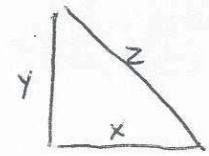
$\frac{dV}{dt} = \frac{\pi}{3} r^2 \frac{dh}{dt} + \frac{\pi}{3} h \cdot 2r \frac{dr}{dt}$

$= \frac{\pi}{3}(100)(-1) + \frac{\pi}{3}(10)(2)(10)(-1)$

$= -100\pi$

Volume is decreasing at the rate of $100\pi\,cm/hr$

15)



$\frac{dx}{dt} = 2\ m/s \quad \frac{dy}{dt} = \text{-ve} \quad \frac{dz}{dt} = 0$

$A = 6 \qquad x = 3,\ y = 4,\ z = 5$

$x^2 + y^2 = z^2$

$A = \frac{1}{2} x \cdot y$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$

$6 = \frac{1}{2}(3)y$

$12 = 3y$

$4 = y$

$3(2) + 4\left(-\frac{8}{3}\right) = 5\frac{dz}{dt}$

$xy = 12$

$6 - \frac{32}{3} = 5\frac{dz}{dt}$

$y\frac{dx}{dt} + x\frac{dy}{dt} = 0$

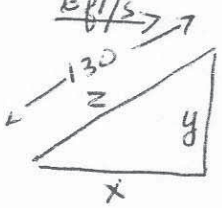
$-\frac{14}{15} = \frac{dz}{dt}$ m/s

$4(2) + 3\left(\frac{dy}{dt}\right) = 0$

$\frac{dy}{dt} = -8/3$

The side y is decreasing at a rate of $\frac{14}{15}$ m/s

16.



$\frac{dx}{dt} = 6\ ft/s \quad \frac{dy}{dt} = 0 \quad \frac{dz}{dt} = ?$

$y = 120, \quad z = 130 \quad x = 50$

$x^2 + y^2 = z^2$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$

$50(6) + 0 = 130\frac{dz}{dt}$

$\frac{dz}{dt} = \frac{300}{130}$ ft/s

String is being let out at a rate of $\frac{30}{13}$ ft/s.

17) $A = \pi r^2$ $\quad \frac{dr}{dt} = .03$ mm/s $\quad \frac{dA}{dt} = ?$

$$\frac{dA}{dt} = \pi \cdot 2r \frac{dr}{dt}$$
$$= \pi(2)(200)(.03)$$
$$= 12\pi \text{ m/s}$$

Area changing at rate 12π mm/s

18) 

$V = \pi r^2 h$ $\quad r = 3,\ h = 4$ $\quad \frac{dh}{dt} = .2$ cm/s $\quad \frac{dV}{dt} = 0$ $\quad \frac{dr}{dt} = ?$

$$\frac{dV}{dt} = \pi r^2 \frac{dh}{dt} + \pi h \cdot 2r \frac{dr}{dt}$$
$$0 = \pi(3)^2(.2) + \pi(4)(2)(3)\left(\frac{dr}{dt}\right)$$
$$0 = 1.8\pi + 24\pi \frac{dr}{dt}$$
$$\frac{dr}{dt} = \frac{-1.8\pi}{24\pi} = \frac{3}{40} \text{ cm/s}$$

Rate of change of radius $\frac{3}{40}$ cm/s.

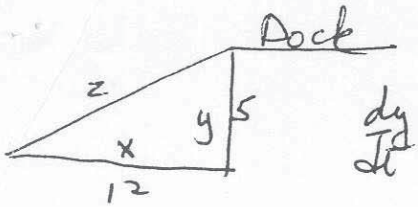
19) $V = \frac{\pi}{3} r^2 h$ $\quad h =$ diameter $\quad h = 2r \quad \frac{h}{2} = r$

$\frac{dV}{dt} = 2 \text{ m}^3/\text{s}$ $\quad \frac{dh}{dt} = ?$ $\quad$ when base is 8, $h = 8$ m

$$V = \frac{\pi}{3}\left(\frac{h}{2}\right)^2 \cdot h$$
$$= \frac{\pi}{3}\frac{h^3}{4} = \frac{\pi h^3}{12}$$
$$\frac{dV}{dt} = \frac{\pi \cdot 3h^2}{12} \cdot \frac{dh}{dt}$$
$$2 = \frac{\pi \cdot 3(8)^2}{12} \frac{dh}{dt}$$
$$\frac{24}{\pi(3)(8)^2} = \frac{1}{8\pi} = \frac{dh}{dt}$$

height increasing at a rate of $\frac{1}{8\pi}$ m/s

20)



$\frac{dy}{dt} = 0 \quad \frac{dz}{dt} = -2 \text{ m/s} \quad \frac{dx}{dt} = ?$

$x^2 + y^2 = z^2 \qquad z = 13$

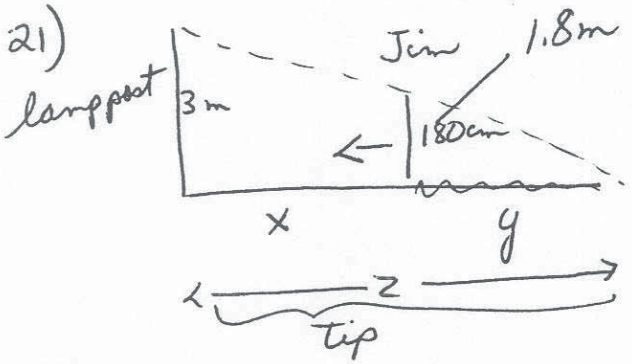
$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$

$(12)\left(\frac{dx}{dt}\right) + 0 = 13(-2)$

$\frac{dx}{dt} = -\frac{26}{12} \text{ m/s}$

Pulled in at rate of $\frac{26}{12}$ or $\underline{\frac{13}{6} \text{ m/s}}$.

21)



$3 \text{ m} = 300 \text{ cm}$

$\frac{dx}{dt} = -2.4 \text{ m/s}$

$\frac{dy}{dt} = ?$

$\frac{dz}{dt}_{(tip)} = ?$

$\frac{y}{x+y} = \frac{1.8}{3}$

$3y = 1.8x + 1.8y$

$1.2y = 1.8x$

$y = \frac{3}{2}x$

$\frac{dy}{dt} = \frac{3}{2}\frac{dx}{dt}$

$= \frac{3}{2}(-2.4)$

$= -3.6 \text{ m/s}$

Shadow decreasing by 3.6 m/s

(b) $\frac{y}{z} = \frac{1.8}{3}$

$3y = 1.8z$

$\frac{3}{1.8}y = z$

$\frac{3}{1.8}\frac{dy}{dt} = \frac{dz}{dt}$

$\frac{3}{1.8}(-3.6) = \frac{dz}{dt}$

-6 m/s

Tip of shadow decreasing by 6 m/s