Chapter 6
Applications of Trigonometry

■ Section 6.1 Vectors in the Plane

Exploration 1

1. Use the HMT rule, which states that if an arrow has initial point (x_1, y_1) and terminal point (x_2, y_2), it represents the vector $\langle x_2 - x_1, y_2 - y_1\rangle$. If the initial point is (2, 3) and the terminal point is (7, 5), the vector is $\langle 7 - 2, 5 - 3\rangle = \langle 5, 2\rangle$.
2. Use the HMT rule, which states that if an arrow has initial point (x_1, y_1) and terminal point (x_2, y_2), it represents the vector $\langle x_2 - x_1, y_2 - y_1\rangle$. If the initial point is (3, 5) and the terminal point is (x_2, y_2), the vector is $\langle x_2 - 3, y_2 - 5\rangle$. Using the given vector $\langle -3, 6\rangle$, we have $x_2 - 3 = -3$ and $y_2 - 5 = 6$.
$x_2 - 3 = -3 \Rightarrow x_2 = 0$; $y_2 - 5 = 6 \Rightarrow y_2 = 11$.
The terminal point is (0, 11).
3. Use the HMT rule, which states that if an arrow has initial point (x_1, y_1) and terminal point (x_2, y_2), it represents the vector $\langle x_2 - x_1, y_2 - y_1\rangle$. If the initial point P is (4, −3) and the terminal point Q is (x_2, y_2), the vector $\overrightarrow{PQ}$ is $\langle x_2 - 4, y_2 - (-3)\rangle$. Using the given vector $\overrightarrow{PQ}\langle 2, -4\rangle$, we have $x_2 - 4 = 2$ and $y_2 + 3 = -4$.
$x_2 - 4 = 2 \Rightarrow x_2 = 6$; $y_2 + 3 = -4 \Rightarrow y_2 = -7$.
The point Q is (6, −7).
4. Use the HMT rule, which states that if an arrow has initial point (x_1, y_1) and terminal point (x_2, y_2), it represents the vector $\langle x_2 - x_1, y_2 - y_1\rangle$. If the initial point P is (x_1, y_1) and the terminal point Q is (4, −3), the vector $\overrightarrow{PQ}$ is $\langle 4 - x_1, -3 - y_1\rangle$. Using the given vector $\overrightarrow{PQ}\langle 2, -4\rangle$, we have $4 - x_1 = 2$ and $-3 - y_1 = -4$.
$4 - x_1 = 2 \Rightarrow x_1 = 2$; $-3 - y_1 = -4 \Rightarrow y_1 = 1$.
The point P is (2, 1).

Quick Review 6.1

1. $x = 9 \cos 30° = \frac{9\sqrt{3}}{2}$, $y = 9 \sin 30° = 4.5$
2. $x = 15 \cos 120° = -7.5$, $y = 15 \sin 120° = \frac{15\sqrt{3}}{2}$
3. $x = 7 \cos 220° \approx -5.36$, $y = 7 \sin 220° \approx -4.5$
4. $x = 6 \cos(-50°) \approx 3.86$, $y = 6 \sin(-50°) \approx -4.60$

For #5 and 6, use a calculator.

5. $\theta \approx 33.85°$
6. $\theta \approx 104.96°$

For #7–9, the angle determined by $P(x, y)$ involves $\tan^{-1}(y/x)$. Since this will always be between −180° and +180°, you may need to add 180° or 360° to put the angle in the correct quadrant.

7. $\theta = \tan^{-1}\left(\frac{9}{5}\right) \approx 60.95°$
8. $\theta = 360° + \tan^{-1}\left(-\frac{7}{5}\right) \approx 305.54°$
9. $\theta = 180° + \tan^{-1}\left(\frac{5}{2}\right) \approx 248.20°$
10. After 3 hours, the ship has traveled (3)(42 sin 40°) naut mi east and (3)(42 cos 40°) naut mi north. Five hours later, it is $(3)(42 \sin 40°) + (5)(42 \sin 125°) \approx 253.013$ naut mi east and $(3)(42 \cos 40°) + (5)(42 \cos 125°) \approx -23.929$ naut mi north (about 23.93 naut mi south) of Port Norfolk.
Bearing: $180° + \tan^{-1}\left(-\frac{253.013}{23.929}\right) \approx 95.40°$
Distance: $\sqrt{(253.013)^2 + (-23.929)^2} \approx 254.14$ naut mi.

Section 6.1 Exercises

For #1–4, recall that two vectors are equivalent if they have the same magnitude and direction. If R has coordinates (a, b) and S has coordinates (c, d), then the magnitude of $\overrightarrow{RS}$ is $|\overrightarrow{RS}| = \sqrt{(c - a)^2 + (d - b)^2} = RS$, the distance from R to S. The direction of $\overrightarrow{RS}$ is determined by the coordinates $(c - a, d - b)$.

1. If $R = (-4, 7)$ and $S = (-1, 5)$, then, using the HMT rule, $\overrightarrow{RS} = \langle -1 - (-4), 5 - 7\rangle = \langle 3, -2\rangle$.
If $P = (0, 0)$ and $Q = (3, -2)$, then, using the HMT rule, $\overrightarrow{PQ} = \langle 3 - 0, -2 - 0\rangle = \langle 3, -2\rangle$.
Both vectors represent $\langle 3, -2\rangle$ by the HMT rule.
2. If $R = (7, -3)$ and $S = (4, -5)$, then, using the HMT rule, $\overrightarrow{RS} = \langle 4 - 7, -5 - (-3)\rangle = \langle -3, -2\rangle$.
If $P = (0, 0)$ and $Q = (-3, -2)$, then, using the HMT rule, $\overrightarrow{PQ} = \langle -3 - 0, -2 - 0\rangle = \langle -3, -2\rangle$.
Both vectors represent $\langle -3, -2\rangle$ by the HMT rule.
3. If $R = (2, 1)$ and $S = (0, -1)$, then, using the HMT rule, $\overrightarrow{RS} = \langle 0 - 2, -1 - 1\rangle = \langle -2, -2\rangle$.
If $P = (1, 4)$ and $Q = (-1, 2)$, then, using the HMT rule, $\overrightarrow{PQ} = \langle -1 - 1, 2 - 4\rangle = \langle -2, -2\rangle$.
Both vectors represent $\langle -2, -2\rangle$ by the HMT rule.
4. If $R = (-2, -1)$ and $S = (2, 4)$, then, using the HMT rule, $\overrightarrow{RS} = \langle 2 - (-2), 4 - (-1)\rangle = \langle 4, 5\rangle$.
If $P = (-3, -1)$ and $Q = (1, 4)$, then, using the HMT rule, $\overrightarrow{PQ} = \langle 1 - (-3), 4 - (-1)\rangle = \langle 4, 5\rangle$.
Both vectors represent $\langle 4, 5\rangle$ by the HMT rule.
5. $\overrightarrow{PQ} = \langle 3 - (-2), 4 - 2\rangle = \langle 5, 2\rangle$,
$|\overrightarrow{PQ}| = \sqrt{5^2 + 2^2} = \sqrt{29}$
6. $\overrightarrow{RS} = \langle 2 - (-2), -8 - 5\rangle = \langle 4, -13\rangle$,
$|\overrightarrow{RS}| = \sqrt{4^2 + (-13)^2} = \sqrt{185}$

7. $\overrightarrow{QR} = \langle -2 - 3, 5 - 4 \rangle = \langle -5, 1 \rangle$,
$|\overrightarrow{QR}| = \sqrt{(-5)^2 + 1^2} = \sqrt{26}$

8. $\overrightarrow{PS} = \langle 2 - (-2), -8 - 2 \rangle = \langle 4, -10 \rangle$,
$|\overrightarrow{PS}| = \sqrt{4^2 + (-10)^2} = \sqrt{116} = 2\sqrt{29}$

9. $2\overrightarrow{QS} = 2\langle 2 - 3, -8 - 4 \rangle = \langle -2, -24 \rangle$,
$|2\overrightarrow{QS}| = \sqrt{(-2)^2 + (-24)^2} = \sqrt{580} = 2\sqrt{145}$

10. $(\sqrt{2})\ \overrightarrow{PR} = \sqrt{2}\ \langle -2 - (-2), 5 - 2 \rangle = \langle 0, 3\sqrt{2} \rangle$,
$|\sqrt{2}\ \overrightarrow{QR}| = \sqrt{0^2 + (3\sqrt{2})^2} = 3\sqrt{2}$

11. $3\overrightarrow{QR} + \overrightarrow{PS} = 3\langle -5, 1 \rangle + \langle 4, -10 \rangle = \langle -11, -7 \rangle$,
$|3\overrightarrow{QR} + \overrightarrow{PS}| = \sqrt{(-11)^2 + (-7)^2} = \sqrt{170}$

12. $\overrightarrow{PS} - 3\overrightarrow{PQ} = \langle 4, -10 \rangle - 3\ \langle 5, 2 \rangle = \langle -11, -16 \rangle$,
$|\overrightarrow{PS} - 3\overrightarrow{PQ}| = \sqrt{(-11)^2 + (-16)^2} = \sqrt{377}$

13. $\langle -1, 3 \rangle + \langle 2, 4 \rangle = \langle 1, 7 \rangle$

14. $\langle -1, 3 \rangle - \langle 2, 4 \rangle = \langle -3, -1 \rangle$

15. $\langle -1, 3 \rangle - \langle 2, -5 \rangle = \langle -3, 8 \rangle$

16. $3\ \langle 2, 4 \rangle = \langle 6, 12 \rangle$

17. $2\ \langle -1, 3 \rangle + 3\ \langle 2, -5 \rangle = \langle 4, -9 \rangle$

18. $2\ \langle -1, 3 \rangle - 4\ \langle 2, 4 \rangle = \langle -10, -10 \rangle$

19. $-2\ \langle -1, 3 \rangle - 3\ \langle 2, 4 \rangle = \langle -4, -18 \rangle$

20. $-\langle -1, 3 \rangle - \langle 2, 4 \rangle = \langle -1, -7 \rangle$

21. $\dfrac{\mathbf{u}}{|\mathbf{u}|} = \left\langle \dfrac{-2}{\sqrt{(-2)^2 + 4^2}}, \dfrac{4}{\sqrt{(-2)^2 + 4^2}} \right\rangle$
$= -\dfrac{1}{\sqrt{5}}\mathbf{i} + \dfrac{2}{\sqrt{5}}\mathbf{j}$

22. $\dfrac{\mathbf{v}}{|\mathbf{v}|} = \left\langle \dfrac{1}{\sqrt{1^2 + (-1)^2}}, \dfrac{-1}{\sqrt{1^2 + (-1)^2}} \right\rangle$
$= \dfrac{1}{\sqrt{2}}\mathbf{i} - \dfrac{1}{\sqrt{2}}\mathbf{j}$

23. $\dfrac{\mathbf{w}}{|\mathbf{w}|} = \left\langle \dfrac{-1}{\sqrt{(-1)^2 + (-2)^2}}, \dfrac{-2}{\sqrt{(-1)^2 + (-2)^2}} \right\rangle$
$= -\dfrac{1}{\sqrt{5}}\mathbf{i} - \dfrac{2}{\sqrt{5}}\mathbf{j}$

24. $\dfrac{\mathbf{w}}{|\mathbf{w}|} = \left\langle \dfrac{5}{\sqrt{5^2 + 5^2}}, \dfrac{5}{\sqrt{5^2 + 5^2}} \right\rangle$
$= \dfrac{1}{\sqrt{2}}\mathbf{i} + \dfrac{1}{\sqrt{2}}\mathbf{j}$

For #25–28, the unit vector in the direction of $\mathbf{v} = \langle a, b \rangle$ is
$\dfrac{1}{|\mathbf{v}|} = \left\langle \dfrac{a}{\sqrt{a^2 + b^2}}, \dfrac{b}{\sqrt{a^2 + b^2}} \right\rangle$
$= \dfrac{a}{\sqrt{a^2 + b^2}}\mathbf{i} + \dfrac{b}{\sqrt{a^2 + b^2}}\mathbf{j}$.

25. **(a)** $\left\langle \dfrac{2}{\sqrt{5}}, \dfrac{1}{\sqrt{5}} \right\rangle$
(b) $\dfrac{2}{\sqrt{5}}\mathbf{i} + \dfrac{1}{\sqrt{5}}\mathbf{j}$

26. **(a)** $\left\langle -\dfrac{3}{\sqrt{13}}, \dfrac{2}{\sqrt{13}} \right\rangle$
(b) $-\dfrac{3}{\sqrt{13}}\mathbf{i} + \dfrac{2}{\sqrt{13}}\mathbf{j}$

27. **(a)** $\left\langle -\dfrac{4}{\sqrt{41}}, -\dfrac{5}{\sqrt{41}} \right\rangle$
(b) $-\dfrac{4}{\sqrt{41}}\mathbf{i} + \left(-\dfrac{5}{\sqrt{41}}\right)\mathbf{j} = -\dfrac{4}{\sqrt{41}}\mathbf{i} - \dfrac{5}{\sqrt{41}}\mathbf{j}$

28. **(a)** $\left\langle \dfrac{3}{5}, -\dfrac{4}{5} \right\rangle$
(b) $\dfrac{3}{5}\mathrm{i} + \left(-\dfrac{4}{5}\right)\mathrm{j} = \dfrac{3}{5}\mathbf{i} - \dfrac{4}{5}\mathbf{j}$

29. $\mathbf{v} = \langle 18 \cos 25°, 18 \sin 25° \rangle \approx \langle 16.31, 7.61 \rangle$

30. $\mathbf{v} = \langle 14 \cos 55°, 14 \sin 55° \rangle \approx \langle 8.03, 11.47 \rangle$

31. $\mathbf{v} = \langle 47 \cos 108°, 47 \sin 108° \rangle \approx \langle -14.52, 44.70 \rangle$

32. $\mathbf{v} = \langle 33 \cos 136°, 33 \sin 136° \rangle \approx \langle -23.74, 22.92 \rangle$

33. $|\mathbf{u}| = \sqrt{3^2 + 4^2} = 5, \alpha = \cos^{-1}\left(\dfrac{3}{5}\right) \approx 53.13°$

34. $|\mathbf{u}| = \sqrt{(-1)^2 + 2^2} = \sqrt{5}, \alpha = \cos^{-1}\left(\dfrac{-1}{\sqrt{5}}\right) \approx 116.57°$

35. $|\mathbf{u}| = \sqrt{3^2 + (-4)^2} = 5, \alpha = 360° - \cos^{-1}\left(\dfrac{3}{5}\right)$
$\approx 306.87°$

36. $|\mathbf{u}| = \sqrt{(-3)^2 + (-5)^2} = \sqrt{34}$,
$\alpha = 360° - \cos^{-1}\left(\dfrac{-3}{\sqrt{34}}\right) \approx 239.04°$

37. Since $(7 \cos 135°)\mathbf{i} + (7 \sin 135°)\mathbf{j} = (|\mathbf{u}| \cos \alpha)\mathbf{i} + (|\mathbf{u}| \sin \alpha)\mathbf{j}$, $|\mathbf{u}| = 7$ and $\alpha = 135°$.

38. Since $(2 \cos 60°)\mathbf{i} + (2 \sin 60°)\mathbf{j} = (|\mathbf{u}| \cos \alpha)\mathbf{i} + (|\mathbf{u}| \sin \alpha)\mathbf{j}$, $|\mathbf{u}| = 2$ and $\alpha = 60°$.

For #39 and 40, first find the unit vector in the direction of **u**. Then multiply by the magnitude of **v**, $|\mathbf{v}|$.

39. $\mathbf{v} = |\mathbf{v}| \cdot \dfrac{\mathbf{u}}{|\mathbf{u}|} = 2\left\langle \dfrac{3}{\sqrt{3^2 + (-3)^2}}, \dfrac{-3}{\sqrt{3^2 + (-3)^2}} \right\rangle$
$= 2\left\langle \dfrac{\sqrt{2}}{2}, \dfrac{-\sqrt{2}}{2} \right\rangle = \langle \sqrt{2}, -\sqrt{2} \rangle$

40. $\mathbf{v} = |\mathbf{v}| \cdot \dfrac{\mathbf{u}}{|\mathbf{u}|} = 5\left\langle \dfrac{-5}{\sqrt{(-5)^2 + 7^2}}, \dfrac{7}{\sqrt{(-5)^2 + 7^2}} \right\rangle$
$\approx 5\langle -0.58, 0.81 \rangle = \langle -2.91, 4.07 \rangle$

41. A bearing of 335° corresponds to a direction angle of 115°. $\mathbf{v} = 530\ \langle \cos 115°, \sin 115° \rangle \approx \langle -223.99, 480.34 \rangle$.

42. A bearing of 170° corresponds to a direction angle of −80°. $\mathbf{v} = 460\ \langle \cos(-80°), \sin(-80°) \rangle \approx \langle 79.88, -453.01 \rangle$.

43. **(a)** A bearing of 340° corresponds to a direction angle of 110°. $\mathbf{v} = 325\ \langle \cos 110°, \sin 110° \rangle \approx \langle -111.16, 305.40 \rangle$.
(b) The wind bearing of 320° corresponds to a direction angle of 130°. The wind vector is $\mathbf{w} = 40\ \langle \cos 130°, \sin 130° \rangle \approx \langle -25.71, 30.64 \rangle$.
Actual velocity vector: $\mathbf{v} + \mathbf{w} \approx \langle -136.87, 336.04 \rangle$.
Actual speed: $\|\mathbf{v} + \mathbf{w}\| \approx \sqrt{136.87^2 + 336.04^2} \approx$ 362.84 mph.

Actual direction: $\theta = 180° + \tan^{-1}\left(\frac{336.04}{-136.87}\right) \approx$ 112.16°, so the bearing is about 337.84°.

44. (a) A bearing of 170° corresponds to a direction angle of −80°:
$\mathbf{v} = 460\langle\cos(-80°), \sin(-80°)\rangle \approx \langle 79.88, -453.01\rangle$.

(b) The wind bearing of 200° corresponds to a direction angle of −110°. The wind vector is $\mathbf{w} = 80\langle\cos(-110°), \sin(-110°)\rangle \approx \langle -27.36, -75.18\rangle$.
Actual velocity vector: $\mathbf{v} + \mathbf{w} \approx \langle 52.52, -528.19\rangle$.
Actual speed: $\|\mathbf{v} + \mathbf{w}\| \approx \sqrt{52.52^2 + 528.19^2} \approx$ 530.79 mph.
Actual direction: $\theta = 180° + \tan^{-1}\left(-\frac{52.52}{528.19}\right) \approx$ −84.32°, so the bearing is about 174.32°.

45. (a) $\mathbf{v} = 10\langle\cos 70°, \sin 70°\rangle \approx \langle 3.42, 9.40\rangle$

(b) The horizontal component is the (constant) horizontal speed of the basketball as it travels toward the basket. The vertical component is the vertical velocity of the basketball, affected by both the initial speed and the downward pull of gravity.

46. (a) $\mathbf{v} = 2.5\langle\cos 15°, \sin 15°\rangle \approx \langle 2.41, 0.65\rangle$

(b) The horizontal component is the force moving the box forward. The vertical component is the force moving the box upward against the pull of gravity.

47. We need to choose $\mathbf{w} = \langle a, b\rangle = k\langle\cos 33°, \sin 33°\rangle$, so that $k\cos(33° - 15°) = k\cos 18° = 2.5$. (Redefine "horizontal" to mean the parallel to the inclined plane; then the towing vector makes an angle of 18° with the "horizontal.") Then $k = \frac{2.5}{\cos 18°} \approx 2.63$ lb, so that $\mathbf{w} \approx \langle 2.20, 1.43\rangle$.

48. Juana's force can be represented by $23\langle\cos 18°, \sin 18°\rangle \approx \langle 21.87, 7.11\rangle$, while Diego's force is $27\langle\cos(-15)°, \sin(-15°)\rangle \approx \langle 26.08, -6.99\rangle$. Their total force is therefore $\langle 47.95, 0.12\rangle$, so Corporal must be pulling with an equal force in the opposite direction: $\langle -47.95, -0.12\rangle$. The magnitude of Corporal's force is about 47.95 lb.

49. $\mathbf{F} = \langle 50\cos 45°, 50\sin 45°\rangle + \langle 75\cos(-30°), 75\sin(-30°)\rangle \approx \langle 100.31, -2.14\rangle$, so $|\mathbf{F}| \approx 100.33$ lb and $\theta \approx -1.22°$.

50. $\mathbf{F} = 100\langle\cos 50°, \sin 50°\rangle + 50\langle\cos 160°, \sin 160°\rangle + 80\langle\cos(-20), \sin(-20°)\rangle \approx \langle 92.47, 66.34\rangle$, so $|\mathbf{F}| \approx 113.81$ lb and $\theta \approx 35.66°$.

51. Ship heading: $\langle 12\cos 90°, 12\sin 90°\rangle = \langle 0, 12\rangle$
Current heading: $\langle 4\cos 225°, 4\sin 225°\rangle \approx \langle -2.83, -2.83\rangle$
The ship's actual velocity vector is $\langle -2.83, 9.17\rangle$, so its speed is $\approx \sqrt{(-2.83)^2 + 9.17^2} \approx 9.6$ mph and the direction angle is $\cos^{-1}\left(\frac{-2.83}{9.6}\right) \approx 107.14°$, so the bearing is about 342.86°.

52. Let $\mathbf{v} = \langle 0, 20\rangle$ be the velocity of the boat and $\mathbf{w} = \langle 8, 0\rangle$ be the velocity vector of the current. If the boat travels t minutes to reach the opposite shore, then its position, in vector form, must be $\langle 0, 20t\rangle + \langle 8t, 0\rangle = \langle 8t, 20t\rangle = \langle 8t, 1\rangle$.
So $20t = 1 \Rightarrow t = \frac{1}{20} \Rightarrow 8t = 0.4$ mi. The boat meets the shore 0.4 mi downstream.

53. Let w be the speed of the ship. The ship's velocity (in still water) is $\langle w\cos 270°, w\sin 270°\rangle = \langle 0, -w\rangle$. Let z be the speed of the current. Then, the current velocity is $\langle z\cos 135°, z\sin 135°\rangle \approx \langle -0.71z, 0.71z\rangle$. The position of the ship after two hours is $\langle 20\cos 240°, 20\sin 240°\rangle \approx \langle -10, -17.32\rangle$. Putting all this together we have:
$2\langle 0, -w\rangle + 2\langle -0.71z, 0.71z\rangle = \langle -10, -17.32\rangle$,
$\langle -1.42z, -2w + 1.42z\rangle = \langle -10, -17.32\rangle$, so $z \approx 7.07$ and $w \approx 13.66$. The speed of the ship is about 13.66 mph, and the speed of the current is about 7.07 mph.

54. Let $\mathbf{u} = \langle u_1, u_2\rangle$, $\mathbf{v} = \langle v_1, v_2\rangle$, and $\mathbf{w} = \langle w_1, w_2\rangle$.

(a) $\mathbf{u} + \mathbf{v} = \langle u_1, u_2\rangle + \langle v_1, v_2\rangle = \langle u_1 + v_1, u_2 + v_2\rangle$
$= \langle v_1 + u_1, v_2 + u_2\rangle = \langle v_1, v_2\rangle + \langle u_1, u_2\rangle = \mathbf{v} + \mathbf{u}$

(b) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \langle u_1 + v_1, u_2 + v_2\rangle + \langle w_1, w_2\rangle$
$= \langle u_1 + v_1 + w_1, u_2 + v_2 + w_2\rangle$
$= \langle u_1, u_2\rangle + \langle v_1 + w_1, v_2 + w_2\rangle$
$= \mathbf{u} + (\mathbf{v} + \mathbf{w})$

(c) $\mathbf{u} + \mathbf{0} = \langle u_1, u_2\rangle + \langle 0, 0\rangle = \langle u_1 + 0, u_2 + 0\rangle$
$= \langle u_1, u_2\rangle = \mathbf{u}$

(d) $\mathbf{u} + (-\mathbf{u}) = \langle u_1, u_2\rangle + \langle -u_1, -u_2\rangle$
$= \langle u_1 + (-u_1), u_2 + (-u_2)\rangle = \langle 0, 0\rangle = \mathbf{0}$

(e) $a(\mathbf{u} + \mathbf{v}) = a\langle u_1 + v_1, u_2 + v_2\rangle$
$= \langle a(u_1 + v_1), a(u_2 + v_2)\rangle$
$= \langle au_1 + av_1, au_2 + av_2\rangle = \langle au_1, au_2\rangle + \langle av_1, av_2\rangle$
$= a\langle u_1, u_2\rangle + a\langle v_1, v_2\rangle = a\mathbf{u} + a\mathbf{v}$

(f) $(a + b)\mathbf{u} = \langle (a + b)u_1, (a + b)u_2\rangle$
$= \langle au_1 + bu_1, au_2 + bu_2\rangle$
$= \langle au_1, au_2\rangle + \langle bu_1, bu_2\rangle = a\langle u_1, u_2\rangle + b\langle u_1, u_2\rangle$
$= a\mathbf{u} + b\mathbf{u}$

(g) $(ab)\mathbf{u} = \langle (ab)u_1, (ab)u_2\rangle = \langle a(bu_1), a(bu_2)\rangle$
$= a\langle bu_1, bu_2\rangle = a(b\mathbf{u})$

(h) $a\mathbf{0} = a\langle 0, 0\rangle = \langle a0, a0\rangle = \langle 0, 0\rangle = \mathbf{0}$
$0\mathbf{u} = 0\langle u_1, u_2\rangle = \langle 0u_1, 0u_2\rangle = \langle 0, 0\rangle = \mathbf{0}$

(i) $(1)\mathbf{u} = \langle (1)u_1, (1)u_2\rangle = \langle u_1, u_2\rangle = \mathbf{u}$
$(-1)\mathbf{u} = \langle (-1)u_1, (-1)u_2\rangle = \langle -u_1, -u_2\rangle = -\mathbf{u}$

(j) $|a\mathbf{u}| = |\langle au_1, au_2\rangle| = \sqrt{(au_1)^2 + (au_2)^2}$
$= \sqrt{a^2u_1^2 + a^2u_2^2} = \sqrt{a^2(u_1^2 + u_2^2)}$
$= |a|\sqrt{u_1^2 + u_2^2} = |a|\,|\mathbf{u}|$

55. True. Vectors $\mathbf{u}$ and $-\mathbf{u}$ have the same length but opposite directions. Thus, the length of $-\mathbf{u}$ is also 1.

56. False. $1/\mathbf{u}$ is not a vector at all.

57. $|\langle 2, -1\rangle| = \sqrt{2^2 + (-1)^2} = \sqrt{5}$
The answer is D.

58. $\mathbf{u} - \mathbf{v} = \langle -2, 3\rangle - \langle 4, -1\rangle$
$= \langle -2 - 4, 3 - (-1)\rangle$
$= \langle -6, 4\rangle$
The answer is E.

59. The x-component is 3 cos 30°, and the y-component is 3 sin 30°. The answer is A.

60. $|\mathbf{v}| = \sqrt{(-1)^2 + 3^2} = \sqrt{10}$, so the unit vector is $\langle -1, 3\rangle/\sqrt{10}$. The answer is C.

61. (a) Let A be the point (a_1, a_2) and B be the point (b_1, b_2). Then $\overrightarrow{OA}$ is the vector $\langle a_1, a_2\rangle$ and $\overrightarrow{OB}$ is the vector $\langle b_1, b_2\rangle$.

So, $\overrightarrow{BA} = \langle a_1 - b_1, a_2 - b_2\rangle$
$= \langle a_1, a_2\rangle - \langle b_1, b_2\rangle$
$= \overrightarrow{OA} - \overrightarrow{OB}$

(b) $x\,\overrightarrow{OA} + y\,\overrightarrow{OB} = x(\overrightarrow{OC} + \overrightarrow{CA}) + y(\overrightarrow{OC} + \overrightarrow{CB})$ (from **(a)**)

$= x\,\overrightarrow{OC} + x\,\overrightarrow{CA} + y\,\overrightarrow{OC} + y\,\overrightarrow{CB}$

$= (x + y)\,\overrightarrow{OC} + x\,\overrightarrow{CA} + y\,\overrightarrow{CB}$

$= \overrightarrow{OC} + x\,\overrightarrow{CA} + y\,\overrightarrow{CB}$ (since $x + y = 1$)

$=\overrightarrow{OC} + y\dfrac{|\overrightarrow{BC}|}{|\overrightarrow{CA}|}\cdot\overrightarrow{CA} + y\,\overrightarrow{CB}$ since $\left(x = y\dfrac{|BC|}{|CA|}\right)$

$= \overrightarrow{OC} + y\left(|\overrightarrow{BC}|\cdot\dfrac{\overrightarrow{CA}}{|\overrightarrow{CA}|} + \overrightarrow{CB}\right)$

$\dfrac{\overrightarrow{CA}}{|\overrightarrow{CA}|}$ is a unit vector, and $\overrightarrow{BC}$ points in the same direction.

$= \overrightarrow{OC} + y\left(|\overrightarrow{BC}|\cdot\dfrac{\overrightarrow{BC}}{|\overrightarrow{BC}|} + \overrightarrow{CB}\right)$

$= \overrightarrow{OC} + y\,(\overrightarrow{BC} + \overrightarrow{CB})$

$= \overrightarrow{OC}$

62. (a) By Exercise 61, $\overrightarrow{OM_1} = x\,\overrightarrow{OA} + y\,\overrightarrow{OB}$, where $x + y = 1$. Since M_1 is the midpoint, $|\overrightarrow{BM_1}| = |\overrightarrow{M}_{1A}|$. We know from Exercise 61, however, that $\dfrac{|\overrightarrow{BM_1}|}{|\overrightarrow{M}_{1A}|} = 1 = \dfrac{x}{y}$. So $x = y = \dfrac{1}{2}$. As a result, $\overrightarrow{OM_1} = \dfrac{1}{2}\,\overrightarrow{OA} + \dfrac{1}{2}\,\overrightarrow{OB}$. The proof for $\overrightarrow{OM_2}$ and $\overrightarrow{OM_3}$ are similar.

(b) $2\,\overrightarrow{OM_1} + \overrightarrow{OC} = 2\left(\dfrac{1}{2}\overrightarrow{OA} + \dfrac{1}{2}\overrightarrow{OB}\right) + \overrightarrow{OC}$

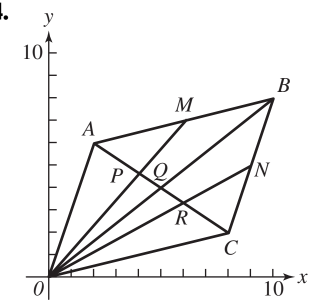
$= \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$. Use the same method for the other proofs.

(c) **(b)** implies that $2\,\overrightarrow{OM_1} + \overrightarrow{OC} = 2\,\overrightarrow{OM_2} + \overrightarrow{OA}$ $= 2\,\overrightarrow{OM_3} + \overrightarrow{OB}$. Each of the three vectors lies along a different median (that is, if nonzero, the three vectors have three different directions). Hence they can only be equal if all are equal to **0**. Thus $2\overrightarrow{OM_1} = -\overrightarrow{OC}$, $2\overrightarrow{OM_2} = -\overrightarrow{OA}$, and $2\overrightarrow{OM_3} = -\overrightarrow{OB}$, so $\dfrac{|\overrightarrow{OM_1}|}{|\overrightarrow{OC}|} = \dfrac{|\overrightarrow{OM_2}|}{|\overrightarrow{OA}|} = \dfrac{|\overrightarrow{OM_3}|}{|\overrightarrow{OB}|} = \dfrac{1}{2}$.

63. Use the result of Exercise 61. First we show that if C is on the line segment AB, then there is a real number t so that $\dfrac{|\overrightarrow{BC}|}{|\overrightarrow{CA}|} = \dfrac{t}{1 - t}$. (Convince yourself that $t = \dfrac{BC}{BC + CA}$ works.) Then $\overrightarrow{OC} = t\,\overrightarrow{OA} + (1 - t)\,\overrightarrow{OB}$. A similar argument can be used in the cases where B is on the line segment AC or A is on the line segment BC.

Suppose there is a real number t so that $\overrightarrow{OC} = t\,\overrightarrow{OA} + (1 - t)\,\overrightarrow{OB}$. We also know $\overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{BC}$ and $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC}$. So we have $t\,\overrightarrow{OA} + (1 - t)\overrightarrow{OB} = \overrightarrow{OB} + \overrightarrow{BC}$ and $t\,\overrightarrow{OA} + (1 - t)\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AC}$. Therefore, $t\,(\overrightarrow{OA} - \overrightarrow{OB}) = \overrightarrow{BC}$ and $(t - 1)(\overrightarrow{OA} - \overrightarrow{OB}) = \overrightarrow{AC}$. Hence $\overrightarrow{BC}$ and $\overrightarrow{AC}$ have the same or opposite directions, so C must lie on the line L through the two points A and B.

64.



The line segment OM is a median of $\triangle ABO$ since M is a midpoint of AB. The line segment AQ is a median of $\triangle ABO$ since diagonals of a parallelogram bisect each other. By the result of Exercise 62, since P is the intersection point of two medians, $\dfrac{AP}{PQ} = \dfrac{2}{1}$. Similarly, $\dfrac{CR}{RQ} = \dfrac{2}{1}$. This implies that $AP = PR = RC$, so the diagonal has been trisected.

■ Section 6.2 Dot Product of Vectors

Exploration 1

1. $\mathbf{u} = \langle -2 - x, 0 - y\rangle = \langle -2 - x, -y\rangle$
   $\mathbf{v} = \langle 2 - x, 0 - y\rangle = \langle 2 - x, -y\rangle$
2. $\mathbf{u}\cdot\mathbf{v} = (-2 - x)(2 - x) + (-y)(-y) = -4 + x^2 + y^2$
   $= -4 + 4 = 0$
   Therefore, $\theta = 90°$.
3. Answers will vary.

Quick Review 6.2

1. $|\mathbf{u}| = \sqrt{2^2 + (-3)^2} = \sqrt{13}$
2. $|\mathbf{u}| = \sqrt{(-3)^2 + (-4)^2} = 5$
3. $|\mathbf{u}| = \sqrt{\cos^2 35° + \sin^2 35°} = 1$
4. $|\mathbf{u}| = 2\sqrt{\cos^2 75° + \sin^2 75°} = 2$
5. $\overrightarrow{AB} = \langle 1 - (-2), \sqrt{3} - 0\rangle = \langle 3, \sqrt{3}\rangle$
6. $\overrightarrow{AB} = \langle 1 - 2, \sqrt{3} - 0\rangle = \langle -1, \sqrt{3}\rangle$
7. $\overrightarrow{AB} = \langle 1 - 2, -\sqrt{3} - 0\rangle = \langle -1, -\sqrt{3}\rangle$
8. $\overrightarrow{AB} = \langle 1 - (-2), -\sqrt{3} - 0\rangle = \langle 3, -\sqrt{3}\rangle$
9. $\mathbf{u} = |\mathbf{u}|\cdot\dfrac{\mathrm{v}}{|v|} = \dfrac{2\cdot \text{k}2, 3\text{l}}{\sqrt{2^2 + 3^2}} = \dfrac{\text{k}4, 6\text{l}}{\sqrt{13}}$
   $= \left\langle \dfrac{4}{\sqrt{13}}, \dfrac{6}{\sqrt{13}}\right\rangle$
10. $\mathbf{u} = |\mathbf{u}|\cdot\dfrac{\mathrm{v}}{|v|} = \dfrac{3\cdot \text{k}{-4}, 3\text{l}}{\sqrt{(-4)^2 + 3^2}} = \dfrac{\text{k}{-12}, 9\text{l}}{5}$
    $= \left\langle -\dfrac{12}{5}, \dfrac{9}{5}\right\rangle$

Section 6.2 Exercises

1. $60 + 12 = 72$

2. $-40 + 26 = -14$

3. $-12 - 35 = -47$

4. $10 - 56 = -46$

5. $12 + 18 = 30$

6. $-16 - 28 = -44$

7. $-14 + 0 = -14$

8. $0 + 33 = 33$

9. $|\mathbf{u}| = \sqrt{\mathrm{u} \cdot \mathrm{u}} = \sqrt{25 + 144} = 13$

10. $|\mathbf{u}| = \sqrt{\mathrm{u} \cdot \mathrm{u}} = \sqrt{64 + 225} = 17$

11. $|\mathbf{u}| = \sqrt{\mathrm{u} \cdot \mathrm{u}} = \sqrt{16} = 4$

12. $|\mathbf{u}| = \sqrt{\mathrm{u} \cdot \mathrm{u}} = \sqrt{9} = 3$

13. $\mathbf{u} \cdot \mathbf{v} = 4 - 15 = -11, |\mathbf{u}| = \sqrt{16 + 9} = 5,$

$|\mathbf{v}| = \sqrt{1 + 25} = \sqrt{26}, \theta = \cos^{-1}\left(\frac{-11}{5\sqrt{26}}\right) \approx 115.6°$

14. $\mathbf{u} \cdot \mathbf{v} = -6 + 6 = 0, \theta = 90°$

15. $\mathbf{u} \cdot \mathbf{v} = -6 + 15 = 9,$

$|\mathbf{u}| = \sqrt{4 + 9} = \sqrt{13},$

$|\mathbf{u}| = \sqrt{9 + 25} = \sqrt{34},$

$\theta = \cos^{-1}\left(\frac{9}{\sqrt{13} \cdot \sqrt{34}}\right) \approx 64.65°$

16. $\mathbf{u} \cdot \mathbf{v} = -30 - 2 = -32, |\mathbf{u}| = \sqrt{25 + 4} = \sqrt{29},$

$|\mathrm{v}| = \sqrt{36 + 1} = \sqrt{37},$

$\theta = \cos^{-1}\left(\frac{-32}{\sqrt{29} \cdot \sqrt{37}}\right) \approx 167.66°$

17. $\mathbf{u} \cdot \mathbf{v} = -6 - 6\sqrt{3}, |\mathrm{u}| = \sqrt{9 + 9} = \sqrt{18},$

$|\mathbf{v}| = \sqrt{4 + 12} = \sqrt{16} = 4,$

$\theta = \cos^{-1}\left(\frac{-6 - 6\sqrt{3}}{\sqrt{18} \cdot 4}\right) = 165°$

18. $\mathbf{u} \cdot \mathbf{v} = 0, \theta = 90°$

19. **u** has direction angle $\frac{\pi}{4}$ and **v** has direction angle $\frac{3\pi}{2}$ $\left(\text{which is equivalent to } -\frac{\pi}{2}\right)$, so the angle between the vectors is $\frac{\pi}{4} - \left(-\frac{\pi}{2}\right) = \frac{3\pi}{4}$ or 135°.

20. **u** has direction angle $\frac{\pi}{3}$ and **v** has direction angle $\frac{5\pi}{6}$, so the angle between the vectors is $\frac{5\pi}{6} - \frac{p}{3} = \frac{\pi}{2}$ or 90°.

21. $\mathbf{u} \cdot \mathbf{v} = -24 + 20 = -4, |\mathrm{u}| = \sqrt{64 + 25} = \sqrt{89},$

$|\mathrm{v}| = \sqrt{9 + 16} = 5,$

$\theta = \cos^{-1}\left(\frac{-4}{5\sqrt{89}}\right) \approx 94.86°$

22. $\mathbf{u} \cdot \mathbf{v} = 3 - 72 = -69, |\mathrm{u}| = \sqrt{9 + 64} = \sqrt{73},$

$|\mathrm{v}| = \sqrt{1 + 81} = \sqrt{82},$

$\theta = \cos^{-1}\left(\frac{-69}{\sqrt{73} \cdot \sqrt{82}}\right) \approx 153.10°$

23. $\mathbf{u} \cdot \mathbf{v} = \langle 2, 3 \rangle \cdot \left\langle \frac{3}{2}, -1 \right\rangle = 2\left(\frac{3}{2}\right) + 3(-1)$

$= 3 - 3 = 0$

Since $\mathbf{u} \cdot \mathbf{v} = 0$, **u** and **v** are orthogonal.

24. $\mathbf{u} \cdot \mathbf{v} = \langle -4, -1 \rangle \cdot \langle 1, -4 \rangle = -4(1) + (-1)(-4)$

$= -4 + 4 = 0$

Since $\mathbf{u} \cdot \mathbf{v} = 0$, **u** and **v** are orthogonal.

For #25–28, first find $\text{proj}_{\mathbf{v}}\mathbf{u}$. Then use the fact that $\mathbf{u} \cdot \mathbf{v} = 0$ when **u** and **v** are orthogonal.

25. $\text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\langle -8, 3 \rangle \cdot \langle -6, -2 \rangle}{36 + 4}\right)\langle -6, -2 \rangle$

$= \left(\frac{42}{40}\right)\langle -, -2 \rangle = \frac{21}{20}\langle -6, -2 \rangle$

$= -\frac{21}{10}\langle 3, 1 \rangle$

$\mathbf{u} = -\frac{21}{10}\langle 3, 1 \rangle + \frac{17}{10}\langle -1, 3 \rangle$

26. $\text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\langle 3, -7 \rangle \cdot \langle -2, -6 \rangle}{4 + 36}\right)\langle -2, -6 \rangle$

$= \left(\frac{36}{40}\right)\langle -2, -6 \rangle = -\frac{9}{5}\langle 1, 3 \rangle$

$\mathbf{u} = -\frac{9}{5}\langle 1, 3 \rangle + \frac{8}{5}\langle 3, -1 \rangle$

27. $\text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\langle 8, 5 \rangle \cdot \langle -9, -2 \rangle}{81 + 4}\right)\langle -9, -2 \rangle$

$= \left(\frac{-82}{85}\right)\langle -9, -2 \rangle = \frac{82}{85}\langle 9, 2 \rangle$

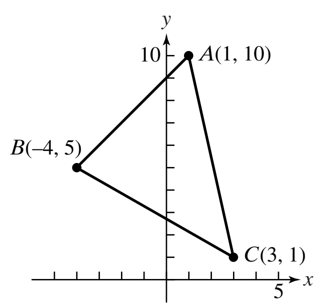
$\mathbf{u} = \frac{82}{85}\langle 9, 2 \rangle + \frac{29}{85}\langle -2, 9 \rangle$

28. $\text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\langle -2, 8 \rangle \cdot \langle 9, -3 \rangle}{81 + 9}\right)\langle 9, -3 \rangle$

$= \left(\frac{-42}{90}\right)\langle 9, -3 \rangle = \frac{7}{5}\langle -3, 1 \rangle$

$\mathbf{u} = \frac{7}{5}\langle -3, 1 \rangle + \frac{1}{5}\langle 11, 33 \rangle$

29.



$\overrightarrow{CA} \cdot \overrightarrow{CB} = \langle 1 - 3, 10 - 1 \rangle \cdot \langle -4 - 3, 5 - 1 \rangle$

$= \langle -2, 9 \rangle \cdot \langle -7, 4 \rangle = 14 + 36 = 50,$

$|\overrightarrow{CA}| = \sqrt{4 + 81} = \sqrt{85}, |\overrightarrow{CB}| = \sqrt{49 + 16} = \sqrt{65},$

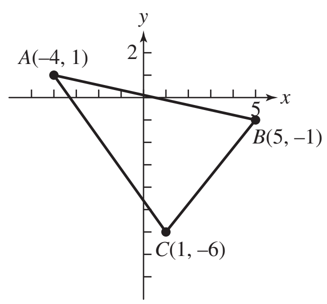
$C = \cos^{-1}\left(\frac{40}{\sqrt{85} \cdot \sqrt{65}}\right) \approx 47.73°$

$\overrightarrow{BC} \cdot \overrightarrow{BA} = \langle 7, -4 \rangle \cdot \langle 1 - (-4), 10 - 5 \rangle$

$= \langle 7, -4 \rangle \cdot \langle 5, 5 \rangle = 35 - 20 = 15,$

$|\overrightarrow{BC}| = \sqrt{65}, |\overrightarrow{BA}| = \sqrt{50}, B = \cos^{-1}\left(\frac{15}{\sqrt{65} \cdot \sqrt{50}}\right)$
$\approx 74.74°$
$A = 180° - B - C \approx 180° - 74.74° - 47.73° = 57.53°$

30.



$\overrightarrow{CA} \cdot \overrightarrow{CB} = \langle -4 - 1, 1 - (-6)\rangle \cdot \langle 5 - 1, -1 - (-6)\rangle$
$= \langle -5, 7\rangle \cdot \langle 4, 5\rangle = -20 + 35 = 15,$
$|\overrightarrow{CA}| = \sqrt{25 + 49} = \sqrt{74}, |\overrightarrow{CB}| = \sqrt{16 + 25} = \sqrt{41},$
$C = \cos^{-1}\left(\frac{15}{\sqrt{74} \cdot \sqrt{41}}\right) \approx 74.20°$
$\overrightarrow{BA} \cdot \overrightarrow{BC} = \langle -4 - 5, 1 - (-1)\rangle \cdot \langle 1 - 5, -6 - (-1)\rangle$
$= \langle -9, 2\rangle \cdot \langle -4, -5\rangle = 36 - 10 = 26,$
$|\overrightarrow{BA}| = \sqrt{81 + 4} = \sqrt{85}, |\overrightarrow{BC}| = \sqrt{41},$
$B = \cos^{-1}\left(\frac{26}{\sqrt{85} \cdot \sqrt{41}}\right) \approx 63.87°$
$A = 180° - B - C \approx 180° - 63.87° - 74.20° = 41.93°$

For #31 and 32, use the relationship $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos \theta$.

31. $\mathbf{u} \cdot \mathbf{v} = 3 \cdot 8 \cos 150° \approx -20.78$

32. $\mathbf{u} \cdot \mathbf{v} = 12 \cdot 40 \cos\left(\frac{\pi}{3}\right) = 240$

For #33–38, vectors are orthogonal if $\mathbf{u} \cdot \mathbf{v} = \mathbf{0}$ and are parallel if $\mathbf{u} = k\mathbf{v}$ for some constant k.

33. Parallel: $-2\left\langle -\frac{10}{4}, -\frac{3}{2}\right\rangle = \left\langle \frac{10}{2}, 3\right\rangle = \langle 5, 3\rangle$

34. Neither: $\mathbf{u} \cdot \mathbf{v} = \frac{40}{3} \neq 0$ and
$\frac{3}{5}\mathbf{v} = \frac{3}{5}\left\langle \frac{10}{3}, \frac{4}{3}\right\rangle\left\langle 2, \frac{4}{5}\right\rangle \neq \mathbf{u}$

35. Neither: $\mathbf{u} \cdot \mathbf{v} = -120 \neq 0$ and
$\frac{-15}{4}\mathbf{v} = \frac{-15}{4}\langle -4, 5\rangle\left\langle 15, \frac{-75}{4}\right\rangle \neq \mathbf{u}$

36. Orthogonal: $\mathbf{u} \cdot \mathbf{v} = -60 + 60 = 0$

37. Orthogonal: $\mathbf{u} \cdot \mathbf{v} = -60 + 60 = 0$

38. Parallel: $-\frac{1}{2}\mathbf{v} = -\frac{1}{2}\langle -4, 14\rangle = \langle 2, -7\rangle = \mathbf{u}$

For #39–42 **(b)**, first find the direction(s) of $\overrightarrow{AP}$ and then find the unit vectors. Then find P by adding the coordinates of A to the components of a unit vector.

39. (a) A is $(4, 0)$ and B is $(0, -3)$.

(b) The line is parallel to
$\overrightarrow{AB} = \langle 0 - 4, -3 - 0\rangle = \langle -4, -3\rangle$, so the direction of $\overrightarrow{AP}$ is $\mathbf{u} = \langle 3, -4\rangle$ or $\mathbf{v} = \langle -3, 4\rangle$.
$\overrightarrow{AP} = \frac{\mathbf{u}}{|\mathbf{u}|} = \frac{\langle 3, -4\rangle}{\sqrt{9 + 16}} = \left\langle \frac{3}{5}, -\frac{4}{5}\right\rangle$ or
$\overrightarrow{AP} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\langle -3, 4\rangle}{\sqrt{9 + 16}} = \left\langle -\frac{3}{5}, \frac{4}{5}\right\rangle.$
So, P is $(4.6, -0.8)$ or $(3.4, 0.8)$.

40. (a) A is $(-5, 0)$ and B is $(0, 2)$.

(b) The line is parallel to
$\overrightarrow{AB} = \langle 0 - (-5), 2 - 0\rangle = \langle 5, 2\rangle$, so the direction of $\overrightarrow{AP}$ is $\mathbf{u} = \langle -2, 5\rangle$ or $\mathbf{v} = \langle 2, -5\rangle$.
$\overrightarrow{AP} = \frac{\mathbf{u}}{|\mathbf{u}|} = \frac{\langle -2, 5\rangle}{\sqrt{4 + 25}} = \left\langle -\frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}}\right\rangle$ or
$\overrightarrow{AP} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\langle 2, -5\rangle}{\sqrt{29}} = \left\langle \frac{2}{\sqrt{29}}, -\frac{5}{\sqrt{29}}\right\rangle.$
So P is $\left(-5 - \frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}}\right) \approx (-5.37, 0.93)$ or
$\left(-5 + \frac{2}{\sqrt{29}}, -\frac{5}{\sqrt{29}}\right) \approx (-4.63, -0.93).$

41. (a) A is $(7, 0)$ and B is $(0, -3)$.

(b) The line is parallel to
$\overrightarrow{AB} = \langle 0 - 7, -3 - 0\rangle = \langle -7, -3\rangle$, so the direction of $\overrightarrow{AP}$ is $\mathbf{u} = \langle 3, -7\rangle$ or $\mathbf{v} = \langle -3, 7\rangle$.
$\overrightarrow{AP} = \frac{\mathbf{u}}{|\mathbf{u}|} = \frac{\langle 3, -7\rangle}{\sqrt{9 + 49}} = \left\langle \frac{3}{\sqrt{58}}, -\frac{7}{\sqrt{58}}\right\rangle$ or
$\overrightarrow{AP} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\langle -3, 7\rangle}{\sqrt{58}} = \left\langle -\frac{3}{\sqrt{58}}, \frac{7}{\sqrt{58}}\right\rangle.$
So P is $\left(7 + \frac{3}{\sqrt{58}}, -\frac{7}{\sqrt{58}}\right) \approx (-7.39, 0.92)$ or
$\left(7 - \frac{3}{\sqrt{58}}, \frac{7}{\sqrt{58}}\right) \approx (6.61, 0.92).$

42. (a) A is $(6, 0)$ and B is $(0, 3)$.

(b) The line is parallel to
$\overrightarrow{AB} = \langle 0 - 6, 3 - 0\rangle = \langle -6, 3\rangle$, so the direction of $\overrightarrow{AP}$ is $\mathbf{u} = \langle 3, 6\rangle$ or $\mathbf{v} = \langle -3, -6\rangle$.
$\overrightarrow{AP} = \frac{\mathbf{u}}{|\mathbf{u}|} = \frac{\langle 3, 6\rangle}{\sqrt{9 + 36}} = \frac{\langle 3, 6\rangle}{3\sqrt{5}} = \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right\rangle$ or
$\overrightarrow{AP} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\langle -3, -6\rangle}{\sqrt{9 + 36}} = \frac{\langle -3, -6\rangle}{3\sqrt{5}}$
$= \left\langle -\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right\rangle$. So P is $\left(6 + \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$
$\approx (6.45, 0.89)$ or $\left(6 - \frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right) \approx (5.55, -0.89).$

43. $2v_1 + 3v_2 = 10, v_1^2 + v_2^2 = 17$. Since $v_1 = 5 - \frac{3}{2}v_2$,
$\left(5 - \frac{3}{2}v_2\right)^2 + v_2^2 = 17, 25 - 15v_2 + \frac{9}{4}v_2^2 + v_2^2 = 17,$
$\frac{13}{4}v_2^2 - 15v_2 + 8 = 0, 13v_2^2 - 60v_2 + 32 = 0,$
$(v_2 - 4)(13v_2 - 8) = 0$, so $v_2 = 4$ or $v_2 = \frac{8}{13}$.

Therefore, $\mathbf{v} \approx \langle -1, 4\rangle$ or $\mathbf{v} = \left\langle \frac{53}{13}, \frac{8}{13} \right\rangle \approx \langle 4.07, 0.62\rangle$.

44. $-2v_1 + 5v_2 = -11, v_1^2 + v_2^2 = 10$. Since $v_1 = \frac{5}{2}v_2 + \frac{11}{2}$,

$\left(\frac{1}{2}(5v_2 + 11)\right)^2 + v_2^2 = 10,$

$\frac{25v_2^2}{4} + \frac{110v_2}{4} + \frac{121}{4} + v_2^2 = 10,$

$\frac{29}{4}v_2^2 + \frac{110}{4}v_2 + \frac{81}{4} = 0, 29v_2^2 + 110v_2 + 81 = 0,$

$(v_2 + 1)(29v_2 + 81) = 0$, so $v_2 = -1$ or $v_2 = -\frac{81}{29}$.

Therefore, $\mathbf{v} = \langle 3, -1\rangle$ or $\mathbf{v} = \left\langle -\frac{43}{29}, -\frac{81}{29} \right\rangle$

$\approx \langle -1.48, -2.79\rangle$.

45. $\mathbf{v} = (\cos 60°)\mathbf{i} + (\sin 60°)\mathbf{j} = \frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$

$\mathbf{F}_1 = \text{proj}_\mathbf{v}\mathbf{F} = (\mathbf{F}\cdot\mathbf{v})\mathbf{v} = \left(-160\,\mathbf{j}\cdot\left(\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}\right)\right)\mathbf{v}$

$= -80\sqrt{3}\left(\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}\right) = -40\sqrt{3}\,\mathbf{i} - 120\,\mathbf{j}.$

The magnitude of the force is

$|\mathbf{F}_1| = \sqrt{(-40\sqrt{3})^2 + (-120)^2} = \sqrt{19{,}200}$

≈ 138.56 pounds.

46. In this case, $\mathbf{F} = -125\,\mathbf{j}$ and $\mathbf{v}$ remains the same as in Example 6.

$\mathbf{F}_1 = \text{proj}_\mathbf{v}\mathbf{F} = (\mathbf{F}\cdot\mathbf{v})\mathbf{v} = -125\left(\frac{\sqrt{2}}{2}\right)\mathbf{v} = 62.5\,(\mathbf{i} + \mathbf{j}).$

The magnitude of the force is $|\mathbf{F}_1| = 62.5\sqrt{2} \approx 88.39$ pounds.

47. **(a)** $\mathbf{v} = (\cos 12°)\mathbf{i} + (\sin 12°)\mathbf{j}$

$\mathbf{F} = -2000\mathbf{j}$

$\mathbf{F}_1 = \text{proj}_\mathbf{v}\mathbf{F} = (\mathbf{F}\cdot\mathbf{v})\mathbf{v}$

$= (\langle 0, -2000\rangle \cdot \langle \cos 12°, \sin 12°\rangle)\,\langle \cos 12°, \sin 12°\rangle$

$= (-2000 \sin 12°)\langle \cos 12°, \sin 12°\rangle$.

Since $\langle \cos 12°, \sin 12°\rangle$ is a unit vector,the magnitude of the force being extended is

$|\mathbf{F}_1| = 2000 \sin 12° \approx 415.82$ pounds.

(b) We are looking for the gravitational force exerted perpendicular to the street. A unit vector perpendicular to the street is $\mathbf{w} = \langle \cos(-78°), \sin(-78°)\rangle$,

so $\mathbf{F}_2 = \text{proj}_\mathbf{w}\mathbf{F} = (\mathbf{F}\cdot\mathbf{w})\mathbf{w}$

$= (-2000 \sin(-78°))\,\langle \cos(-78°), \sin(-78°)\rangle$

Since $\langle \cos(-78°), \sin(-78°)\rangle$ is a unit vector, the magnitude of the force perpendicular to the street is $-2000 \sin(-78°) \approx 1956.30$ pounds.

48. We want to determine "how much" of the 60 pound force is projected along the inclined plane.

$\mathbf{F} = 60\,\langle \cos 43°, \sin 43°\rangle \approx \langle 43.88, 40.92\rangle$ and

$\mathbf{v} = \langle \cos 18°, \sin 18°\rangle \approx \langle 0.95, 0.31\rangle$

$\text{proj}_\mathbf{v}\mathbf{F} = \frac{(\langle 43.88, 40.92\rangle \cdot \langle 0.95, 0.31\rangle)\,\langle 0.95, 0.31\rangle}{(\sqrt{1})^2}$

$\approx \frac{54.38\,\langle 0.95, 0.31\rangle}{1} \approx \langle 51.72, 16.80\rangle$. The magnitude of this force is $|\mathbf{F}_1| = \sqrt{(51.72)^2 + (16.80)^2}$

≈ 54.38 pounds. Of note, it is also possible to evaluate this problem considering the x-axis parallel to the inclined plane and the y-axis perpendicular to the plane. In this case $\mathbf{F} = 60\,\langle \cos 25°, \sin 25°\rangle \approx \langle 54.38, 25.36\rangle$. Since we only want the force in the x-direction, we immediately find our answer of about 54.38 pounds.

49. Since the car weighs 2600 pounds, the force needed to lift the car is $\langle 0, 2600\rangle$.

$W = \mathbf{F}\cdot\overrightarrow{AB} = \langle 0, 2600\rangle \cdot \langle 0, 5.5\rangle = 14{,}300$ foot-pounds

50. Since the potatoes weigh 100 pounds, the force needed to lift the potatoes is $\langle 0, 100\rangle$.

$W = \mathbf{F}\cdot\overrightarrow{AB} = \langle 0, 100\rangle \cdot \langle 0, 3\rangle = 300$ foot-pounds

51. $\mathbf{F} = 12 \cdot \frac{\langle 1, 2\rangle}{|\langle 1, 2\rangle|} = \frac{12}{\sqrt{5}}\langle 1, 2\rangle$

$W = \mathbf{F}\cdot\overrightarrow{AB} = \frac{12}{\sqrt{5}}\langle 1, 2\rangle \cdot \langle 4, 0\rangle = \frac{48}{\sqrt{5}}$

≈ 21.47 foot-pounds

52. $\mathbf{F} = 24 \cdot \frac{\langle 4, 5\rangle}{|\langle 4, 5\rangle|} = \frac{24}{\sqrt{4^2 + 5^2}}\langle 4, 5\rangle \approx \frac{24}{\sqrt{41}}\langle 4, 5\rangle$

$W = \mathbf{F}\cdot\overrightarrow{AB} = \frac{24}{\sqrt{41}}\langle 5, 0\rangle = \frac{120}{\sqrt{41}}$

$= \mathbf{F}\cdot\overrightarrow{AB} = \frac{24}{\sqrt{41}}\langle 4, 5\rangle \cdot \langle 5, 0\rangle = \frac{480}{\sqrt{41}}$

≈ 74.96 foot-pounds

53. $\mathbf{F} = 30 \cdot \frac{\langle 2, 2\rangle}{|\langle 2, 2\rangle|} = \frac{30}{\sqrt{2^2 + 2^2}}\langle 2, 2\rangle = 15\sqrt{2}\,\langle 1, 1\rangle$

Since we want to move 3 feet along the line $y = \frac{1}{2}x$, we solve for x and y by using the Pythagorean theorem:

$x^2 + y^2 = 3^2, x^2 + \left(\frac{1}{2}x\right)^2 = 9, \frac{5}{4}x^2 = 9,$

$x = \frac{6}{\sqrt{5}}, y = \frac{3}{\sqrt{5}}$

$\overrightarrow{AB} = \left\langle \frac{6}{\sqrt{5}}, \frac{3}{\sqrt{5}} \right\rangle$

$W = \mathbf{F}\cdot\overrightarrow{AB} = \langle 15\sqrt{2}, 15\sqrt{2}\rangle \cdot \left\langle \frac{6}{\sqrt{5}}, \frac{3}{\sqrt{5}} \right\rangle$

$= 135\sqrt{\frac{2}{5}} = 27\sqrt{10} \approx 85.38$ foot-pounds

54. $\mathbf{F} = 50 \cdot \frac{\langle 2, 3\rangle}{|\langle 2, 3\rangle|} = \frac{50}{\sqrt{2^2 + 3^2}}\langle 2, 3\rangle = \frac{50}{\sqrt{13}}\langle 2, 3\rangle$

Since we want to move the object 5 feet along the line $y = x$, we solve for x and y by using the Pythagorean theorem: $x^2 + y^2 = 5^2, x^2 + x^2 = 25, 2x^2 = 25,$

$x = 2.5\sqrt{2}, y = 2.5\sqrt{2}$.

$\overrightarrow{AB} = \langle 2.5\sqrt{2}, 2.5\sqrt{2}\rangle$.

$W = \mathbf{F}\cdot\overrightarrow{AB} = \frac{50}{\sqrt{13}}\langle 2, 3\rangle \cdot \langle 2.5\sqrt{2}, 2.5\sqrt{2}\rangle = 625\sqrt{\frac{2}{13}}$

≈ 245.15 foot-pounds

55. $W = \mathbf{F}\cdot\overrightarrow{AB} = |\mathbf{F}|\,|\overrightarrow{AB}| \cos\theta = 200\sqrt{13}\cos 30°$

$= 200\sqrt{13}\cdot\frac{\sqrt{3}}{2} = 100\sqrt{39} \approx 624.5$ foot-pounds

56. $\overrightarrow{AB} = \langle 4, 3\rangle - \langle -1, 1\rangle = \langle 5, 2\rangle$

$W = \mathbf{F}\cdot\overrightarrow{AB} = |\mathbf{F}|\,|\overrightarrow{AB}| \cos\theta = 75\sqrt{29}\cos 60°$

$= 75\sqrt{29}\cdot\frac{1}{2} = \frac{75\sqrt{29}}{2} \approx 201.94$ foot-pounds

57. (a) Let $\mathbf{u} = \langle u_1, u_2\rangle$, $\mathbf{v} = \langle v_1, v_2\rangle$, and $\mathbf{w} = \langle w_1, w_2\rangle$.
$\mathbf{0} \cdot u = \langle 0, 0\rangle \cdot \langle u_1, u_2\rangle = 0 \cdot u_1 + 0 \cdot u_2 = 0$

(b) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \langle u_1, u_2\rangle \cdot (\langle v_1, v_2\rangle + \langle w_1, w_2\rangle)$
$= \langle u_1, u_2\rangle \cdot \langle v_1 + w_1, v_2 + w_2\rangle$
$= u_1(v_1 + w_1) + u_2(v_2 + w_2)$
$= u_1v_1 + u_2v_2 + u_1w_1 + u_2w_2$
$= \langle u_1, u_2\rangle \cdot \langle v_1, v_2\rangle + \langle u_1, u_2\rangle \cdot \langle w_1, w_2\rangle$
$= \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$

(c) $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = (\langle u_1, u_2\rangle + \langle v_1, v_2\rangle) \cdot \langle w_1, w_2\rangle$
$= \langle u_1 + v_1, u_2 + v_2\rangle \cdot \langle w_1, w_2\rangle$
$= (u_1 + v_1)\, w_1 + (u_2 + v_2)w_2$
$= u_1w_1 + u_2w_2 + v_1w_1 + v_2w_2$
$= \langle u_1, u_2\rangle \cdot \langle w_1, w_2\rangle + \langle v_1, v_2\rangle \cdot \langle w_1, w_2\rangle$
$= \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$

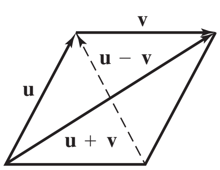
(d) $(c\mathbf{u}) \cdot \mathbf{v} = (c\langle u_1, u_2\rangle) \cdot \langle v_1, v_2\rangle = \langle cu_1, cu_2\rangle \cdot \langle v_1, v_2\rangle$
$= cu_1v_1 + cu_2v_2 = \langle u_1, u_2\rangle \cdot \langle cv_1, cv_2\rangle = \mathbf{u} \cdot (c\mathbf{v})$
$= c(u_1v_1 + u_2v_2) = c(\langle u_1, u_2\rangle \cdot \langle v_1, v_2\rangle) = c(\mathbf{u} \cdot \mathbf{v})$

58. (a) When we evaluate the projection of **u** onto **v** we are actually trying to determine "how much" of **u** is "going" in the direction of **v**. Using Figure 6.19, imagine that **v** runs along our x-axis, with the y-axis perpendicular to it. Written in component form, $\mathbf{u} = (|\mathbf{u}|\cos\theta, |\mathbf{u}|\sin\theta)$ and we see that the projection of **u** onto **v** is exactly $|\mathbf{u}|\cos\theta$ times **v**'s unit vector $\frac{\mathbf{v}}{|\mathbf{v}|}$. Thus, $\text{proj}_{\mathbf{v}}\mathbf{u} = |\mathbf{u}|\cos\theta \cdot \frac{\mathbf{v}}{|\mathbf{v}|}$

$$= |\mathbf{u}|\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right)\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{(\mathbf{u}\cdot\mathbf{v})}{|\mathbf{v}|}\frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}$$

(b) Recall Figure 6.19 and let $\mathbf{w}_1$ be $\overrightarrow{PR} = \text{proj}_{\mathbf{v}}\mathbf{u}$ and $\mathbf{w}_2 = \overrightarrow{RQ} = \mathbf{u} - \text{proj}_{\mathbf{v}}\mathbf{u}$. Then, $(\mathbf{u} - \text{proj}_{\mathbf{v}}\mathbf{u}) \cdot (\text{proj}_{\mathbf{v}}\mathbf{u}) = \mathbf{w}_2 \cdot \mathbf{w}_1$. Since $\mathbf{w}_1$ and $\mathbf{w}_2$ are perpendicular, $\mathbf{w}_1 \cdot \mathbf{w}_2 = 0$.

59.



As the diagram indicates, the long diagonal of the parallelogram can be expressed as the vector $\mathbf{u} + \mathbf{v}$, while the short diagonal can be expressed as the vector $\mathbf{u} - \mathbf{v}$. The sum of the squares of the diagonals is
$|\mathbf{u} + \mathbf{v}|^2 + |\mathbf{u} - \mathbf{v}|^2 = (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v})$
$+ (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = \mathbf{u}\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v}$
$+ \mathbf{u}\cdot\mathbf{u} - \mathbf{u}\cdot\mathbf{v} - \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v} = 2|\mathbf{u}|^2 + 2|\mathbf{v}|^2$
$+ 2\mathbf{u}\cdot\mathbf{v} - 2\mathbf{u}\cdot\mathbf{v} = 2|\mathbf{u}|^2 + 2|\mathbf{v}|^2$, which is the sum of the squares of the sides.

60. Let $\mathbf{u} = \langle u_1, u_2\rangle$.
$(\mathbf{u}\cdot\mathbf{i})\mathbf{i} + (\mathbf{u}\cdot\mathbf{j})\mathbf{j}$
$= (\langle u_1, u_2\rangle \cdot \langle 1, 0\rangle)\mathbf{i} + (\langle u_1, u_2\rangle \cdot \langle 0, 1\rangle)\mathbf{j}$
$= (u_1)\mathbf{i} + (u_2)\mathbf{j}$
$= u_1\mathbf{i} + u_2\mathbf{j}$
$= \langle u_1, u_2\rangle$
$= \mathbf{u}$

61. False. If either **u** or **v** is the zero vector, then $\mathbf{u}\cdot\mathbf{v} = 0$ and so **u** and **v** are orthogonal, but they do not count as perpendicular.

62. True. $\mathbf{u}\cdot\mathbf{u} = |\mathbf{u}|^2 = (1)^2 = 1$.

63. $\mathbf{u}\cdot\mathbf{v} = 0$, so the vectors are perpendicular. The answer is D.

64. $\mathbf{u}\cdot\mathbf{v} = \langle 4, -5\rangle \cdot \langle -2, -3\rangle$
$= 4(-2) + (-5)(-3)$
$= -8 + 15$
$= 7$
The answer is C.

65. $\text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}$
$= \left(\frac{3 + 0}{2^2}\right)\langle 2, 0\rangle$
$= \left(\frac{3}{4}\right)\langle 2, 0\rangle$
$= \left\langle \frac{3}{2}, 0\right\rangle$
The answer is A.

66. The unit vector in the direction of **u** is $\frac{1}{\sqrt{2}}\langle -1, 1\rangle$. The force is represented by 5 times the unit vector. The answer is B.

67. (a) $2\cdot 0 + 5\cdot 2 = 10$ and $2\cdot 5 + 5\cdot 0 = 10$

(b) $\overrightarrow{AP} = \langle 3 - 0, 7 - 2\rangle = \langle 3, 5\rangle$
$\overrightarrow{AB} = \langle 5 - 0, 0 - 2\rangle = \langle 5, -2\rangle$
$\mathbf{w}_1 = \text{proj}_{\overrightarrow{AB}}\overrightarrow{AB} = \left(\frac{\langle 3, 5\rangle \cdot \langle 5, -27\rangle}{5^2 + (-2)^2}\right)\langle 5, -2\rangle$
$= \left(\frac{15 - 10}{29}\right)\langle 5, -2\rangle = \frac{5}{29}\langle 5, -2\rangle$
$\mathbf{w}_2 = \overrightarrow{AP} - \text{proj}_{\overrightarrow{AB}}\overrightarrow{AP}$
$= \langle 3, 5\rangle - \frac{5}{29}\langle 5, -2\rangle$
$= \left\langle 3 - \frac{25}{29} 5 + \frac{10}{29}\right\rangle = \frac{1}{29}\langle 62, 155\rangle$

(c) $\mathbf{w}_2$ is a vector from a point on $\overrightarrow{AB}$ to point P. Since $\mathbf{w}_2$ is perpendicular to $\overrightarrow{AB}$, $|\mathbf{w}_2|$ is the shortest distance from $\overrightarrow{AB}$ to P.
$$|\mathbf{w}_2| = \sqrt{\left(\frac{62}{29}\right)^2 + \left(\frac{155}{29}\right)^2} = \sqrt{\frac{27{,}869}{29^2}} = \frac{31\sqrt{29}}{29}$$

(d) Consider Figure 6.19. To find the distance from a point P to a line L, we must first find $\mathbf{u}_1 = \text{proj}_{\mathbf{v}}\mathbf{u}$. In this case,
$\text{proj}_{\overrightarrow{AB}}\overrightarrow{AP} = \left(\frac{\langle x_0, y_0 - 2\rangle \cdot \langle 5, -2\rangle}{(\sqrt{5^2 + (-2)^2})^2}\right)\langle 5, -2\rangle$
$= \left(\frac{5x_0 - 2y_0 + 4}{29}\right)\langle 5, -2\rangle$
$= \left\langle \frac{25x_0 - 10y_0 + 20}{29}, \frac{-10x_0 + 4y_0 - 8}{29}\right\rangle$ and
$\overrightarrow{AP} - \text{proj}_{\overrightarrow{AB}}\overrightarrow{AP} = \langle x_0, y_0 - 2\rangle$
$- \left\langle \frac{25x_0 - 10y_0 + 20}{29}, \frac{-10x_0 + 4y_0 - 8}{29}\right\rangle$
$= \frac{1}{29}(\langle 29x_0, 29\,(y_0 - 2)\rangle$
$- \langle 25x_0 - 10y_0 + 20, -10x_0 + 4y_0 - 8\rangle)$
$= \frac{1}{29}\langle 4x_0 + 10y_0 - 20, 10x_0 + 25y_0 - 50\rangle$

So, the distance is the magnitude of this vector.

$$d = \frac{1}{29}\sqrt{(4x_0 + 10y_0 - 20)^2 + (10x_0 + 25y_0 - 50)^2}$$
$$= \frac{\sqrt{2^2(2x_0 + 5y_0 - 10)^2 + 5^2(2x_0 + 5y_0 - 10)^2}}{29}$$
$$= \frac{\sqrt{29(2x_0 + 5y_0 - 10)^2}}{29}$$
$$= \frac{|(2x_0 + 5y_0 - 10)|}{\sqrt{29}}$$

(e) In the general case, $\overrightarrow{AB} = \left\langle \frac{c}{a}, \frac{-c}{b} \right\rangle$ and

$\overrightarrow{AP} = \left\langle x_0, y_0 - \frac{c}{b} \right\rangle$, so $\text{proj}_{\overrightarrow{AB}}\ \overrightarrow{AP}$s

$$= \frac{\left(\frac{x_0 c}{a} - \frac{(bcy_0 - c^2)}{b^2}\right)\left\langle \frac{c}{a}, -\frac{c}{b} \right\rangle}{\left|\frac{c}{a}\right|^2 + \left|-\frac{c}{b}\right|^2}$$
$$= \left\langle \frac{b^2x_0 - aby_0 + ac}{a^2 + b^2}, \frac{-abx_0 + a^2y_0 - \frac{a^2c}{b}}{a^2 + b^2} \right\rangle$$

$$|\overrightarrow{AP} - \text{proj}_{\overrightarrow{AB}}\ \overrightarrow{AP}| = \left\langle x_0, y_0 - \frac{c}{b} \right\rangle$$
$$- \left\langle \frac{b^2x_0 - aby_0 + ac}{a^2 + b^2}, \frac{-abx_0 + a^2y_0 - \frac{a^2c}{b}}{a^2 + b^2} \right\rangle$$
$$= \left\langle \frac{a^2x_0 + aby_0 - ac}{a^2 + b^2}, \frac{abx_0 + b^2 y_0 - bc}{a^2 + b^2} \right\rangle$$

The magnitude of this vector, $|\overrightarrow{AP} - \text{proj}_{\overrightarrow{AB}}\ \overrightarrow{AP}|$, is the distance from point P to L: $\frac{|ax_0 + by_0 - c|}{\sqrt{a^2 + b^2}}$.

68. **(a)** Yes, if $\mathbf{v} = \langle 0, 0 \rangle$ or $t = n\pi$, $n =$ any integer.

(b) Yes, if $\mathbf{u} = \langle 0, 0 \rangle$ or $t = \frac{n\pi}{2}$, $n =$ odd integer.

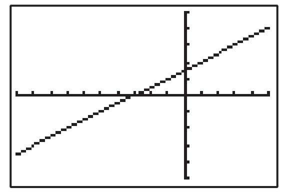
(c) Generally, no, because $\sin t \neq \cos t$ for most t. Exceptions, however, would occur when $t = \frac{\pi}{4} + n\pi$, $n =$ any integer, or if $\mathbf{u} = \langle 0, 0 \rangle$ and/or $\mathbf{v} = \langle 0, 0 \rangle$.

69. One possible answer:
If $a\mathbf{u} + b\mathbf{v} = c\mathbf{u} + d\mathbf{v}$
$a\mathbf{u} - c\mathbf{u} + b\mathbf{v} - d\mathbf{v} = 0$
$(a - c)\mathbf{u} + (b - d)\mathbf{v} = 0$
Since $\mathbf{u}$ and $\mathbf{v}$ are not parallel, the only way for this equality to hold true for all vectors $\mathbf{u}$ and $\mathbf{v}$ is if $(a - c) = 0$ and $(b - d) = 0$, which indicates that $a = c$ and $b = d$.

■ Section 6.3 Parametric Equations and Motion

Exploration 1

1.



$[-10, 5]$ by $[-5, 5]$

2. $0.5(17) + 1.5 = 10$, so the point $(17, 10)$ is on the graph, $t = -8$.

3. $0.5(-23) + 1.5 = -10$, so the point $(-23, -10)$ is on the graph, $t = 12$.

4. $x = a = 1 - 2t$, $2t = 1 - a$, $t = \frac{1}{2} - \frac{a}{2}$. Alternatively, $b = 2 - t$, so $t = 2 - b$.

5. Choose Tmin and Tmax so that Tmin ≤ -2 and Tmax ≥ 5.5.

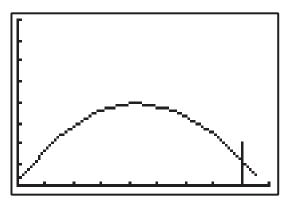
Exploration 2

1. It looks like the line in Figure 6.32.

2. The graph is a vertical line segment that extends from $(400, 0)$ to $(400, 20)$.

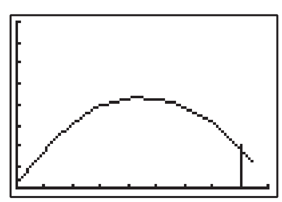
3. For 19° and 20°, the ball does not clear the fence, as shown below.

19°:



$[0, 450]$ by $[0, 80]$

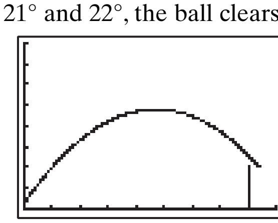
20°:



$[0, 450]$ by $[0, 80]$

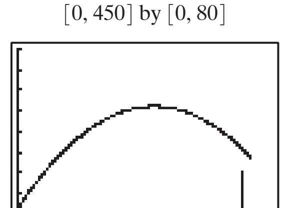
For 21° and 22°, the ball clears the fence, as shown below.

21°



$[0, 450]$ by $[0, 80]$

22°:



$[0, 450]$ by $[0, 80]$

Quick Review 6.3

1. (a) $\overrightarrow{OA} = \langle -3, -2 \rangle$

(b) $\overrightarrow{OB} = \langle 4, 6 \rangle$

(c) $\overrightarrow{AB} = \langle 4 - (-3), 6 - (-2) \rangle = \langle 7, 8 \rangle$

2. (a) $\overrightarrow{OA} = \langle -1, 3 \rangle$

(b) $\overrightarrow{OB} = \langle 4, -3 \rangle$

(c) $\overrightarrow{AB} = \langle 4 - (-1), -3 - 3 \rangle = \langle 5, -6 \rangle$

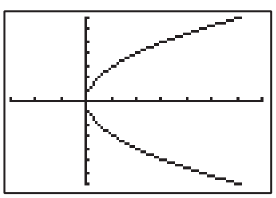
3. $m = \dfrac{6 - (-2)}{4 - (-3)} = \dfrac{8}{7}$

$y + 2 = \dfrac{8}{7}(x + 3)$ or $y - 6 = \dfrac{8}{7}(x - 4)$

4. $m = \dfrac{-3 - 3}{4 - (-1)} = -\dfrac{6}{5}$

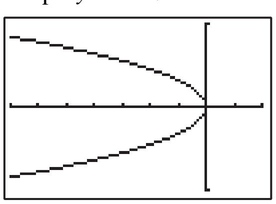
$y - 3 = -\dfrac{6}{5}(x + 1)$ or $y + 3 = -\dfrac{6}{5}(x - 4)$

5. Graph $y = \pm\sqrt{8x}$.



$[-3, 7]$ by $[-7, 7]$

6. Graph $y = \pm\sqrt{-5x}$.



$[-7, 2]$ by $[-7, 7]$

7. $x^2 + y^2 = 4$

8. $(x + 2)^2 + (y - 5)^2 = 9$

9. $\dfrac{600 \text{ rotations}}{1 \text{ min}} \cdot \dfrac{1 \text{ min}}{60 \text{ sec}} \cdot \dfrac{2\pi \text{ rad}}{1 \text{ rotation}} = 20\pi \text{ rad/sec}$

10. $\dfrac{700 \text{ rotations}}{1 \text{ min}} \cdot \dfrac{1 \text{ min}}{60 \text{ sec}} \cdot \dfrac{2\pi \text{ rad}}{1 \text{ rotation}} = \dfrac{70}{3}\pi \text{ rad/sec}$

Section 6.3 Exercises

1. (b) $[-5, 5]$ by $[-5, 5]$

2. (d) $[-5, 5]$ by $[-5, 5]$

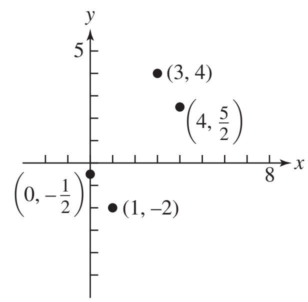
3. (a) $[-5, 5]$ by $[-5, 5]$

4. (c) $[-10, 10]$ by $[-12, 10]$

5. (a)

t	-2	-1	0	1	2
x	0	1	2	3	4
y	$-\frac{1}{2}$	-2	undef.	4	$\frac{5}{2}$

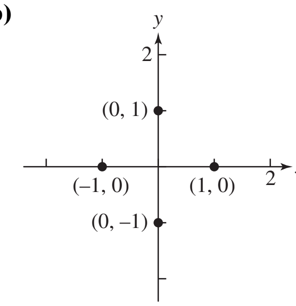
(b)



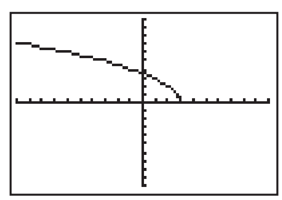
6. (a)

t	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
x	1	0	-1	0	1
y	0	1	0	-1	0

(b)

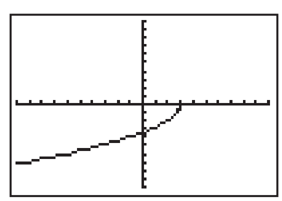


7.



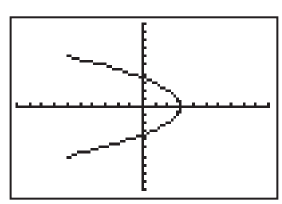
$[-10, 10]$ by $[-10, 10]$

8.



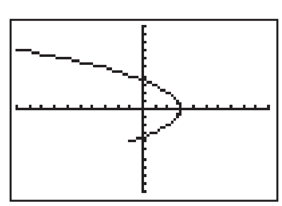
$[-10, 10]$ by $[-10, 10]$

9.



$[-10, 10]$ by $[-10, 10]$

10.



$[-10, 10]$ by $[-10, 10]$

11. $x = 1 + y$, so $y = x - 1$: line through $(0, -1)$ and $(1, 0)$

12. $t = y - 5$, so $x = 2 - 3(y - 5)$; $y = -\frac{1}{3}x + \frac{17}{3}$: line through $\left(0, \frac{17}{3}\right)$ and $(17, 0)$

13. $t = \frac{1}{2}x + \frac{3}{2}$, so $y = 9 - 4\left(\frac{1}{2}x + \frac{3}{2}\right)$; $y = -2x + 3$, $3 \le x \le 7$: line segment with endpoints $(3, -3)$ and $(7, -11)$

14. $t = y - 2$, so $x = 5 - 3(y - 2)$; $y = -\frac{1}{3}x + \frac{11}{3}$, $-4 \le x \le 8$: line segment with endpoints $(8, 1)$ and $(-4, 5)$

15. $x = (y - 1)^2$: parabola that opens to right with vertex at $(0, 1)$

16. $y = x^2 - 3$: parabola that opens upward with vertex at $(0, -3)$

17. $y = x^3 - 2x + 3$: cubic polynomial

18. $x = 2y^2 - 1$; parabola that opens to right with vertex at $(0, -1)$

19. $x = 4 - y^2$; parabola that opens to left with vertex at $(4, 0)$

20. $t = 2x$, so $y = 16x^3 - 3$: cubic, $-1 \le x \le 1$

21. $t = x + 3$, so $y = \frac{2}{x + 3}$, on domain: $-8 \le x \le 2, x \ne -3$

22. $t = x - 2$, so $y = \frac{4}{x - 2}$, $x \ge 4$

23. $x^2 + y^2 = 25$, circle of radius 5 centered at $(0, 0)$

24. $x^2 + y^2 = 16$, circle of radius 4 centered at $(0, 0)$

25. $x^2 + y^2 = 4$, three-fourths of a circle of radius 2 centered at $(0, 0)$ (not in Quadrant II)

26. $x^2 + y^2 = 9$, semicircle of radius 3, $y \ge 0$ only

27. $\overrightarrow{OA} = \langle -2, 5 \rangle, \overrightarrow{OB}\langle 4, 2 \rangle, \overrightarrow{OP} = \langle x, y \rangle$
$\overrightarrow{OP} - \overrightarrow{OA} = t\left(\overrightarrow{OB} - \overrightarrow{OA}\right)$
$\langle x + 2, y - 5 \rangle = t\langle 6, -3 \rangle$
$x + 2 = 6t \Rightarrow x = 6t - 2$
$y - 5 = 3t \Rightarrow y = -3t + 5$

28. $\overrightarrow{OA}\langle -3, -3 \rangle, \overrightarrow{OB} = \langle 5, 1 \rangle, \overrightarrow{OP} = \langle x, y \rangle$
$\overrightarrow{OP} - \overrightarrow{OA} = t\left(\overrightarrow{OB} - \overrightarrow{OA}\right)$
$\langle x + 3, y + 3 \rangle = t\langle 8, 4 \rangle$
$x + 3 = 8t \Rightarrow x = 8t - 3$
$y + 3 = 4t \Rightarrow y = 4t - 3$

For #29–32, many answers are possible; one or two of the simplest are given.

29. Two possibilities are $x = t + 3$, $y = 4 - \frac{7}{3}t, 0 \le t \le 3$, or $x = 3t + 3, y = 4 - 7t, 0 \le t \le 1$.

30. Two possibilities are $x = 5 - t, y = 2 - \frac{6}{7}t$, $0 \le t \le 7$, or $x = 5 - 7t, y = 2 - 6t, 0 \le t \le 1$.

31. One possibility is $x = 5 + 3 \cos t, y = 2 + 3 \sin t$, $0 \le t \le 2\pi$.

32. One possibility is $x = -2 + 2 \cos t, y = -4 + 2 \sin t$, $0 \le t \le 2\pi$.

33. In Quadrant I, we need $x > 0$ and $y > 0$, so $2 - |t| > 0$ and $t - 0.5 > 0$. Then $-2 < t < 2$ and $t > 0.5$, so $0.5 < t < 2$. This is not changed by the additional requirement that $-3 \le t \le 3$.

34. In Quadrant II, we need $x < 0$ and $y > 0$, so $2 - |t| < 0$ and $t - 0.5 > 0$. Then ($t < -2$ or $t > 2$) and t > 0.5, so $t > 2$. With the additional requirement that $-3 \le t \le 3$, this becomes $2 < t \le 3$.

35. In Quadrant III, we need $x < 0$ and $y < 0$, so $2 - |t| < 0$ and $t - 0.5 < 0$. Then ($t < -2$ or $t > 2$) and $t < 0.5$, so $t < -2$. With the additional requirement that $-3 \le t \le 3$, this becomes $-3 \le t < -2$.

36. In Quadrant IV, we need $x > 0$ and $y < 0$, so $2 - |t| > 0$ and $t - 0.5 < 0$. Then $-2 < t < 2$ and $t < 0.5$, so $-2 < t < 0.5$. This is not changed by the additional requirement that $-3 \le t \le 3$.

37. (a) One good window is $[-20, 300]$ by $[-1, 10]$. If your grapher allows, use "Simultaneous" rather than "Sequential" plotting. Note that 100 yd is 300 ft. To show the whole race, use $0 \le t \le 13$ (upper limit may vary), since Ben finishes in $12.91\overline{6}$ sec. Note that it is the *process* of graphing (during which one observes Ben passing Jerry and crossing "the finish line" first), not the final product (which is two horizontal lines) which is needed; for that reason, no graph is shown here.

(b) After 3 seconds, Jerry is at $20(3) = 60$ ft and Ben is at $24(3) - 10 = 62$ ft. Ben is ahead by 2 ft.

38. (a) If your grapher allows, use "Simultaneous" rather than "Sequential" plotting. To see the whole race, use $0 \le t \le 5.1$ (upper limit may vary), since the faster runner reaches the flag after 5.1 sec. Note that it is the *process* of graphing, not the final product (which shows a horizontal line) which is needed; for that reason, no graph is shown here.

(b) The faster runner (who is coming from the left in the simulation) arrives at $t = 5.1$ sec. At this instant, the slower runner is 4.1 ft away from the flag; the slower runner doesn't reach the flag until $t = 5.\overline{5}$ sec. This can be observed from the simulation, or by solving algebraically $x_1 = 50$ and $x_2 = 50$.

39. (a) $y = -16t^2 + v_0t + s_0 = -16t^2 + 0t + 1000$
$= -16t^2 + 1000$

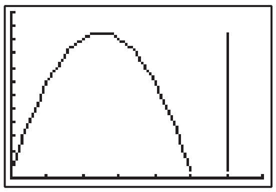
(b) Graph and trace: $x = 1$ and $y = -16t^2 + 1000$ with $0 \le t \le 6$, on the window $[0, 2]$ by $[0, 1200]$. Use something like 0.2 or less for Tstep. This graph will appear as a vertical line from $(1, 424)$ to $(1, 1000)$; it is not shown here because the simulation is accomplished by the *tracing,* not by the *picture.*

(c) When $t = 4$, $y = -16(4)^2 + 1000 = 744$ ft; the food containers are 744 ft above the ground after 4 sec.

40. (a) $y = -16t^2 + v_0t + s_0 = -16t^2 + 80t + 5$

(b) Graph and trace: $x = 6$ and $y = -16t^2 + 80t + 5$ with $0 \le t \le 5.1$ (upper limit may vary) on $[0, 7]$ by $[0, 120]$. This graph will appear as a vertical line from about $(6, 0)$ to about $(6, 105)$. Tracing shows how the ball begins at a height of 5 ft, rises to over 100 ft, then falls back to the ground.

(c) Graph $x = t$ and $y = -16t^2 + 80t + 5$ with $0 \le t \le 5.1$ (upper limit may vary).



[0, 7] by [0, 120]

(d) When $t = 4$, $y = -16(4)^2 + 80(4) + 5 = 69$ ft. The ball is 69 ft above the ground after 4 sec.

(e) From the graph in (b), when $t = 2.5$ sec, the ball is at its maximum height of 105 ft.

41. Possible answers:

(a) $0 < t < \frac{\pi}{2}$ (t in radians)

(b) $0 < t < \pi$

(c) $\frac{\pi}{2} < t < \frac{3\pi}{2}$

42. **(a)** Both pairs of equations can be changed to $x^2 + y^2 = 9$ — a circle centered at the origin with radius 3. Also, when one chooses a point on this circle and swaps the x- and y-coordinates, one obtains another point on the same circle.

(b) The first begins at the right side (when $t = 0$) and traces the circle counterclockwise. The second begins at the top (when $t = 0$) and traces the circle clockwise.

43. **(a)** $x = 400$ when $t \approx 2.80$ — about 2.80 sec.

(b) When $t \approx 2.80$ sec, $y \approx 7.18$ ft.

(c) Reaching up, the outfielder's glove should be at or near the height of the ball as it approaches the wall. If hit at an angle of 20°, the ball would strike the wall about 19.74 ft up (after 2.84 sec) — the outfielder could not catch this.

44. **(a)** No: $x = (120 \cos 30°)t$; this equals 350 when $t \approx 3.37$. At this time, the ball is at a height of $y = -16t^2 + (120 \sin 30°)t + 4 \approx 24.59$ ft.

(b) The ball hits the wall about 24.59 ft up when $t \approx 3.37$ (see **(a)**) — not catchable.

45. **(a)** Yes: $x = (5 + 120 \cos 30°)t$; this equals 350 when $t \approx 3.21$. At this time, the ball is at a height of $y = -16t^2 + (120 \sin 30°)t + 4 \approx 31.59$ ft.

(b) The ball clears the wall with about 1.59 ft to spare (when $t \approx 3.21$).

46. For Linda's ball, $x_1 = (45 \cos 44°)t$ and $y_1 = -16t^2 + (45 \sin 44°)t + 5$. For Chris's ball, $x_2 = 78 - (41 \cos 39°)t$ and $y_2 = -16t^2 + (41 \sin 39°)t + 5$. Find (graphically) the minimum of $d(t) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. It occurs when $t \approx 1.21$ sec; the minimum distance is about 6.60 ft.

47. No: $x = (30 \cos 70°)t$ and $y = -16t^2 + (30 \sin 70°)t + 3$. The dart lands when $y = 0$, which happens when $t \approx 1.86$ sec. At this point, the dart is about 19.11 ft from Tony, just over 10 in. short of the target.

48. Yes: $x = (25 \cos 55°)t$ and $y = -16t^2 + (25 \sin 55°)t + 4$. The dart lands when $y = 0$, which happens when $t \approx 1.45$ sec. At this point, the dart is about 20.82 ft from Sue, inside the target.

49. The parametric equations for this motion are $x = (v + 160 \cos 20°)\,t$ and $y = -16t^2 + (160 \sin 20°)t + 4$, where v is the velocity of the wind (in ft/sec) — it should be positive if the wind is in the direction of the hit, and negative if the wind is against the ball.
To solve this algebraically, eliminate the parameter t as follows:

$$t = \frac{x}{v + 160 \cos 20°}. \text{ So } y = -16\left(\frac{x}{v + 160 \cos 20°}\right)^2 + 160 \sin 20°\left(\frac{x}{v + 160 \cos 20°}\right) + 4.$$

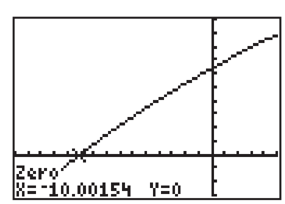
Substitute $x = 400$ and $y = 30$:

$$30 = -16\left(\frac{400}{v + 160 \cos 20°}\right)^2 + 160 \sin 20°\left(\frac{400}{v + 160 \cos 20°}\right) + 4.$$

Let $u = \frac{400}{v + 160 \cos 20°}$, so the equation becomes $-16u^2 + 54.72u - 26 = 0$. Using the quadratic formula, we find that $u = \frac{-54.72 \pm \sqrt{54.72^2 - 4(-16)(-26)}}{-32} \approx 0.57, 2.85$. Solving $0.57 = \frac{400}{v + 160 \cos 20°}$ and $2.85 = \frac{400}{v + 160 \cos 20°}$, $v \approx 551.20$, $v \approx -10.00$. A wind speed of 551 ft/sec (375.7 mph) is unrealistic, so we eliminate that solution. So the wind will be blowing against the ball in order for the ball to hit within a few inches of the top of the wall.
To verify this graphically, graph the equation

$$30 = -16\left(\frac{400}{v + 160 \cos 20°}\right)^2 + 160 \sin 20°\left(\frac{400}{v + 160 \cos 20°}\right) + 4,$$

and find the zero.



[–15, 5] by [–3, 10]

50. Assuming the course is level, the ball hits the ground when $y = -16t^2 + (180 \sin \theta)t$ equals 0, which happens when $t = \frac{180 \sin \theta}{16} = 11.25 \sin \theta$ sec. At that time, the ball has traveled $x = (180 \cos \theta)t = 2025(\cos \theta)(\sin \theta)$ feet. The answers are therefore approximately:

(a) 506.25 ft.

(b) 650.82 ft.

(c) 775.62 ft.

(d) 876.85 ft.

51. $x = 35\cos\left(\frac{\pi}{6}t\right)$ and $y = 50 + 35\sin\left(\frac{\pi}{6}t\right)$

52. $t = \frac{1}{5}x + \frac{2}{5}$, $y = 3 + 3\left(\frac{1}{5}x + \frac{2}{5}\right) = \frac{3}{5}x + \frac{21}{5}$

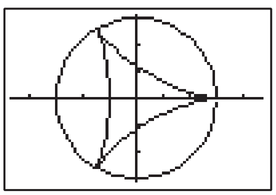
Since $3 = \frac{3}{5}(-2) + \frac{21}{5} = \frac{15}{5} = 3$ and

$6 = \frac{3}{5}(3) + \frac{21}{5} = \frac{30}{5} = 6$, both $(-2, 3)$ and $(3, 6)$ are on the line.

53. **(a)** When $t = \pi$(or 3π, or 5π, etc.), $y = 2$. This corresponds to the highest points on the graph.

(b) The x-intercepts occur where $y = 0$, which happens when $t = 0, 2\pi, 4\pi$, etc. The x-coordinates at those times are (respectively) $0, 2\pi, 4\pi$, etc., so these are 2π units apart.

54. **(a)**



$[-4.7, 4.7]$ by $[-3.1, 3.1]$

(b) All 2s should be changed to 3s.

55. The particle begins at -10, moves right to $+2.25$ (at $t = 1.5$), then changes direction and ends at -4.

56. The particle begins at -5, moves right to $+4$ (at time $t = 2$), then changes direction and returns to -5.

57. The particle begins at -5, moves right to about $+0.07$ (at time $t \approx 0.15$), changes direction and moves left to about -20.81 (at time $t \approx 4.5$), then changes direction and ends at $+7$.

58. The particle begins at -10, moves right to about $+0.88$ (at $t \approx 0.46$), changes direction and moves left to about -6.06 (at time $t \approx 2.9$), then changes direction and ends at $+20$.

59. True. Eliminate t from the first set:

$t = x_1 + 1$
$y_1 = 3(x_1 + 1) + 1$
$y_1 = 3x_1 + 4$

Eliminate t from the second set:

$t = \frac{3}{2}x_2 + 2$
$y_2 = 2\left(\frac{3}{2}x_2 + 2\right)$
$y_2 = 3x_2 + 4$

Both sets correspond to the rectangular equation $y = 3x + 4$.

60. True. $x = 0$ and $y = 1$ when $t = 1$, and $x = 2$ and $y = 5$ when $t = 3$. Eliminating t,

$t = x + 1.$
$y = 2(x + 1) - 1.$
$y = 2x + 1, \quad 0 \le x \le 2, 1 \le y \le 5.$

61. $x = (-1)^2 - 4 = -3$, $y = -1 + \frac{1}{-1} = -2$

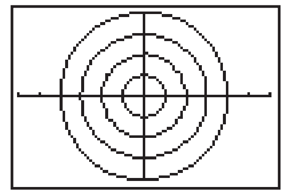
The answer is A.

62. The parametrization describes a circle of radius 2, centered at the origin and t represents the angle traveled counterclockwise from $(1, 0)$. The answer is A.

63. Set $-16t^2 + 80t + 7$ equal to 91 and solve either graphically or using the quadratic formula. The answer is D.

64. The equations are both linear, so the answer is either A, B, or C. Since t has a minimum value and no maximum value, the answer is C.

65. **(a)**

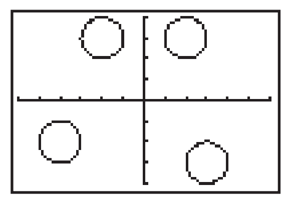


$[-6, 6]$ by $[-4, 4]$

(b) $x^2 + y^2 = (a\cos t)^2 + (a\sin t)^2$
$= a^2\cos^2 t + a^2\sin^2 t$
$= a^2$

The radius of the circles are $a = \{1, 2, 3, 4\}$, centered at $(0, 0)$.

(c)



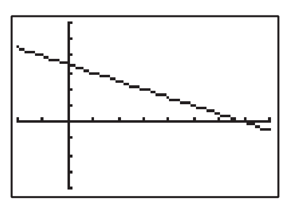
$[-6, 6]$ by $[-4, 4]$

(d) $x - h = a\cos t$ and $y - k = a\sin t$, so
$(x - h)^2 + (y - k)^2$
$= (a\cos t)^2 + (a\sin t)^2$
$= a^2\cos^2 t + a^2\sin^2 t$
$= a^2$

The graph is the circle of radius a centered at (h, k).

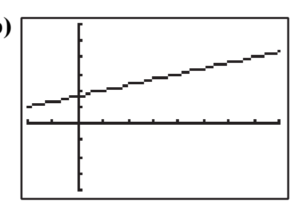
(e) If $(x + 1)^2 + (y - 4)^2 = 9$, then $a = 3$, $h = -1$, and $k = 4$.
As a result, $x = 3\cos t - 1$ and $y = 3\sin t + 4$.

66. **(a)**



$[-2, 8]$ by $[-4, 6]$

(b)



$[-2, 8]$ by $[-4, 6]$

(c) $t = \frac{1}{a}x - \frac{b}{a}$, $y = c\left(\frac{1}{a}x - \frac{b}{a}\right) + d$
$= \frac{c}{a}x + \frac{ad - bc}{a}$, $a \ne 0$.

(d) Slope: $\frac{c}{a}$, if $a \neq 0$;

y-intercept: $\left(0, \frac{-bc + ad}{a}\right)$, if $a \neq 0$;

x-intercept: $\left(\frac{bc - ad}{c}, 0\right)$, if $c \neq 0$.

(e) The line will be horizontal if $c = 0$. The line will be vertical if $a = 0$.

67. (a) Jane is traveling in a circle of radius 20 feet and center (0, 20), which yields $x_1 = 20\cos(nt)$ and $y_1 = 20 + 20\sin(nt)$. Since the ferris wheel is making one revolution (2π) every 12 seconds,

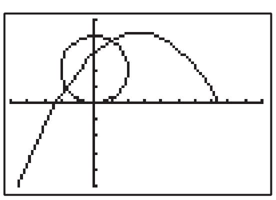
$2\pi = 12n$, so $n = \frac{2\pi}{12} = \frac{\pi}{6}$.

Thus,

$x_1 = 20\cos\left(\frac{\pi}{6}t\right)$ and $y_1 = 20 + 20\sin\left(\frac{\pi}{6}t\right)$ in radian mode.

(b) Since the ball was released at 75 ft in the positive x-direction and gravity acts in the negative y-direction at 16 ft/s^2, we have $x_2 = at + 75$ and $y_2 = -16t^2 + bt$, where a is the initial speed of the ball in the x-direction and b is the initial speed of the ball in the y-direction. The initial velocity vector of the ball is $60\langle\cos 120°, \sin 120°\rangle = \langle -30, 30\sqrt{3}\rangle$, so $a = -30$ and $b = 30\sqrt{3}$. As a result $x_2 = -30t + 75$ and $y_2 = -16t^2 + (30\sqrt{3})t$ are the parametric equations for the ball.

(c)

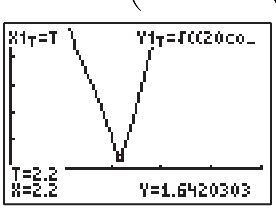


[−50, 100] by [−50, 50]

Our graph shows that Jane and the ball will be close to each other but not at the exact same point at $t = 2.2$ seconds.

(d)
$$d(t) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$
$$= \sqrt{\left(20\cos\left(\frac{\pi}{6}t\right) + 30t - 75\right)^2 + \left(20 + 20\sin\left(\frac{\pi}{6}t\right) + 16t^2 - (30\sqrt{3})t\right)^2}$$

(e)



[0, 5] by [−5, 25]

The minimum distance occurs at $t = 2.2$, when $d(t) = 1.64$ feet.

68. Assuming that the bottom of the ferris wheel and the ball's initial position are at the same height, the position of Matthew is $x_1 = 71\cos\left(\frac{\pi}{10}t\right)$ and $y_1 = 71 + 71\sin\left(\frac{\pi}{10}t\right)$. The ball's position is $x_2 = 90 + (88\cos 100°)t$ and $y_2 = -16t^2 + (88\sin 100°)t$.

Find (graphically) the minimum of $d(t) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. It occurs when $t \approx 2.19$ sec; the minimum distance is about 3.47 ft.

69. Chang's position: $x_1 = 20\cos\left(\frac{\pi}{6}t\right)$ and $y_1 = 20 + 20\sin\left(\frac{\pi}{6}t\right)$. Kuan's position: $x_2 = 15 + 15\cos\left(\frac{\pi}{4}t\right)$ and $y_2 = 15 + 15\sin\left(\frac{\pi}{4}t\right)$.
Find (graphically) the minimum of $d(t) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. It occurs when $t \approx 21.50$ sec; the minimum distance is about 4.11 ft.

70. Chang's position: $x_1 = 20\cos\left(\frac{\pi}{6}t\right)$ and $y_1 = 20 + 20\sin\left(\frac{\pi}{6}t\right)$. Kuan's position: $x_2 = 15 + 15\sin\left(\frac{\pi}{4}t\right)$ and $y_2 = 15 - 15\cos\left(\frac{\pi}{4}t\right)$ Find (graphically) the mini mum of $d(t) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. It occurs when $t \approx 12.32$ sec; the minimum distance is about 10.48 ft.

71. (a) $x(0) = 0c + (1 - 0)a = a$ and $y(0) = 0d + (1 - 0)b = b$
(b) $x(1) = 1c + (1 - 1)a = c$ and $y(1) = 1d + (1 - 1)b = d$

72. $x(0.5) = 0.5c + (1 - 0.5)a = 0.5(a + c) = (a + c)/2$, while $y(0.5) = (b + d)/2$ — the correct coordinates for the midpoint.

73. Since the relationship between x and y is linear and one unit of time ($t = 1$) separates the two points, $t = \frac{1}{3}, \frac{2}{3}$ will divide the segment into three equal pieces. Similarly, $t = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$ will divide the segment into four equal pieces.

■ Section 6.4 Polar Coordinates

Exploration 1

2.
$$\left(2, \frac{\pi}{3}\right) = (1, \sqrt{3})$$
$$\left(-1, \frac{\pi}{2}\right) = (0, -1)$$
$$(2, \pi) = (-2, 0)$$
$$\left(-5, \frac{3\pi}{2}\right) = (0, 5)$$
$$(3, 2\pi) = (3, 0)$$

3. $(-1, -\sqrt{3}) = \left(-2, \frac{\pi}{3}\right)$

$(0, 2) = \left(2, \frac{\pi}{2}\right)$

$(3, 0) = (3, 0)$

$(-1, 0) = (1, \pi)$

$(0, -4) = \left(4, \frac{3\pi}{2}\right)$

Quick Review 6.4

1. **(a)** Quadrant II
   (b) Quadrant III
2. **(a)** Quadrant I
   (b) Quadrant III
3. Possible answers: $7\pi/4, -9\pi/4$
4. Possible answers: $7\pi/3, -5\pi/3$
5. Possible answers: $520°, -200°$
6. Possible answers: $240°, -480°$
7. $(x - 3)^2 + y^2 = 4$
8. $x^2 + (y + 4)^2 = 9$
9. $a^2 = 12^2 + 10^2 - 2(12)(10)\cos 60°$
   $a \approx 11.14$
10. $a^2 = 9^2 + 6^2 - 2(9)(6) \cos 40°$
    $a \approx 5.85$

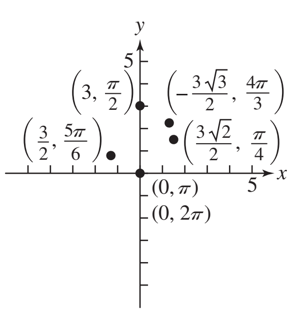
Section 6.4 Exercises

1. $\left(-\frac{3}{2}, \frac{3\sqrt{3}}{2}\right)$
2. $(2\sqrt{2}, 2\sqrt{2})$
3. $(-1, -\sqrt{3})$
4. $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

5. (a)

θ	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{5\pi}{6}$	π	$\frac{4\pi}{3}$	2π
r	$\frac{3\sqrt{2}}{2}$	3	$\frac{3}{2}$	0	$\frac{-3\sqrt{3}}{2}$	0

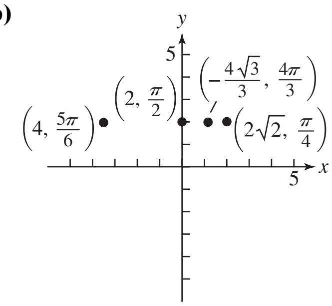
(b)



6. (a)

θ	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{5\pi}{6}$	π	$\frac{4\pi}{3}$	2π
r	$2\sqrt{2}$	2	4	undefined	$\frac{-4\sqrt{3}}{2}$	undefined

(b)



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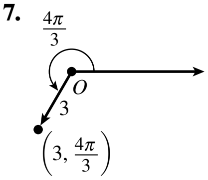

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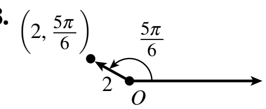

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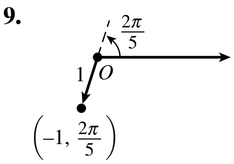

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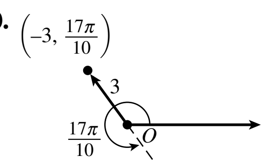

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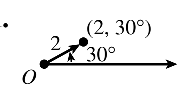
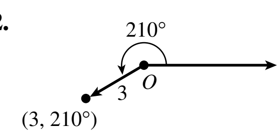


12.



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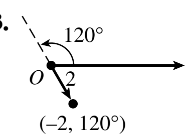

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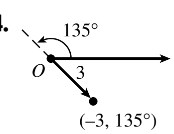


15. $\left(\frac{3}{4}, \frac{3}{4}\sqrt{3}\right)$
16. $\left(\frac{5}{4}\sqrt{2}, \frac{5}{4}\sqrt{2}\right)$
17. $(-2.70, 1.30)$
18. $(1.62, 1.18)$
19. $(2, 0)$
20. $(0, 1)$

21. $(0, -2)$

22. $(-3, 0)$

23. $\left(2, \frac{\pi}{6} + 2n\pi\right)$ and $\left(-2, \frac{\pi}{6} + (2n+1)\pi\right)$, n an integer

24. $\left(1, -\frac{\pi}{4} + 2n\pi\right)$ and $\left(-1, -\frac{\pi}{4} + (2n+1)\pi\right)$, n an integer

25. $(1.5, -20° + 360n°)$ and $(-1.5, 160° + 360n°)$, n an integer

26. $(-2.5, 50° + 360n°)$ and $(2.5, 230° + 360n°)$, n an integer

27. **(a)** $\left(\sqrt{2}, \frac{\pi}{4}\right)$ or $\left(-\sqrt{2}, \frac{5\pi}{4}\right)$

(b) $\left(\sqrt{2}, \frac{\pi}{4}\right)$ or $\left(-\sqrt{2}, -\frac{3\pi}{4}\right)$

(c) The answers from **(a)**, and also $\left(\sqrt{2}, \frac{9\pi}{4}\right)$ or $\left(-\sqrt{2}, \frac{13\pi}{4}\right)$

28. **(a)** $(\sqrt{10}, \tan^{-1} 3) \approx (\sqrt{10}, 1.25)$ or $(-\sqrt{10}, \tan^{-1} 3 + \pi) \approx (-\sqrt{10}, 4.39)$

(b) $(\sqrt{10}, \tan^{-1} 3) \approx (\sqrt{10}, 1.25)$ or $(-\sqrt{10}, \tan^{-1} 3 - \pi) \approx (-\sqrt{10}, -1.89)$

(c) The answers from **(a)**, and also $(\sqrt{10}, \tan^{-1} 3 + 2\pi) \approx (\sqrt{10}, 7.53)$ or $(-\sqrt{10}, \tan^{-1} 3 + 3\pi) \approx (-\sqrt{10}, 10.67)$

29. **(a)** $(\sqrt{29}, \tan^{-1}(-2.5) + \pi) \approx (\sqrt{29}, 1.95)$ or $(-\sqrt{29}, \tan^{-1}(-2.5) + 2\pi) \approx (-\sqrt{29}, 5.09)$

(b) $(-\sqrt{29}, \tan^{-1}(-2.5)) \approx (-\sqrt{29}, -1.19)$ or $(\sqrt{29}, \tan^{-1}(-2.5) + \pi) \approx (\sqrt{29}, 1.95)$

(c) The answers from **(a)**, plus $(\sqrt{29}, \tan^{-1}(-2.5) + 3\pi) \approx (\sqrt{29}, 8.23)$ or $(-\sqrt{29}, \tan^{-1}(-2.5) + 4\pi) \approx (-\sqrt{29}, 11.38)$

30. **(a)** $(-\sqrt{5}, \tan^{-1} 2) \approx (-\sqrt{5}, 1.11)$ or $(\sqrt{5}, \tan^{-1} 2 + \pi) \approx (\sqrt{5}, 4.25)$

(b) $(-\sqrt{5}, \tan^{-1} 2) \approx (-\sqrt{5}, 1.11)$ or $(\sqrt{5}, \tan^{-1} 2 - \pi) \approx (\sqrt{5}, -2.03)$

(c) The answers from **(a)**, plus $(-\sqrt{5}, \tan^{-1} 2 + 2\pi) \approx (-\sqrt{5}, 7.39)$ or $(\sqrt{5}, \tan^{-1} 2 + 3\pi) \approx (\sqrt{5}, 10.53)$

31. **(b)**

32. **(d)**

33. **(c)**

34. **(a)**

35. $x = 3$ — a vertical line

36. $y = -2$ — a horizontal line

37. $r^2 + 3r\sin\theta = 0$, or $x^2 + y^2 + 3y = 0$. Completing the square gives $x^2 + \left(y + \frac{3}{2}\right)^2 = \frac{9}{4}$ — a circle centered at $\left(0, -\frac{3}{2}\right)$ with radius $\frac{3}{2}$.

38. $r^2 + 4r\cos\theta = 0$, or $x^2 + y^2 + 4x = 0$. Completing the square gives $(x+2)^2 + y^2 = 4$ — a circle centered at $(-2, 0)$ with radius 2

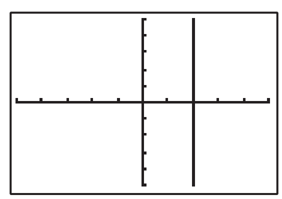
39. $r^2 - r\sin\theta = 0$, or $x^2 + y^2 - y = 0$. Completing the square gives $x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$ — a circle centered at $\left(0, \frac{1}{2}\right)$ with radius $\frac{1}{2}$.

40. $r^2 - 3r\cos\theta = 0$, or $x^2 + y^2 - 3x = 0$. Completing the square gives $\left(x - \frac{3}{2}\right)^2 + y^2 = \frac{9}{4}$ — a circle centered at $\left(\frac{3}{2}, 0\right)$ with radius $\frac{3}{2}$.

41. $r^2 - 2r\sin\theta + 4r\cos\theta = 0$, or $x^2 + y^2 - 2y + 4x = 0$. Completing the square gives $(x+2)^2 + (y-1)^2 = 5$ — a circle centered at $(-2, 1)$ with radius $\sqrt{5}$.

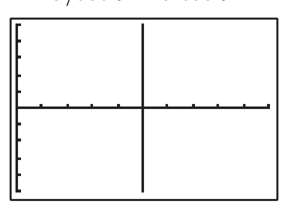
42. $r^2 - 4r\cos\theta + 4r\sin\theta = 0$, or $x^2 + y^2 - 4x + 4y = 0$. Completing the square gives $(x-2)^2 + (y+2)^2 = 8$ — a circle centered at $(2, -2)$ with radius $2\sqrt{2}$.

43. $r = 2/\cos\theta = 2\sec\theta$ — a vertical line



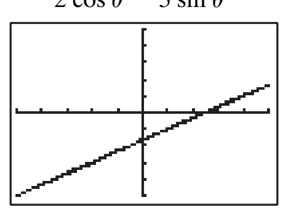
$[-5, 5]$ by $[-5, 5]$

44. $r = 5/\cos\theta = 5\sec\theta$



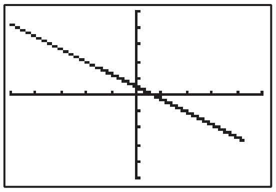
$[0, 10]$ by $[-5, 5]$

45. $r = \dfrac{5}{2\cos\theta - 3\sin\theta}$



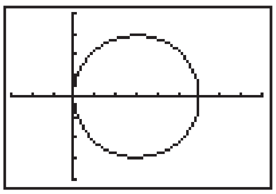
$[-5, 5]$ by $[-5, 5]$

46. $r = \dfrac{2}{3\cos\theta + 4\sin\theta}$



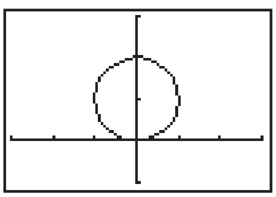
$[-5, 5]$ by $[-5, 5]$

47. $r^2 - 6r\cos\theta = 0$, so $r = 6\cos\theta$



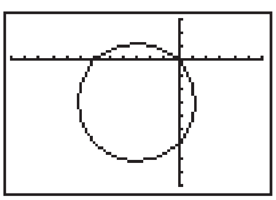
$[-3, 9]$ by $[-4, 4]$

48. $r^2 - 2r\sin\theta = 0$, so $r = 2\sin\theta$



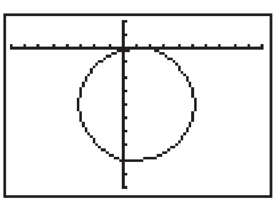
$[-3, 3]$ by $[-1, 3]$

49. $r^2 + 6r\cos\theta + 6r\sin\theta = 0$, so $r = -6\cos\theta - 6\sin\theta$



$[-12, 6]$ by $[-9, 3]$

50. $r^2 - 2r\cos\theta + 8r\sin\theta = 0$, so $r = 2\cos\theta - 8\sin\theta$



$[-8, 10]$ by $[-10, 2]$

51. $d = \sqrt{3^2 + 5^2 - 2 \cdot 3 \cdot 5\cos(170° - 150°)}$
$= \sqrt{34 - 30\cos 20°} \approx 3.46$ mi

52. $d = \sqrt{4^2 + 2^2 - 2 \cdot 4 \cdot 2\cos(12° - 72°)}$
$= \sqrt{20 - 16\cos 60°} = \sqrt{12} = 2\sqrt{3} \approx 2.41$ mi

53. Using the Pythagorean theorem, the center-to-vertex distance is $\dfrac{a}{\sqrt{2}}$. The four vertices are then $\left(\dfrac{a}{\sqrt{2}}, \dfrac{\pi}{4}\right)$, $\left(\dfrac{a}{\sqrt{2}}, \dfrac{3\pi}{4}\right)$, $\left(\dfrac{a}{\sqrt{2}}, \dfrac{5\pi}{4}\right)$, and $\left(\dfrac{a}{\sqrt{2}}, \dfrac{7\pi}{4}\right)$. Other polar coordinates for these points are possible, of course.

54. The vertex on the x-axis has polar coordinates $(a, 0)$. All other vertices must also be a units from the origin; their coordinates are $\left(a, \dfrac{2\pi}{5}\right)$, $\left(a, \dfrac{4\pi}{5}\right)$, $\left(a, \dfrac{6\pi}{5}\right)$, and $\left(a, \dfrac{8\pi}{5}\right)$. Other polar coordinates for these points are possible, of course.

55. False. Point (r, θ) is the same as point $(r, \theta + 2n\pi)$ for any integer n. So each point has an infinite number of distinct polar coordinates.

56. True. For (r_1, θ) and $(r_2, \theta + \pi)$ to represent the same point, $(r_2, \theta + \pi)$ has to be the reflection across the origin of $(r_1, \theta + \pi)$, and this is accomplished by setting $r_2 = -r_1$.

57. For point (r, θ), changing the sign on r and adding 3π to θ constitutes a twofold reflection across the origin. The answer is C.

58. The rectangular coordinates are
$(-2\cos(-\pi/3), -2\sin(-\pi/3)) = (-1, \sqrt{3})$.
The answer is C.

59. For point (r, θ), changing the sign on r and subtracting $180°$ from θ constitutes a twofold reflection across the origin. The answer is A.

60. $(-2, 2)$ lies in Quadrant III, whereas $(-2\sqrt{2}, 135°)$ lies in Quadrant IV. The answer is E.

61. **(a)** If $\theta_1 - \theta_2$ is an odd integer multiple of π, then the distance is $|r_1 + r_2|$. If $\theta_1 - \theta_2$ is an even integer multiple of π, then the distance is $|r_1 - r_2|$.

(b) Consider the triangle formed by O_1, P_1, and P_2 (ensuring that the angle at the origin is less than $180°$), then by the law of cosines,
$\overline{P_1P_2}^{\,2} = \overline{OP_1}^{\,2} + \overline{OP_2}^{\,2} - 2 \cdot \overline{OP_1} \cdot \overline{OP2}\cos\theta$,
where θ is the angle between $\overline{OP_1}$ and $\overline{OP_2}$. In polar coordinates, this formula translates very nicely into $d^2 = r_1{}^2 + r_2{}^2 - 2r_1r_2\cos(\theta_2 - u_1)$ (or $\cos(\theta_1 - \theta_2)$ since $\cos(\theta_2 - \theta_1) = \cos(\theta_1 - \theta_2)$), so
$d = \sqrt{r_1{}^2 + r_2{}^2 - 2r_1 r_2\cos(\theta_2 - u_1)}$.

(c) Yes. If $\theta_1 - \theta_2$ is an odd integer multiple of π, then $\cos(\theta_1 - \theta_2) = -1 \Rightarrow d = \sqrt{r_1{}^2 + r_2{}^2 + 2r_1r_2}$ $= |r_1 + r_2|$. If $\theta_1 - \theta_2$ is an even integer multiple of π, then $\cos(\theta_1 - u_2) = 1 \Rightarrow$
$d = \sqrt{r_1{}^2 + r_2{}^2 - 2r_1r_2} = |r_1 - r_2|$.

62. **(a)** The right half of a circle centered at $(0, 2)$ of radius 2

(b) Three quarters of the same circle, starting at $(0, 0)$ and moving counterclockwise

(c) The full circle (plus another half circle found through the TRACE function)

(d) 4 counterclockwise rotations of the same circle

63. $d = \sqrt{2^2 + 5^2 - 2(2)(5)\cos 120°} \approx 6.24$

64. $d = \sqrt{4^2 + 6^2 - 2(4)(6)\cos 45°} \approx 4.25$

65. $d = \sqrt{(-3)^2 + (-5)^2 - 2(-3)(-5)\cos 135°} \approx 7.43$

66. $d = \sqrt{6^2 + 8^2 - 2(6)(8)\cos 30°} \approx 4.11$

67. Since $x = r\cos\theta$ and $y = r\sin\theta$, the parametric equation would be $x = f(\theta)\cos(\theta)$ and $y = f(\theta)\sin(\theta)$.

68. $x = 2\cos^2\theta$
$y = 2(\cos\theta)(\sin\theta)$

69. $x = 5(\cos\theta)(\sin\theta)$ $y = 5\sin^2\theta$

70. $x = 2(\cos\theta)(\sec\theta) = 2$
$y = 2(\sin\theta)(\sec\theta) = 2\tan\theta$

71. $x = 4(\cos\theta)(\csc\theta) = 4\cot\theta$
$y = 4(\sin\theta)(\csc\theta) = 4$

■ Section 6.5 Graphs of Polar Equations

Exploration 1

Answers will vary.

Exploration 2

1. If $r^2 = 4\cos(2\theta)$, then r does not exist when $\cos(2\theta) < 0$. Since $\cos(2\theta) < 0$ whenever θ is in the interval $\left(\frac{\pi}{4} + n\pi, \frac{3\pi}{4} + n\pi\right)$, n is any integer, the domain of r does not include these intervals.

2. $-r\sqrt{\cos(2\theta)}$ draws the same graph, but in the opposite direction.

3. $(r)^2 - 4\cos(-2\theta) = r^2 - 4\cos(2\theta)$
(since $\cos(\theta) = \cos(-\theta)$)

4. $(-r)^2 - 4\cos(-2\theta) = r^2 - 4\cos(2\theta)$

5. $(-r)^2 - 4\cos(2\theta) = r^2 - 4\cos(2\theta)$

Quick Review 6.5

For #1–4, use your grapher's TRACE function to solve.

1. Minimum: −3 at $x = \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$; Maximum: 3 at $x = \{0, \pi, 2\pi\}$

2. Minimum: −1 at $x = \pi$; Maximum: 5 at $x = \{0, 2\pi\}$

3. Minimum: 0 at $x = \left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}$; Maximum: 2 at $x = \{0, \pi, 2\pi\}$

4. Minimum: 0 at $x = \frac{\pi}{2}$; Maximum: 6 at $x = \frac{3\pi}{2}$

5. **(a)** No **(b)** No **(c)** Yes

6. **(a)** No **(b)** Yes **(c)** No

7. $\sin(\pi - \theta) = \sin\theta$

8. $\cos(\pi - \theta) = -\cos\theta$

9. $\cos 2(\pi + \theta) = \cos(2\pi + 2\theta) = \cos 2\theta$
$= \cos^2\theta - \sin^2\theta$

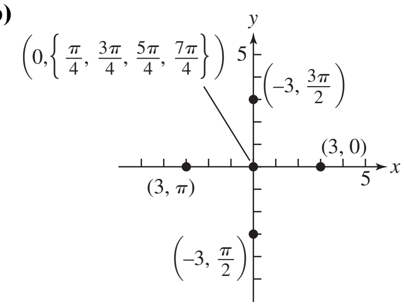
10. $\sin 2(\pi + \theta) = \sin(2\pi + 2\theta) = \sin 2\theta$
$= 2\sin\theta\cos\theta$

Section 6.5 Exercises

1. (a)

θ	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$
r	3	0	−3	0	3	0	−3	0

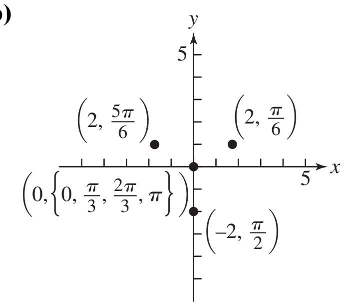
(b)



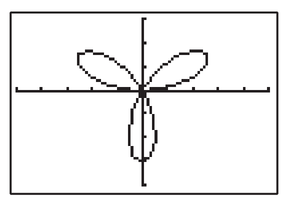
2. (a)

θ	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π
r	0	2	0	−2	0	2	0

(b)

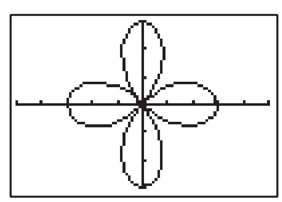


3. $k = \pi$



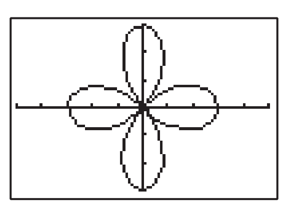
[−5, 5] by [−4, 3]

4. $k = 2\pi$



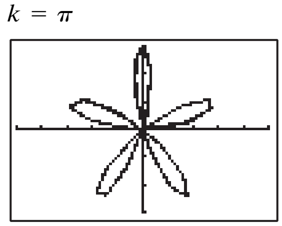
[−5, 5] by [−3, 3]

5. $k = 2\pi$



[−5, 5] by [−3, 3]

6. $k = \pi$



[−5, 5] by [−3, 3]

7. r_1 is not shown (this is a 12-petal rose). r_2 is not shown (this is a 6-petal rose), r_3 is graph **(b)**.

8. $6 \cos 2\theta \sin 2\theta = 3(2 \cos u \sin u)$ where $u = 2\theta$; this equals $3 \sin 2u = 3 \sin 4\theta$. $r = 3 \sin 4\theta$ is the equation for the 8-petal rose shown in graph **(a)**.

9. Graph **(b)** is $r = 2 - 2 \cos \theta$: Taking $\theta = 0$ and $\theta = \frac{\pi}{2}$, we get $r = 2$ and $r = 4$ from the first equation, and $r = 0$ and $r = 2$ from the second. No graph matches the first of these (r, θ) pairs, but **(b)** matches the latter (and any others one might choose).

10. Graph **(c)** is $r = 2 + 3 \cos \theta$: Taking $\theta = 0$, we get $r = -1$ from the other equation, which matches nothing. Any (r, θ) pair from the first equation matches **(c)**, however.

11. Graph **(a)** is $r = 2 - 2 \sin \theta$ — where $\theta = \frac{\pi}{2}$, $2 + 2 \cos \theta = 2$, but $\left(2, \frac{\pi}{2}\right)$ is clearly not on graph **(a)**; meanwhile $2 - 2 \sin \frac{\pi}{2} = 0$, and $\left(0, \frac{\pi}{2}\right)$ (the origin) is part of graph **(a)**.

12. Graph **(d)** is $r = 2 - 1.5 \sin \theta$ — where $\theta = \frac{\pi}{2}$, $2 + 1.5 \cos \theta = 2$, but $\left(2, \frac{\pi}{2}\right)$ is clearly not on graph **(d)**; meanwhile $2 - 1.5 \sin \frac{\pi}{2} = 0.5$, and $\left(0.5, \frac{\pi}{2}\right)$ is part of graph **(d)**.

13. Symmetric about the y-axis: Replacing (r, θ) with $(r, \pi - \theta)$ gives the same equation, since $\sin(\pi - \theta) = \sin \theta$.

14. Symmetric about the x-axis: Replacing (r, θ) with $(r, -\theta)$ gives the same equation, since $\cos(-\theta) = \cos \theta$.

15. Symmetric about the x-axis: Replacing (r, θ) with $(r, -\theta)$ gives the same equation, since $\cos(-\theta) = \cos \theta$.

16. Symmetric about the y-axis: Replacing (r, θ) with $(r, \pi - \theta)$ gives the same equation, since $\sin(\pi - \theta) = \sin \theta$.

17. All three symmetries. Polar axis: Replacing (r, θ) with $(r, -\theta)$ gives the same equation, since $\cos(-2\theta) = \cos 2\theta$. y-axis: replacing (r, θ) with $(r, \pi - \theta)$ gives the same equation, since $\cos[2(\pi - \theta)] = \cos(2\pi - 2\theta) = \cos(-2\theta) = \cos 2\theta$. Pole: Replacing (r, θ) with $(r, \theta + \pi)$ gives the same equation, since $\cos[2(\theta + \pi)] = \cos(2\theta + 2\pi) = \cos 2\theta$.

18. Symmetric about the y-axis: Replacing (r, θ) with $(-r, -\theta)$ gives the same equation, since $\sin(-3\theta) = -\sin 3\theta$.

19. Symmetric about the y-axis: Replacing (r, θ) with $(r, \pi - \theta)$ gives the same equation, since $\sin(\pi - \theta) = \sin \theta$.

20. Symmetric about the x-axis: Replacing (r, θ) with $(r, -\theta)$ gives the same equation, since $\cos(-\theta) = \cos \theta$.

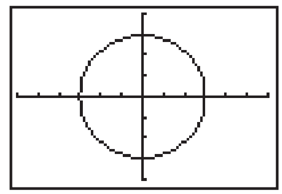
21. Maximum $|r|$ is 5 — when $\theta = 2n\pi$ for any integer n.

22. Maximum $|r|$ is 5 (when $r = -5$) — when $\theta = \frac{3\pi}{2} + 2n\pi$ for any integer n.

23. Maximum $|r|$ is 3 (when $r = \pm 3$) — when $\theta = 2n\pi/3$ for any integer n.

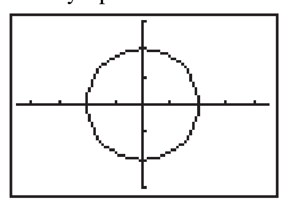
24. Maximum $|r|$ is 4 (when $r = \pm 4$) — when $\theta = n\pi/4$ for any odd integer n.

25. Domain: $(-\infty, \infty)$
Range: $r = 3$
Symmetric about the x-axis, y-axis, and origin
Continuous
Bounded
Maximum $|r|$ value: 3
No asymptotes



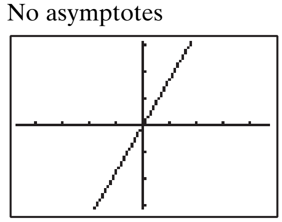
[–6, 6] by [–4, 4]

26. Domain: $(-\infty, \infty)$
Range: $r = 2$
Symmetric about the x-axis, y-axis, and origin
Continuous
Bounded
Maximum $|r|$ value: 2
No asymptotes



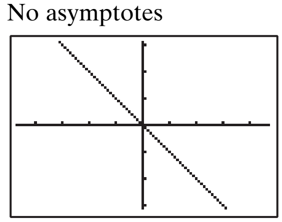
[–4.5, 4.5] by [–3, 3]

27. Domain: $\theta = \pi/3$
Range: $(-\infty, \infty)$
Symmetric about the origin
Continuous
Unbounded
Maximum $|r|$ value: none
No asymptotes



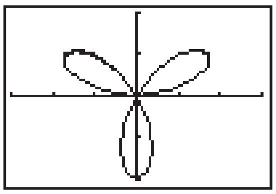
[−4.7, 4.7] by [−3.1, 3.1]

28. Domain: $\theta = -\pi/4$
Range: $(-\infty, \infty)$
Symmetric about the origin
Continuous
Unbounded
Maximum $|r|$ value: none
No asymptotes



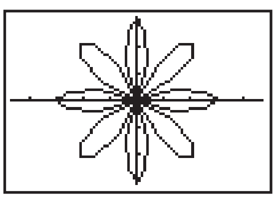
[−4.7, 4.7] by [−3.1, 3.1]

29. Domain: $(-\infty, \infty)$
Range: $[-2, 2]$
Symmetric about the y-axis
Continuous
Bounded
Maximum $|r|$ value: 2
No asymptotes



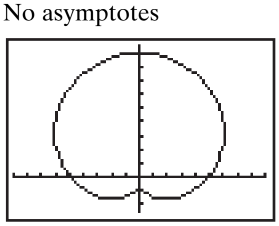
[–3, 3] by [–2, 2]

30. Domain: $(-\infty, \infty)$
Range: $[-3, 3]$
Symmetric about the x-axis, y-axis, and origin
Continuous
Bounded
Maximum $|r|$ value: 3
No asymptotes



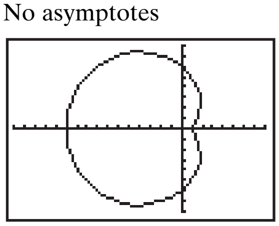
[–4.7, 4.7] by [–3.1, 3.1]

31. Domain: $(-\infty, \infty)$
Range: $[1, 9]$
Symmetric about the y-axis
Continuous
Bounded
Maximum $|r|$ value: 9
No asymptotes



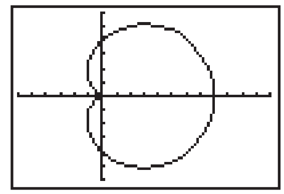
[–9, 9] by [–2.5, 9.5]

32. Domain: $(-\infty, \infty)$
Range: $[1, 11]$
Symmetric about the x-axis
Continuous
Bounded
Maximum $|r|$ value: 11
No asymptotes



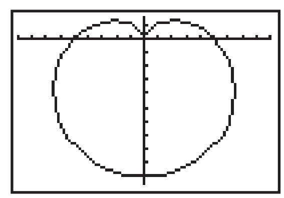
[–16, 8] by [–8, 8]

33. Domain: $(-\infty, \infty)$
Range: $[0, 8]$
Symmetric about the x-axis
Continuous
Bounded
Maximum $|r|$ value: 8
No asymptotes



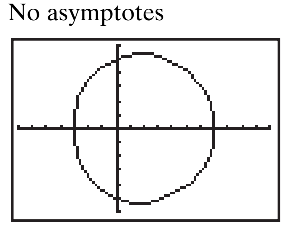
[–6, 12] by [–6, 6]

34. Domain: $(-\infty, \infty)$
Range: $[0, 10]$
Symmetric about the y-axis
Continuous
Bounded
Maximum $|r|$ value: 10
No asymptotes



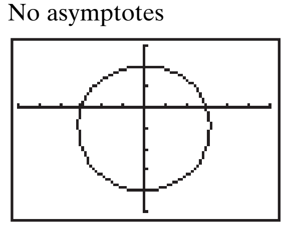
[–9, 9] by [–10.5, 1.5]

35. Domain: $(-\infty, \infty)$
Range: $[3, 7]$
Symmetric about the x-axis
Continuous
Bounded
Maximum $|r|$ value: 7
No asymptotes



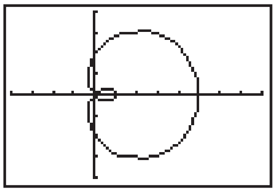
[–7, 11] by [–6, 6]

36. Domain: $(-\infty, \infty)$
Range: $[2, 4]$
Symmetric about the y-axis
Continuous
Bounded
Maximum $|r|$ value: 4
No asymptotes

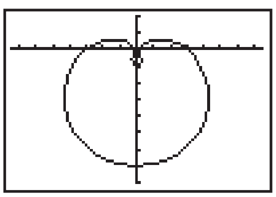


[–6, 6] by [–6, 3]

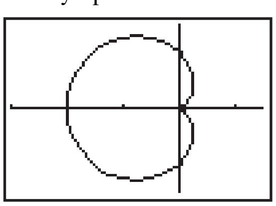
37. Domain: $(-\infty, \infty)$
Range: $[-3, 7]$
Symmetric about the x-axis
Continuous
Bounded
Maximum $|r|$ value: 7
No asymptotes


[–4, 8] by [–4, 4]

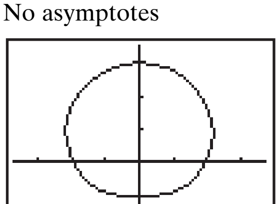
38. Domain: $(-\infty, \infty)$
Range: $[-1, 7]$
Symmetric about the y-axis
Continuous
Bounded
Maximum $|r|$ value: 7
No asymptotes


[–7.5, 7.5] by [–8, 2]

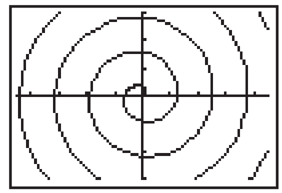
39. Domain: $(-\infty, \infty)$
Range: $[0, 2]$
Symmetric about the x-axis
Continuous
Bounded
Maximum $|r|$ value: 2
No asymptotes


[–3, 1.5] by [–1.5, 1.5]

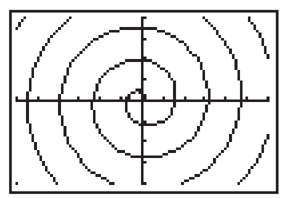
40. Domain: $(-\infty, \infty)$
Range: $[1, 3]$
Symmetric about the y-axis
Continuous
Bounded
Maximum $|r|$ value: 3
No asymptotes


[–3.75, 3.75] by [–1.5, 3.5]

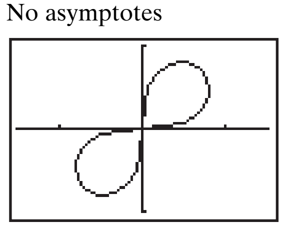
41. Domain: $(-\infty, \infty)$
Range: $[0, \infty)$
Continuous
No symmetry
Unbounded
Maximum $|r|$ value: none
No asymptotes
Graph for $\theta \geq 0$:


[–45, 45] by [–30, 30]

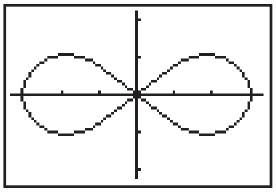
42. Domain: $(-\infty, \infty)$
Range: $[0, \infty)$
Continuous
No symmetry
Unbounded
Maximum $|r|$ value: none
No asymptotes
Graph for $\theta \geq 0$:


[–6, 6] by [–4, 4]

43. Domain: $\left[0, \frac{\pi}{2}\right] \cup \left[\pi, \frac{3\pi}{2}\right]$
Range: $[0, 1]$
Symmetric about the origin
Continuous on each interval in domain
Bounded
Maximum $|r|$ value: 1
No asymptotes


[–1.5, 1.5] by [–1, 1]

44. Domain: $\left[0, \frac{\pi}{4}\right] \cup \left[\frac{3\pi}{4}, \frac{5\pi}{4}\right] \cup \left[\frac{7\pi}{4}, 2\pi\right]$
Range: $[0, 3]$
Symmetric about the x-axis, y-axis, and origin
Continuous on each interval in domain
Bounded
Maximum $|r|$ value: 3
No asymptotes



[–3.3, 3.3] by [–2.2, 2.2]

For #45–48, recall that the petal length is the maximum $|r|$ value over the interval that creates the petal.

45. $r = -2$ when $\theta = \left\{\frac{3\pi}{4}, \frac{7\pi}{4}\right\}$ and $r = 6$ when $\theta = \left\{\frac{\pi}{4}, \frac{5\pi}{4}\right\}$. There are four petals with lengths {6, 2, 6, 2}.

46. $r = -2$ when $\theta = \{0, \pi\}$ and $r = 8$ when $\theta = \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$. There are four petals with lengths {2, 8, 2, 8}.

47. $r = -3$ when $\theta = \left\{0, \frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5}\right\}$ and $r = 5$ when $\theta = \left\{\frac{\pi}{5}, \frac{3\pi}{5}, \pi, \frac{7\pi}{5}, \frac{9\pi}{5}\right\}$.

There are ten petals with lengths {3, 5, 3, 5, 3, 5, 3, 5, 3, 5}.

48. $r = 7$ when $\theta = \left\{\frac{\pi}{10}, \frac{\pi}{2}, \frac{9\pi}{10}, \frac{13\pi}{10}, \frac{17\pi}{10}\right\}$ and $r = -1$ when $\theta = \left\{\frac{3\pi}{10}, \frac{7\pi}{10}, \frac{11\pi}{10}, \frac{3\pi}{2}, \frac{19\pi}{10}\right\}$.

There are ten petals with lengths {7, 1, 7, 1, 7, 1, 7, 1, 7, 1}.

49. r_1 and r_2 produce identical graphs — r_1 begins at (1, 0) and r_2 begins at (−1, 0).

50. r_1 and r_3 produce identical graphs — r_1 begins at (3, 0) and r_2 begins at (1, 0).

51. r_2 and r_3 produce identical graphs — r_1 begins at (3, 0) and r_3 begins at (−3, 0).

52. r_1 and r_2 produce identical graphs — r_1 begins at (2, 0) and r_2 begins at (−2, 0).

53. **(a)** A 4-petal rose curve with 2 short petals of length 1 and 2 long petals of length 3.

(b) Symmetric about the origin.

(c) Maximum $|r|$ value: 3.

54. **(a)** A 4-petal rose curve with petals of about length 1, 3.3, and 4 units.

(b) Symmetric about the y-axis.

(c) Maximum $|r|$ value: 4.

55. **(a)** A 6-petal rose curve with three short petals of length 2 and three long petals of length 4.

(b) Symmetric about the x-axis.

(c) Maximum $|r|$ value: 4.

56. **(a)** A 6-petal rose curve with three short petals of length 2 and three long petals of length 4.

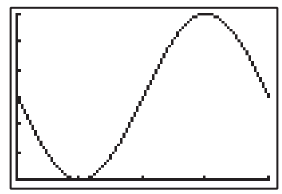
(b) Symmetric about the y-axis.

(c) Maximum $|r|$ value: 4.

57. Answers will vary but generally students should find that a controls the length of the rose petals and n controls both the number of rose petals and symmetry. If n is odd, n rose petals are formed, with the cosine curve symmetric about the polar x-axis and sine curve symmetric about the y-axis. If n is even, $2n$ rose petals are formed, with both the cosine and sine functions having symmetry about the polar x-axis, y-axis, and origin.

58. Symmetry about y-axis: $r - 3\sin(4\theta) = 0 \Rightarrow -r - 3\sin(4(-\theta)) = -r + 3\sin(4\theta)$ (since $\sin(\theta)$ is odd, i.e., $\sin(-\theta) = -\sin(\theta)$) $= r - 3\sin(4\theta) = 0$.
Symmetry about the origin: $r - 3\sin(4\theta) = 0 \Rightarrow r - 3\sin(4\theta + 4\pi) = r - 3\sin(4\theta) = 0$.

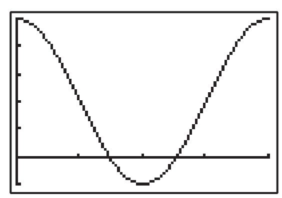
59.



$[0, 2\pi]$ by $[0, 6]$

$y = 3 - 3\sin x$ has minimum and maximum values of 0 and 6 on $[0, 2\pi]$. So the range of the polar function $r = 3 - 3\sin\theta$ is also [0, 6].

60.



$[0, 2\pi]$ by $[-1, 5]$

$y = 2 + 3\cos x$ has minimum and maximum values of −1 and 5 on $[0, 2\pi]$. So the range of the polar function $r = 2 + 3\cos\theta$ is also [−1, 5].

In general, this works because any polar graph can also be plotted using rectangular coordinates. Here, we have y representing r and x representing θ on a rectangular coordinate graph. Since y is exactly equal to r, the range of y and range of r will be exactly the same.

61. False. The spiral $r = \theta$ is unbounded, since a point on the curve can be found at any arbitrarily large distance from the origin by setting θ numerically equal to that distance.

62. True. If point (r, θ) satisfies the equation $r = 2 + \cos\theta$, then point $(r, -\theta)$ does also, since $2 + \cos(-\theta) = 2 + \cos\theta = r$.

63. With $r = a\cos n\theta$, if n is even there are $2n$ petals. The answer is D.

64. The four petals lie along the x- and y-axis, because $\cos 2\theta$ takes on its extreme values at multiples of $\pi/2$. The answer is D.

65. When $\cos\theta = -1$, $r = 5$. The answer is B.

66. With $r = a\sin n\theta$, if n is odd there are n petals. The answer is B.

67. **(a)** Symmetry about the polar x-axis: $r - a\cos(n\theta) = 0 \Rightarrow r - a\cos(-n\theta) = r - a\cos(n\theta)$ (since $\cos(\theta)$ is even, i.e., $\cos(\theta) = \cos(-\theta)$ for all θ) $= 0$.

(b) No symmetry about y-axis: $r - a\cos(n\theta) = 0 \Rightarrow -r - a\cos(-n\theta) = -r - a\cos(n\theta)$ (since $\cos(\theta)$ is even) $\neq r - a\cos(n\theta)$ unless $r = 0$. As a result, the equation is not symmetric about the y-axis.

(c) No symmetry about origin: $r - a\cos(n\theta) = 0 \Rightarrow -r - a\cos(n\theta) \neq -r - a\cos(n\theta)$ unless $r = 0$. As a result, the equation is not symmetric about the origin.

(d) Since $|\cos(n\theta)| \leq 1$ for all θ, the maximum $|r|$ value is $|a|$.

(e) Domain: $(-\infty, \infty)$
Range: $[-|a|, |a|]$
Symmetric about the x-axis
Continuous
Bounded
Maximum $|r|$ value: $|a|$
No asymptotes

68. (a) Symmetry about the y-axis: $r - a\sin(n\theta) = 0 \Rightarrow -r - a\sin(-n\theta) = -r + a\sin(n\theta)$ (since $\sin(\theta)$ is odd, $\sin(-\theta) = -\sin(\theta)) = -1(r - a\sin(n\theta)) = (-1)(0) = 0$.

(b) Not symmetric about polar x-axis: $r - a\sin(n\theta) = 0 \Rightarrow r - a\sin(-n\theta) = r + a\sin(n\theta)$. The two functions are equal only when $-\sin(n\theta) = \sin(n\theta) = 0$, or $\theta = \{0, \pi\}$, so $r - a\sin(n\theta)$ is not symmetric about the polar x-axis.

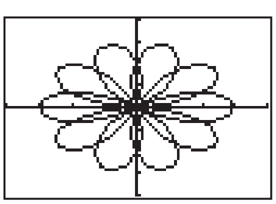
(c) Not symmetric about origin: $r - a\sin(n\theta) \Rightarrow -r - a\sin(n\theta) = -(r + a\sin(n\theta))$. The two functions are equal only when $r = 0$, so $r - a\sin(n\theta))$ is not symmetric about the origin.

(d) Since $|\sin(n\theta)| \leq 1$ for all θ, the maximum $|r|$ value is $|a|$.

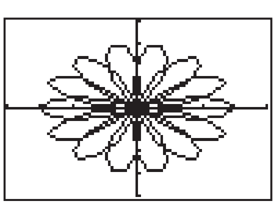
(e) Domain: $(-\infty, \infty)$
Range: $[-|a|, |a|]$
Symmetric about y-axis
Continuous
Bounded
Maximum $|r|$ value: $|a|$
No asymptotes

69. (a) For r_1: $0 \leq \theta \leq 4\pi$ (or any interval that is 4π units long). For r_2: same answer.

(b) r_1: 10 (overlapping) petals. r_2: 14 (overlapping) petals.



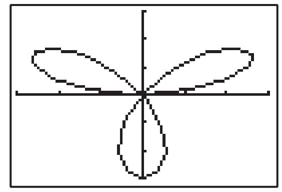
[–4, 4] by [–4, 4]



[–4, 4] by [–4, 4]

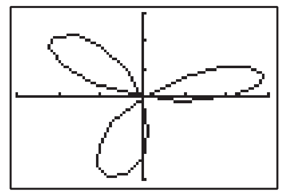
70. Starting with the graph of r_1, if we rotate clockwise (centered at the origin) by $\pi/12$ radians (15°), we get the graph of r_2; rotating r_1 clockwise by $\pi/4$ radians (45°) gives the graph of r_3.

(a)



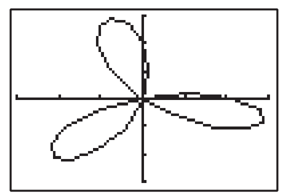
$[-3, 3]$ by $[-3, 3]$

(b)



$[-3, 3]$ by $[-3, 3]$

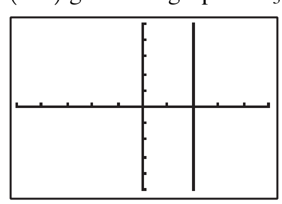
(c)



$[-3, 3]$ by $[-3, 3]$

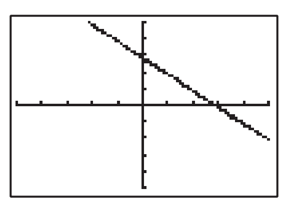
71. Starting with the graph of r_1, if we rotate counterclockwise (centered at the origin) by $\pi/4$ radians (45°), we get the graph of r_2; rotating r_1 counterclockwise by $\pi/3$ radians (60°) gives the graph of r_3.

(a)



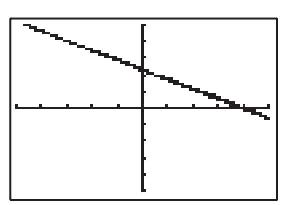
$[-5, 5]$ by $[-5, 5]$

(b)



$[-5, 5]$ by $[-5, 5]$

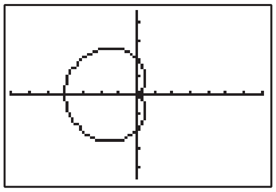
(c)



$[-5, 5]$ by $[-5, 5]$

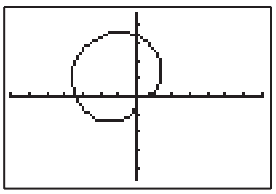
72. Starting with the graph of r_1, if we rotate clockwise (centered at the origin) by $\pi/4$ radians (45°), we get the graph of r_2; rotating r_1 clockwise by $\pi/3$ radians (60°) gives the graph of r_3.

(a)



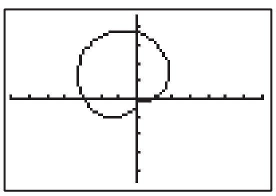
[−7.05, 7.05] by [−4.65, 4.65]

(b)



[−7.05, 7.05] by [−4.65, 4.65]

(c)



[−7.05, 7.05] by [−4.65, 4.65]

73. The second graph is the result of rotating the first graph clockwise (centered at the origin) through an angle of α. The third graph results from rotating the first graph counterclockwise through the same angle. One possible explanation: the radius r achieved, for example, when $\theta = 0$ in the first equation is achieved instead when $\theta = -\alpha$ for the second equation, and when $\theta = \alpha$ for the third equation.

■ Section 6.6 DeMoivre's Theorem and *n*th Roots

Quick Review 6.6

1. Using the quadratic equation to find the roots of $x^2 + 13 = 4x$, we have $x^2 - 4x + 13 = 0$ with $a = 1$, $b = -4$, and $c = 13$.

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(13)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{16 - 52}}{2} = \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm 6i}{2}$$

$$x = \frac{4 + 6i}{2} = \frac{4}{2} + \frac{6i}{2} = 2 + 3i \text{ and}$$

$$x = \frac{4 - 6i}{2} = \frac{4}{2} - \frac{6i}{2} = 2 - 3i$$

The roots are $2 + 3i$ and $2 - 3i$.

2. Using the quadratic equation to find the roots of $5(x^2 + 1) = 6x$, we have $5x^2 - 6x + 5 = 0$ with $a = 5$, $b = -6$, and $c = 5$.

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(5)(5)}}{2(5)}$$

$$= \frac{6 \pm \sqrt{36 - 100}}{10} = \frac{6 \pm \sqrt{-64}}{10} = \frac{6 \pm 8i}{10}$$

$$x = \frac{6 + 8i}{10} = \frac{6}{10} + \frac{8i}{10} = 0.6 + 0.8i \text{ and}$$

$$x = \frac{6 - 8i}{10} = \frac{6}{10} + \frac{8i}{10} = 0.6 - 0.8i$$

The roots are $0.6 + 0.8i$ and $0.6 - 0.8i$.

3. $(1 + i)^5 = (1 + i) \cdot [(1 + i)^2]^2 = (1 + i) \cdot (2i)^2 = -4(1 + i) = -4 - 4i$

4. $(1 - i)^4 = [(1 - i)^2]^2 = (-2i)^2 = -4 = -4 + 0i$

For #5–8, use the given information to find a point P on the terminal side of the angle, which in turn determines the quadrant of the terminal side.

5. $P(-\sqrt{3}, 1)$, in Quadrant II: $\theta = \frac{5\pi}{6}$

6. $P(1, -1)$, in Quadrant IV: $\theta = \frac{7\pi}{4}$

7. $P(-1, -\sqrt{3},)$, in Quadrant III: $\theta = \frac{4\pi}{3}$

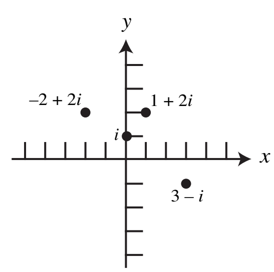
8. $P(-1, -1)$, in Quadrant III: $\theta = \frac{5\pi}{4}$

9. $x^3 = 1$ when $x = 1$

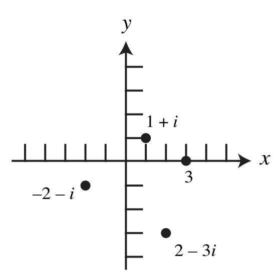
10. $x^4 = 1$ when $x = \pm 1$

Section 6.6 Exercises

1.



2.



For #3–12, $a + bi = r(\cos\theta + i\sin\theta)$, where $r = |a + bi| = \sqrt{a^2 + b^2}$ and θ is chosen so that $\cos\theta = \frac{a}{\sqrt{a^2 + b^2}}$ and $\sin\theta = \frac{b}{\sqrt{a^2 + b^2}}$.

3. $r = |3i| = 3$; $\cos\theta = 0$ and $\sin\theta = 1$, so $\theta = \frac{\pi}{2}$:

$$3i = 3\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

4. $r = |-2i| = 2$; $\cos\theta = 0$ and $\sin\theta = -1$, so $\theta = \frac{3\pi}{2}$: $-2i = 2\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$

5. $r = |2 + 2i| = 2\sqrt{2}$; $\cos\theta = \frac{\sqrt{2}}{2}$ and $\sin\theta = \frac{\sqrt{2}}{2}$, so $\theta = \frac{\pi}{4}$: $2 + 2i = 2\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

6. $r = |\sqrt{3} + i| = 2$; $\cos\theta = \frac{\sqrt{3}}{2}$ and $\sin\theta = \frac{1}{2}$, so $\theta = \frac{\pi}{6}$: $\sqrt{3} + i = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

7. $r = |-2 + 2i\sqrt{3}| = 4$; $\cos\theta = -\frac{1}{2}$ and $\sin\theta = \frac{\sqrt{3}}{2}$, so $\theta = \frac{2\pi}{3}$: $-2 + 2i\sqrt{3} = 4\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

8. $r = |3 - 3i| = 3\sqrt{2}$; $\cos\theta = \frac{\sqrt{2}}{2}$ and $\sin\theta = -\frac{\sqrt{2}}{2}$, so $\theta = \frac{7\pi}{4}$: $3 - 3i = 3\sqrt{2}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$

9. $r = |3 + 2i| = \sqrt{13}$; $\cos\theta = \frac{3}{\sqrt{13}}$ and $\sin\theta = \frac{2}{\sqrt{13}}$, so $\theta \approx 0.588$: $3 + 2i \approx \sqrt{13}(\cos 0.59 + i\sin 0.59)$

10. $r = |4 - 7i| = \sqrt{65}$; $\cos\theta = \frac{4}{\sqrt{65}}$ and $\sin\theta = -\frac{7}{\sqrt{65}}$, so $\theta \approx 5.232$: $4 - 7i \approx \sqrt{65}\,(\cos 5.23 + i\sin 5.23)$

11. $r = 3$; $30^\circ = \frac{\pi}{6}$; $3\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

12. $r = 3$; $225^\circ = \frac{5\pi}{4}$; $4\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$

13. $3\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = \frac{3\sqrt{3}}{2} - \frac{3}{2}i$

14. $8\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = -4\sqrt{3} - 4i$

15. $5\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = \frac{5}{2} - \frac{5\sqrt{3}}{2}i$

16. $5\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right) = \frac{5\sqrt{2}}{2} + \frac{5\sqrt{2}}{2}i$

17. $\sqrt{2}\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = \frac{\sqrt{6}}{2} - \frac{\sqrt{2}}{2}i$

18. $\approx 2.56 + 0.68i$

19. $(2 \cdot 7)[\cos(25^\circ + 130^\circ) + i\sin(25^\circ + 130^\circ)] = 14\,(\cos 155^\circ + i\sin 155^\circ)$

20. $(\sqrt{2} \cdot 0.5)[\cos(188^\circ - 19^\circ) + i\sin(118^\circ - 19^\circ)] = \frac{\sqrt{2}}{2}\,(\cos 99^\circ + i\sin 99^\circ)$

21. $(5 \cdot 3)\left[\cos\left(\frac{\pi}{4} + \frac{5\pi}{3}\right) + i\sin\left(\frac{\pi}{4} + \frac{5\pi}{3}\right)\right] = 15\left(\cos\frac{23\pi}{12} + i\sin\frac{23\pi}{12}\right)$

22. $\left(\sqrt{3} \cdot \frac{1}{3}\right)\left[\cos\left(\frac{3\pi}{4} + \frac{\pi}{6}\right) + i\sin\left(\frac{3\pi}{4} + \frac{\pi}{6}\right)\right] = \frac{\sqrt{3}}{3}\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right)$

23. $\frac{2}{3}[\cos(30^\circ - 60^\circ) + i\sin(30^\circ - 60^\circ)] = \frac{2}{3}[\cos(-30^\circ) + i\sin(-30^\circ)] = \frac{2}{3}(\cos 30^\circ - i\sin 30^\circ)$

24. $\frac{5}{2}[\cos(220^\circ - 115^\circ) + i\sin(220^\circ - 115^\circ)] = \frac{5}{2}(\cos 105^\circ + i\sin 105^\circ)$

25. $\frac{6}{3}[\cos(5\pi - 2\pi) + i\sin(5\pi - 2\pi)] = 2(\cos 3\pi + i\sin 3\pi) = 2(\cos\pi + i\sin\pi)$

26. $1\left[\cos\left(\frac{\pi}{2} - \frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{2} - \frac{\pi}{4}\right)\right] = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$

27. **(a)** $3 - 2i \approx \sqrt{13}\,[\cos(5.695) + i\sin(5.695)]$ and $1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + \text{i}\sin\frac{\pi}{4}\right)$, so

$\sqrt{13}\,[\cos(5.695) + i\sin(5.695)] \cdot \sqrt{2}\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right]$

$= \sqrt{26}\left[\cos\left(5.695 + \frac{\pi}{4}\right) + i\sin\left(5.695 + \frac{\pi}{4}\right)\right] = 5 + i$

$\frac{\sqrt{13}[\cos(5.695) + i\sin(5.695)]}{\sqrt{2}[\cos(\pi/4) + \text{i}\sin(\pi/4)]}$

$\approx \sqrt{6.5}\left[\cos\left(5.695 - \frac{\pi}{4}\right) + i\sin\left(5.695 - \frac{\pi}{4}\right)\right]$

$= \frac{1}{2} - \frac{5}{2}i$

(b) $(3 - 2i)(1 + i) = 3 + 3i - 2i - 2i^2 = 5 + i$

$\frac{3 - 2i}{1 + i} = \frac{3 - 2i}{1 + i} \cdot \frac{1 - i}{1 - i} = \frac{1 - 5i}{2} = \frac{1}{2} - \frac{5}{2}i$

28. **(a)** $1 - i = \sqrt{2}\,(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4})$ and $\sqrt{3} + i = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$, so

$\sqrt{2}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right) \cdot 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

$= 2\sqrt{2}\left(\cos\frac{23\pi}{12} + \sin\frac{23\pi}{12}\right) \approx 2.73 - 0.73i$

$\frac{\sqrt{2}[\cos(7\pi/4) + i\sin(7\pi/4)]}{2[\cos(\pi/6) + i\sin(\pi/6)]}$

$= \frac{1}{\sqrt{2}}\left(\cos\frac{19\pi}{12} + i\sin\frac{19\pi}{12}\right) \approx 0.18 - 0.68i$

(b) $(1 - i)(\sqrt{3} + i) = \sqrt{3} + i - \sqrt{3}i - i^2$
$(1 + \sqrt{3}) + (1 - \sqrt{3})i \approx 2.73 - 0.73i$
$\frac{1 - i}{\sqrt{3} + i} = \frac{1 - i}{\sqrt{3} + i} \cdot \frac{\sqrt{3} - i}{\sqrt{3} - i} = \frac{(1 - i)(\sqrt{3} - i)}{4}$
$= \frac{1}{4}\left[\sqrt{3} - 1 - (\sqrt{3} + 1)i\right] \approx 0.18 - 0.68i$

29. (a) $3 + i \approx \sqrt{10}[\cos(0.321) + i\sin(0.321)]$ and $5 - 3i \approx \sqrt{34}[\cos(-0.540) + i\sin(-0.540)]$, so $\sqrt{10}[\cos(0.321) + i\sin(0.321)] \cdot \sqrt{34}[\cos(-0.540) + i\sin(-0.540)]$
$= 2\sqrt{85}[\cos(-0.219) + i\sin(-0.219)] = 18 - 4i$
$\frac{\sqrt{10}[\cos(0.321) + i\sin(0.321)]}{\sqrt{34}[\cos(-0.540) + i\sin(-0.540)]}$
$\approx \sqrt{\frac{5}{17}}[\cos(0.862) + i\sin(0.862)]$
$\approx 0.35 + 0.41i$

(b) $(3 + i)(5 - 3i) = 15 - 9i + 5i - 3i^2 = 18 - 4i$
$\frac{3 + i}{5 - 3i} = \frac{3 + i}{5 - 3i} \cdot \frac{5 + 3i}{5 + 3i} = \frac{(3 + i)(5 + 3i)}{34}$
$\frac{1}{17}(6 + 7i) \approx 0.35 + 0.41i$

30. (a) $2 - 3i \approx \sqrt{13}[\cos(-0.982) + i\sin(-0.982)]$, and $1 - \sqrt{3}i = 2\left[\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right]$, so
$\sqrt{13}[\cos(-0.983) + i\sin(-0.983)] \cdot \left[\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right] = 2\sqrt{13}\left[\cos\left(-0.983 - \frac{\pi}{3}\right) + i\sin\left(-0.983 - \frac{\pi}{3}\right)\right]$
$\approx -3.20 - 6.46i$
$\frac{\sqrt{13}[\cos(-0.983) + i\sin(-0.983)]}{2[\cos(-\pi/3) + i\sin(-\pi/3)]}$
$\approx \frac{\sqrt{13}}{2}[\cos(0.064) + i\sin(0.064)]$
$\approx 1.80 + 0.12i$

(b) $(2 - 3i)(1 - \sqrt{3}i) = 2 - 2\sqrt{3}i - 3i + 3\sqrt{3}i^2$
$= (2 - 3\sqrt{3}) - (2\sqrt{3} + 3)i \approx -3.196 - 6.464i$
$\frac{2 - 3i}{1 - \sqrt{3}i} = \frac{2 - 3i}{1 - \sqrt{3}i} \cdot \frac{1 + \sqrt{3}i}{1 + \sqrt{3}i}$
$= \frac{(2 - 3i)(1 + \sqrt{3}i)}{4}$
$= \frac{1}{4}\left[2 + 3\sqrt{3} + (2\sqrt{3} - 3)i\right] \approx 1.80 + 0.12i$

31. $\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^3 = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}$
$= -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

32. $\left[3\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)\right]^5 = 243\left(\cos\frac{15\pi}{2} + i\sin\frac{15\pi}{2}\right)$
$= -243i$

33. $\left[2\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)\right]^3 = 8\left(\cos\frac{9\pi}{4} + i\sin\frac{9\pi}{4}\right)$
$= 4\sqrt{2} + 4\sqrt{2}i$

34. $\left[6\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)\right]^4$
$= 1296\left(\cos\frac{20\pi}{6} + i\sin\frac{20\pi}{6}\right)$
$= -648 - 648\sqrt{3}i$

35. $(1 + i)^5 = \left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^5$
$= (\sqrt{2})^5\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$
$= 4\sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = -4 - 4i$

36. $(3 + 4i)^{20}$
$= \left\{5\left[\cos\tan^{-1}\left(\frac{4}{3}\right) + i\sin\tan^{-1}\left(\frac{4}{3}\right)\right]\right\}^{20}$
$= 5^{20}\left\{\cos\left[20\tan^{-1}\left(\frac{4}{3}\right)\right] + i\sin\left[20\tan^{-1}\left(\frac{4}{3}\right)\right]\right\}$
$= 5^{20}[\cos(5.979) + i\sin(5.979)] \approx 5^{20}(0.95 - 0.30i)$

37. $(1 - \sqrt{3}i)^3 = \left[2\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)\right]^3$
$= 8(\cos 5\pi + i\sin 5\pi) = 8(\cos\pi + i\sin\pi)$
$= -8 + 0i = -8$

38. $\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^3 = \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^3$
$= \cos\pi + i\sin\pi = -1 + 0i = -1$

For #39–44, the cube roots of $r(\cos\theta + i\sin\theta)$ are
$\sqrt[3]{r} = \left(\cos\frac{\theta + 2k\pi}{3} + i\sin\frac{\theta + 2k\pi}{3}\right), k = 0, 1, 2.$

39. $\sqrt[3]{2}\left(\cos\frac{2k\pi + 2\pi}{3} + i\sin\frac{2k\pi + 2\pi}{3}\right)$
$= \sqrt[3]{2}\left(\cos\frac{2\pi(k + 1)}{3} + i\sin\frac{2\pi(k + 1)}{3}\right)$,
$k = 0, 1, 2$:
$\sqrt[3]{2}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = \sqrt[3]{2}\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$
$= \frac{-1 + \sqrt{3}\,i}{\sqrt[3]{4}}$,
$\sqrt[3]{2}\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$
$= \sqrt[3]{2}\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = \frac{-1 - \sqrt{3}\,i}{\sqrt[3]{4}}$,
$\sqrt[3]{2}(\cos 2\pi + i\sin 2\pi) = \sqrt[3]{2}$

40. $\sqrt[3]{2}\left(\cos\frac{2k\pi + \pi/4}{3} + i\sin\frac{2k\pi + \pi/4}{3}\right)$
$= \sqrt[3]{2}\left(\cos\frac{\pi(8k + 1)}{12} + i\sin\frac{\pi(8k + 1)}{12}\right)$,
$k = 0, 1, 2$:

$\sqrt[3]{2}\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)$,

$\sqrt[3]{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$

$\sqrt[3]{2}\left(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)=\frac{-1+i}{\sqrt[6]{2}}$,

$\sqrt[3]{2}\left(\cos\frac{17\pi}{12}+i\sin\frac{17\pi}{12}\right)$

41. $\sqrt[3]{3}\left(\cos\frac{2k\pi+4\pi/3}{3}+i\sin\frac{2k\pi+4\pi/3}{3}\right)$

$=\sqrt[3]{3}\left(\cos\frac{2\pi(3k+2)}{9}+i\sin\frac{2\pi(3k+2)}{9}\right)$,

$k=0,1,2$:

$\sqrt[3]{3}\left(\cos\frac{4\pi}{9}+i\sin\frac{4\pi}{9}\right)$, $\sqrt[3]{3}\left(\cos\frac{10\pi}{9}+i\sin\frac{10\pi}{9}\right)$,

$\sqrt[3]{3}\left(\cos\frac{16\pi}{9}+i\sin\frac{16\pi}{9}\right)$

42. $\sqrt[3]{27}\left(\cos\frac{2k\pi+11\pi/6}{3}+i\sin\frac{2k\pi+11\pi/6}{3}\right)$

$=3\left(\cos\frac{\pi(12k+11)}{18}+i\sin\frac{\pi(12k+11)}{18}\right)$,

$k=0,1,2$:

$3\left(\cos\frac{11\pi}{18}+i\sin\frac{11\pi}{18}\right)$, $3\left(\cos\frac{23\pi}{18}+i\sin\frac{23\pi}{18}\right)$,

$3\left(\cos\frac{35\pi}{18}+i\sin\frac{35\pi}{18}\right)$

43. $3-4i\approx 5(\cos 5.355+i\sin 5.355)$

$\sqrt[3]{5}\left(\cos\frac{2k\pi+5.355}{3}+i\sin\frac{2k\pi+5.355}{3}\right)$

$k=0,1,2$:

$\approx\sqrt[3]{5}(\cos 1.79+i\sin 1.79)$,

$\approx\sqrt[3]{5}(\cos 3.88+i\sin 3.88)$,

$\approx\sqrt[3]{5}(\cos 5.97+i\sin 5.97)$

44. $-2+2i=2\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$. Note that

$\sqrt[3]{2\sqrt{2}}=\sqrt{2}$.

$\sqrt{2}\left(\cos\frac{2k\pi+3\pi/4}{3}+i\sin\frac{2k\pi+3\pi/4}{3}\right)$

$=\sqrt{2}\left(\cos\frac{\pi(8k+3)}{12}+i\sin\frac{\pi(8k+3)}{12}\right)$,

$k=0,1,2$:

$\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)=\sqrt{2}\left(\frac{\sqrt{2}}{2}+\frac{i\sqrt{2}}{2}\right)=1+i$,

$\sqrt{2}\left(\cos\frac{11\pi}{12}+i\sin\frac{11\pi}{12}\right)$,

$\sqrt{2}\left(\cos\frac{19\pi}{12}+i\sin\frac{19\pi}{12}\right)$

For #45–50, the fifth roots of $r(\cos\theta+i\sin\theta)$ are

$\sqrt[5]{r}\left(\cos\frac{\theta+2k\pi}{5}+i\sin\frac{\theta+2k\pi}{5}\right)$, $k=0,1,2,3,4$.

45. $\cos\frac{2k\pi+\pi}{5}+i\sin\frac{2k\pi+\pi}{5}$

$=\cos\frac{\pi(2k+1)}{5}+i\sin\frac{\pi(2k+1)}{5}$, $k=0,1,2,3,4$:

$\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}$, $\cos\frac{3\pi}{5}+i\sin\frac{3\pi}{5}$, -1,

$\cos\frac{7\pi}{5}+i\sin\frac{7\pi}{5}$, $\cos\frac{9\pi}{5}+i\sin\frac{9\pi}{5}$

46. $\sqrt[5]{32}\left(\cos\frac{2k\pi+\pi/2}{5}+i\sin\frac{2k\pi+\pi/2}{5}\right)$

$=2\left(\cos\frac{\pi(4k+1)}{10}+i\sin\frac{\pi(4k+1)}{10}\right)$,

$k=0,1,2,3,4$:

$2\left(\cos\frac{\pi}{10}+i\sin\frac{\pi}{10}\right)$, $2\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)=2i$,

$2\left(\cos\frac{9\pi}{10}+i\sin\frac{9\pi}{10}\right)$, $2\left(\cos\frac{13\pi}{10}+i\sin\frac{13\pi}{10}\right)$,

$2\left(\cos\frac{17\pi}{10}+i\sin\frac{17\pi}{10}\right)$

47. $\sqrt[5]{2}\left(\cos\frac{2k\pi+\pi/6}{5}+i\sin\frac{2k\pi+\pi/6}{5}\right)$

$\sqrt[5]{2}\left(\cos\frac{\pi(12k+1)}{30}+\sin\frac{\pi(12k+1)}{30}\right)$,

$k=0,1,2,3,4$:

$\sqrt[5]{2}\left(\cos\frac{\pi}{30}+i\sin\frac{\pi}{30}\right)$, $\sqrt[5]{2}\left(\cos\frac{13\pi}{30}+i\sin\frac{13\pi}{30}\right)$,

$\sqrt[5]{2}\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)$, $\sqrt[5]{2}\left(\cos\frac{37\pi}{30}+i\sin\frac{37\pi}{30}\right)$,

$\sqrt[5]{2}\left(\cos\frac{49\pi}{30}+i\sin\frac{49\pi}{30}\right)$

48. $\sqrt[5]{2}\left(\cos\frac{2k\pi+\pi/4}{5}+i\sin\frac{2k\pi+\pi/4}{5}\right)$

$=\sqrt[5]{2}\left(\cos\frac{\pi(8k+1)}{20}+i\sin\frac{\pi(8k+1)}{20}\right)$,

$k=0,1,2,3,4$:

$\sqrt[5]{2}\left(\cos\frac{\pi}{20}+i\sin\frac{\pi}{20}\right)$, $\sqrt[5]{2}\left(\cos\frac{9\pi}{20}+i\sin\frac{9\pi}{20}\right)$,

$\sqrt[5]{2}\left(\cos\frac{17\pi}{20}+i\sin\frac{17\pi}{20}\right)$, $\sqrt[5]{2}\left(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right)=$

$\sqrt[5]{2}\left(-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i\right)=\frac{-2^{1/5}-2^{1/5}i}{2^{1/2}}=$

$\frac{-1-i}{2^{3/10}}=\frac{-1-i}{\sqrt[10]{8}}$,

$\sqrt[5]{2}\left(\cos\frac{33\pi}{20}+i\sin\frac{33\pi}{20}\right)$

49. $\sqrt[5]{2}\left(\cos\frac{2k\pi+\pi/2}{5}+i\sin\frac{2k\pi+\pi/2}{5}\right)$

$=\sqrt[5]{2}\left(\cos\frac{\pi(4k+1)}{10}+i\sin\frac{\pi(4k+1)}{10}\right)$,

$k=0,1,2,3,4$:

$\sqrt[5]{2}\left(\cos\frac{\pi}{10}+i\sin\frac{\pi}{10}\right)$, $\sqrt[5]{2}\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)=\sqrt[5]{2}i$,

$\sqrt[5]{2}\left(\cos\frac{9\pi}{10}+i\sin\frac{9\pi}{10}\right)$, $\sqrt[5]{2}\left(\cos\frac{13\pi}{10}+i\sin\frac{13\pi}{10}\right)$,

$\sqrt[5]{2}\left(\cos\frac{17\pi}{10}+i\sin\frac{17\pi}{10}\right)$

50. $\sqrt[5]{2}\left(\cos\frac{2k\pi + \pi/3}{5} + i\sin\frac{2k\pi + \pi/3}{5}\right)$
$= \sqrt[5]{2}\left(\cos\frac{\pi(6k + 1)}{15} + i\sin\frac{\pi(6k + 1)}{15}\right),$
$k = 0, 1, 2, 3, 4:$
$\sqrt[5]{2}\left(\cos\frac{\pi}{15} + i\sin\frac{\pi}{15}\right), \sqrt[5]{2}\left(\cos\frac{7\pi}{15} + i\sin\frac{7\pi}{15}\right),$
$\sqrt[5]{2}\left(\cos\frac{13\pi}{15} + i\sin\frac{13\pi}{15}\right), \sqrt[5]{2}\left(\cos\frac{19\pi}{15} + i\sin\frac{19\pi}{15}\right),$
$\sqrt[5]{2}\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right) = 2^{1/5}\left(\frac{1}{2} - \frac{\sqrt{3}i}{2}\right) =$
$\frac{1 - \sqrt{3}i}{2^{4/5}} = \frac{1 - \sqrt{3}i}{\sqrt[5]{16}}$

For #51–56, the nth roots of $r(\cos\theta + i\sin\theta)$are
$\sqrt[n]{r}\left(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right), k = 0, 1, 2, \ldots, n - 1.$

51. $1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$, so the roots are
$\sqrt[4]{2}\left(\cos\frac{2k\pi + \pi/4}{4} + i\sin\frac{2k\pi + \pi/4}{4}\right)$
$= \sqrt[8]{2}\left(\cos\frac{\pi(8k + 1)}{16} + i\sin\frac{\pi(8k + 1)}{16}\right),$
$k = 0, 1, 2, 3:$
$\sqrt[8]{2}\left(\cos\frac{\pi}{16} + i\sin\frac{\pi}{16}\right), \sqrt[8]{2}\left(\cos\frac{9\pi}{16} + i\sin\frac{9\pi}{16}\right),$
$\sqrt[8]{2}\left(\cos\frac{17\pi}{16} + i\sin\frac{17\pi}{16}\right), \sqrt[8]{2}\left(\cos\frac{25\pi}{16} + i\sin\frac{25\pi}{16}\right)$

52. $1 - i = \sqrt{2}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$, so the roots are
$\sqrt[6]{2}\left(\cos\frac{2k\pi + 7\pi/4}{6} + i\sin\frac{2k\pi + 7\pi/4}{6}\right)$
$\sqrt[12]{2}(\cos\frac{\pi(8k + 7)}{24} + i\sin\frac{\pi(8k + 7)}{24}\Big),$
$k = 0, 1, 2, 3, 4, 5:$
$\sqrt[12]{2}\left(\cos\frac{7\pi}{24} + i\sin\frac{7\pi}{24}\right), \sqrt[12]{2}\left(\cos\frac{5\pi}{8} + i\sin\frac{5\pi}{8}\right),$
$\sqrt[12]{2}\left(\cos\frac{23\pi}{24} + i\sin\frac{23\pi}{24}\right), \sqrt[12]{2}\left(\cos\frac{31\pi}{24} + i\sin\frac{31\pi}{24}\right),$
$\sqrt[12]{2}\left(\cos\frac{13\pi}{8} + i\sin\frac{13\pi}{8}\right), \sqrt[12]{2}\left(\cos\frac{47\pi}{24} + i\sin\frac{47\pi}{24}\right)$

53. $2 + 2i = 2\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$, so the roots are
$\sqrt[3]{2}\left(\cos\frac{2k\pi + p/4}{3} + i\sin\frac{2k\pi + \pi/4}{3}\right)$
$= \sqrt{2}\left(\cos\frac{\pi(8k + 1)}{12} + i\sin\frac{\pi(8k + 1)}{12}\right), k = 0, 1, 2:$
$\sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right), -1 + i,$
$\sqrt{2}\left(\cos\frac{17\pi}{12} + i\sin\frac{17\pi}{12}\right)$

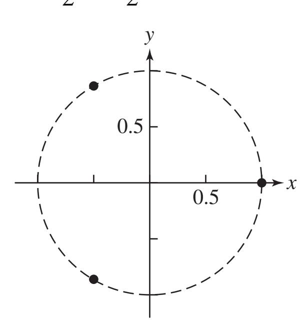
54. $-2 + 2i = 2\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$, so the roots are
$\sqrt[4]{2\sqrt{2}}\left(\cos\frac{2k\pi + 3\pi/4}{4} + i\sin\frac{2k\pi + 3\pi/4}{4}\right)$
$\sqrt[8]{8}\left(\cos\frac{\pi(8k + 3)}{16} + i\sin\frac{\pi(8k + 3)}{16}\right),$
$k = 0, 1, 2, 3:$
$\sqrt[8]{8}\left(\cos\frac{3\pi}{16} + i\sin\frac{3\pi}{16}\right), \sqrt[8]{8}\left(\cos\frac{11\pi}{16} + i\sin\frac{11\pi}{16}\right),$
$\sqrt[8]{8}\left(\cos\frac{19\pi}{16} + i\sin\frac{19\pi}{16}\right), \sqrt[8]{8}\left(\cos\frac{27\pi}{16} + i\sin\frac{27\pi}{16}\right)$

55. $-2i = 2\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$, so the roots are
$\sqrt[6]{2}\left(\cos\frac{2k\pi + 3\pi/2}{6} + i\sin\frac{2k\pi + 3\pi/2}{6}\right)$
$= \sqrt[6]{2}\left(\cos\frac{\pi(4k + 3)}{12} + i\sin\frac{\pi(4k + 3)}{12}\right),$
$k = 0, 1, 2, 3, 4, 5:$
$\frac{1 + i}{\sqrt[6]{4}}, \sqrt[6]{2}\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right),$
$\sqrt[6]{2}\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right), \sqrt[6]{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right),$
$\sqrt[6]{2}\left(\cos\frac{19\pi}{12} + i\sin\frac{19\pi}{12}\right), \sqrt[6]{2}\left(\cos\frac{23\pi}{12} + i\sin\frac{23\pi}{12}\right)$

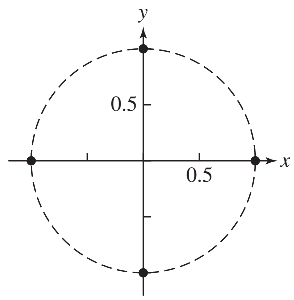
56. $32 = 32(\cos 0 + i\sin 0)$, so the roots are
$\sqrt[5]{32}\left(\cos\frac{2k\pi + 0}{5} + i\sin\frac{2k\pi + 0}{5}\right)$
$= 2\left(\cos\frac{2k\pi}{5} + i\sin\frac{2k\pi}{5}\right), k = 0, 1, 2, 3, 4:$
$2(\cos 0 + i\sin 0) = 2, 2\left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right),$
$2\left(\cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}\right), 2\left(\cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}\right),$
$2\left(\cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}\right)$

For #57–60, the nth roots of unity are
$\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n},\ k = 0, 1, 2, \ldots, n - 1.$

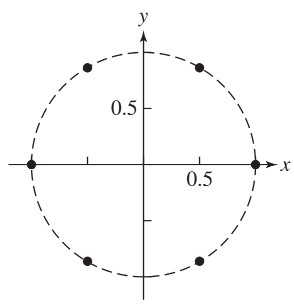
57. $1, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$



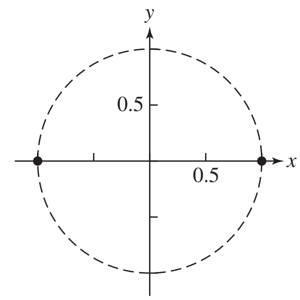
58. $\pm 1, \pm i$



59. $\pm 1, \frac{1}{2} \pm \frac{\sqrt{3}}{2}i, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$



60. $1, -1$



61. $z = (1 + \sqrt{3}i)^3 = \left[2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right]^3$
$= 8(\cos\pi + i\sin\pi) = -8$; the cube roots are -2 and $1 \pm \sqrt{3}i$.

62. $z = (-2 - 2i)^4 = \left[2\sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)\right]^4$
$= 64(\cos 5\pi + i\sin 5\pi) = -64$; the fourth roots are $2 \pm 2i$ and $-2 \pm 2i$.

63. $\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1}{r_2} \cdot \frac{\cos\theta_1 + i\sin\theta_1}{\cos\theta_2 + i\sin\theta_2} \cdot \frac{\cos\theta_2 - i\sin\theta_2}{\cos\theta_2 - i\sin\theta_2}$

$= \frac{r_1}{r_2} \cdot \frac{\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)}{(\cos\theta_2)^2 + (\sin\theta_2)^2}$

$= \frac{r_1}{r_2} \cdot [\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)]$.

Now use the angle difference formulas:
$\cos(\theta_1 - \theta_2) = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2$ and $\sin(\theta_1 - \theta_2) = \sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2$.

So $\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$.

64. The n nth roots are given by $\sqrt[n]{r}\left(\cos\frac{2k\pi + \theta}{n} + i\sin\frac{2k\pi + \theta}{n}\right)$, $k = 0, 1, 2, \ldots n - 1$:

$\sqrt[n]{r}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right), \sqrt[n]{r}\left(\cos\frac{\theta + 2\pi}{n} + i\sin\frac{\theta + 2\pi}{n}\right),$

$\sqrt[n]{r}\left(\cos\frac{\theta + 4\pi}{n} + i\sin\frac{\theta + 4\pi}{n}\right), \ldots, \sqrt[n]{r}\left(\cos\frac{\theta + 2(n-1)\pi}{n} + i\sin\frac{\theta + 2(n-1)\pi}{n}\right)$.

The angles between successive values in this list differ by $\frac{2\pi}{n}$ radians, while the first and last roots differ by $2\pi - \frac{2\pi}{n}$, which also makes the angle between them $\frac{2\pi}{n}$. Also, the modulus of each root is $\sqrt[n]{r}$, placing it at that distance from the origin, on the circle with radius $\sqrt[n]{r}$.

65. False. If $z = r(\cos\theta + i\sin\theta)$, then it is also true that $z = r[\cos(\theta + 2n\pi) + i\sin(\theta + 2n\pi)]$ for any integer n.
For example, here are two trigonometric forms for $1 + i$: $\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right), \sqrt{2}\left(\cos\frac{9\pi}{4} + i\sin\frac{9\pi}{4}\right)$.

66. True. $i^3 = i^2 \cdot i = -i$, so i is a cube root of $-i$.

67. $2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = 2\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = -1 + \sqrt{3}i$

The answer is B.

68. Any complex number has *n* distinct *n*th roots, so $1 + i$ has five 5th roots. The answer is E.

69. $\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\cdot\sqrt{2}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$

$= (\sqrt{2}\cdot\sqrt{2})\left[\cos\left(\frac{\pi}{4} + \frac{7\pi}{4}\right) + i\sin\left(\frac{\pi}{4} + \frac{7\pi}{4}\right)\right]$

$= 2(\cos 2\pi + i\sin 2\pi)$

$= 2$

The answer is A.

70. $(\sqrt{i})^4 = [(\sqrt{i})^2]^2 = i^2 = -1 \neq 1$. The answer is E.

71. **(a)** $a + bi = r(\cos\theta + i\sin\theta)$, where $r = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1}\left(\frac{b}{a}\right)$. Then $a + (-bi) = r'(\cos\theta' + i\sin\theta')$, where $r' = \sqrt{a^2 + (-b)^2}$ and $\theta' = \tan^{-1}\left(\frac{-b}{a}\right)$. Since $r' = \sqrt{a^2 + b^2} = r$ and $\theta' = \tan^{-1}\left(\frac{-b}{a}\right) = -\tan^{-1}\left(\frac{b}{a}\right) = -\theta$, we have $a - bi = r(\cos(-\theta) + i\sin(-\theta))$

(b) $z\cdot\bar{z} = r[\cos\theta + i\sin\theta]\cdot r[\cos(-\theta) + i\sin(-\theta)]$

$= r^2[\cos\theta\cos(-\theta) + i(\sin(-\theta))(\cos\theta) + i(\sin\theta)(\cos(-\theta)) - (\sin\theta)(\sin(-\theta))]$

Since $\sin\theta$ is an odd function (i.e., $\sin(-\theta) = -\sin(\theta)$) and $\cos\theta$ is an even function (i.e., $\cos(-\theta) = \cos\theta$), we have

$z\cdot\bar{z} = r^2[\cos^2\theta - i(\sin\theta)(\cos\theta) + i(\sin\theta)(\cos\theta) + \sin^2\theta]$

$= r^2[\cos^2\theta + \sin^2\theta]$

$= r^2$

(c) $\frac{z}{\bar{z}} = \frac{r[\cos\theta + i\sin\theta]}{r[\cos(-\theta) + i\sin(-\theta)]} = \cos[\theta - (-\theta)] + i\sin[\theta - (-\theta)]$

$= \cos(2\theta) + i\sin(2\theta)$

(d) $-z = -(a + bi) = -a + (-bi) = r(\cos\theta + i\sin\theta)$, where $r = \sqrt{(-a^2) + (-b)^2}$ and $\theta = \tan^{-1}\left(\frac{-b}{-a}\right) = \tan^{-1}\left(\frac{b}{a}\right)$

Recall, however, that $(-a, -b)$ is in the quadrant directly opposite the quadrant that holds (a, b) (i.e., if (a, b) is in Quadrant I, $(-a, -b)$ is in Quadrant III, and if (a, b) is in Quadrant II, $(-a, -b)$ is in Quadrant IV). Thus, $-z = \sqrt{a^2 + b^2}(\cos(\theta + \pi) + i\sin(\theta + \pi)) = r(\cos(\theta + \pi) + i\sin(\theta + \pi))$.

72. **(a)** $|z| = \sqrt{(r\cos\theta)^2 + (r\sin\theta)^2}$

$= \sqrt{r^2\cos^2\theta + r^2\sin^2\theta}$

$= |r|\sqrt{\cos^2\theta + \sin^2\theta}$.

$= |r|$

(b) $|z_1\cdot z_2| = |[r_1\cos\theta_1 + (r_1\sin\theta_1)\mathrm{i}]\cdot[\mathrm{r}_2\cos\theta_2 + (r_2\sin\theta_2)i]|$

$= |r_1r_2\cos\theta_1\cos\theta_2 + (r_1r_2\cos\theta_1\sin\theta_2)i + (r_1r_2\sin\theta_1\cos\theta_2)i + (r_1r_2\sin\theta_1 u_2)i^2|$

$= |r_1r_2[\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + (\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)i]|$

$= |r_1r_2[\cos(\theta_1 + \theta_2) + (\sin(\theta_1 + \theta_2))i]|$

$= \sqrt{(r_1r_2)^2[\cos^2(\theta_1 + \theta_2) + \sin^2(\theta_1 + \theta_2)]}$

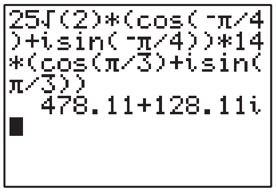
$= \sqrt{(r_1r_2)^2}$

$= |r_1|\cdot|r_2|$

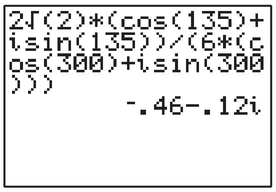
$= |z_1|\cdot|z_1|$ by **(a)**

73. Set the calculator for rounding to 2 decimal places. In **(b)**, use Degree mode.

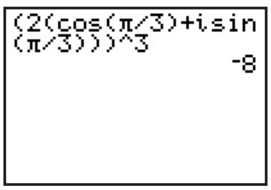
(a)



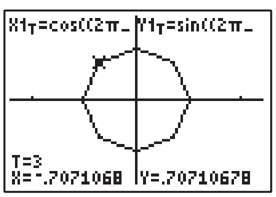
(b)



(c)



74. (a)



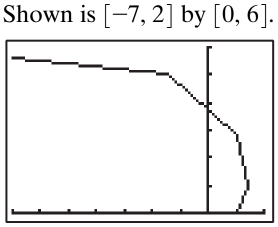
$[-2.4, 2.4]$ by $[-1.6, 1.6]$

(b) Yes. $6\pi/8$, $10\pi/8$, $14\pi/8$

(c) For the fifth and seventh roots of unity, all of the roots except the complex number 1 generate the corresponding roots of unity. For the sixth roots of unity, only $2\pi/6$ and $10\pi/6$ generate the sixth roots of unity.

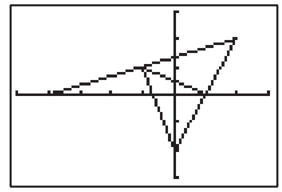
(d) $2\pi k/n$ generates the nth roots of unity if and only if k and n have no common factors other than 1.

75. Using $\tan^{-1}\left(\frac{1}{\sqrt{2}}\right) \approx 0.62$, we have $\sqrt{2} + i \approx \sqrt{3}\,(\cos 0.62 + i \sin 0.62)$, so graph $x(t) = (\sqrt{3})^t \cos(0.62t)$ and $y(t) = (\sqrt{3})^t \sin(0.62t)$.
Use Tmin = 0, Tmax = 4, Tstep = 1.
Shown is $[-7, 2]$ by $[0, 6]$.



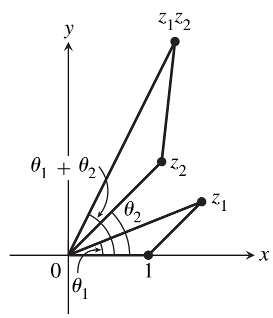
$[-7, 2]$ by $[0, 6]$

76. $-1 + i = \sqrt{2}\left(\cos\frac{3\pi}{4} + i \sin\frac{3\pi}{4}\right)$, so graph
$x(t) = (\sqrt{2})^t \cos(0.75\pi t)$ and $y(t) = (\sqrt{2})^t \sin(0.75\pi t)$
Use Tmin = 0, Tmax = 4, Tstep = 1.
Shown is $[-5, 3]$ by $[-3, 3]$.



$[-5, 3]$ by $[-3, 3]$

77. Suppose that $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. Each triangle's angle at the origin has the same measure: For the smaller triangle, this angle has measure θ_1; for the larger triangle, the side from the origin out to z_2 makes an angle of θ_2 with the x-axis, while the side from the origin to z_1z_2 makes an angle of $\theta_1 + \theta_2$, so that the angle between is θ_1 as well.
The corresponding side lengths for the sides adjacent to these angles have the same ratio: the two longest sides have lengths $|z_1| = r_1$ (for the smaller) and $|z_1z_2| = r_1r_2$, for a ratio of r_2. For the other two sides, the lengths are 1 and r_2, again giving a ratio of r_2. Finally, the law of sines can be used to show that the remaining side have the same ratio.



78. Construct an angle with vertex at z_2, and one ray from z_2 to 0, congruent to the angle formed by 0, 1, and z_1, with vertex 1. Also, be sure that this new angle is oriented in the appropriate direction: e.g., if z_1 is located "counterclockwise" from 1 then the points on this new ray should also be located counterclockwise from z_2. Now similarly construct an angle with vertex at 0, and one ray from 0 to z_2, congruent to the angle formed by 1, 0, and z_1, with vertex 0. The intersection of the two newly constructed rays is z_1z_2.

79. The solutions are the cube roots of 1:
$\cos\left(\frac{2\pi k}{3}\right) + i \sin\left(\frac{2\pi k}{3}\right)$, $k = 0, 1, 2$ or
$1, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

80. The solutions are the fourth roots of 1:
$\cos\left(\frac{\pi k}{2}\right) + i \sin\left(\frac{\pi k}{2}\right)$, $k = 0, 1, 2, 3$ or $-1, 1, -i, i$

81. The solutions are the cube roots of -1:
$\cos\left(\frac{\pi + 2\pi k}{3}\right) + i \sin\left(\frac{\pi + 2\pi k}{3}\right)$, $k = 0, 1, 2$ or
$-1, \frac{1}{2} + \frac{\sqrt{3}}{2}i, \frac{1}{2} - \frac{\sqrt{3}}{2}i$

82. The solutions are the fourth roots of −1:
$\cos\left(\frac{\pi + 2\pi k}{4}\right) + i\sin\left(\frac{\pi + 2\pi k}{4}\right), k = 0, 1, 2, 3$ or
$\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i, -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$

83. The solutions are the fifth roots of −1:
$\cos\left(\frac{\pi + 2\pi k}{5}\right) + i\sin\left(\frac{\pi + 2\pi k}{5}\right), k = 0, 1, 2, 3, 4$, or
$-1, \approx 0.81 + 0.59i, 0.81 - 0.59i, -0.31 + 0.95i, -0.31 - 0.95i$

84. The solutions are the fifth roots of 1:
$\cos\left(\frac{2\pi k}{5}\right) + i\sin\left(\frac{2\pi k}{5}\right), k = 0, 1, 2, 3, 4$ or
$1, \approx 0.31 + 0.95i, 0.31 - 0.95i, -0.81 + 0.59i, -0.81 - 0.59i$

■ Chapter 6 Review

1. $\mathbf{u} - \mathbf{v} = \langle 2 - 4, -1 - 2\rangle = \langle -2, -3\rangle$

2. $2\mathbf{u} - 3\mathbf{w} = \langle 4 - 3, -2 + 9\rangle = \langle 1, 7\rangle$

3. $|\mathbf{u} + \mathbf{v}| = \sqrt{(2 + 4)^2 + (-1 + 2)^2} = \sqrt{37}$

4. $|\mathbf{w} - 2\mathbf{u}| = \sqrt{(1 - 4)^2 + (-3 + 2)^2} = \sqrt{10}$

5. $\mathbf{u} \cdot \mathbf{v} = 8 - 2 = 6$

6. $\mathbf{u} \cdot \mathbf{w} = 2 + 3 = 5$

7. $3\overrightarrow{AB} = 3\langle 3 - 2, 1 - (-1)\rangle = \langle 3, 6\rangle$;
$|3\overrightarrow{AB}| = \sqrt{3^2 + 6^2} = \sqrt{45} = 3\sqrt{5}$

8. $\overrightarrow{AB} + \overrightarrow{CD} = \langle 3 - 2, 1 - (-1)\rangle + \langle 1 - (-4), -5 - 2\rangle = \langle 6, -5\rangle$; $|\overrightarrow{AB} + \overrightarrow{CD}| = \sqrt{6^2 + 5^2} = \sqrt{61}$

9. $\overrightarrow{AC} + \overrightarrow{BD} = \langle -4 - 2, 2 - (-1)\rangle + \langle 1 - 3, -5 - 1\rangle = \langle -8, -3\rangle$; $|\overrightarrow{AC} + \overrightarrow{BD}| = \sqrt{8^2 + 3^2} = \sqrt{73}$

10. $\overrightarrow{CD} - \overrightarrow{AB} = \langle 1 - (-4), -5 - 2\rangle - \langle 3 - 2, 1 - (-1)\rangle = \langle 4, -9\rangle$; $|\overrightarrow{CD} + \overrightarrow{AB}| = \sqrt{4^2 + 9^2} = \sqrt{97}$

11. (a) $\frac{\overrightarrow{AB}}{|\overrightarrow{AB}|} = \frac{\langle -2, 1\rangle}{\sqrt{(-2)^2 + 1^2}} = \frac{\langle -2, 1\rangle}{\sqrt{5}} = \left\langle -\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right\rangle$

(b) $-3 \cdot \frac{\overrightarrow{AB}}{|\overrightarrow{BA}|} = -3\left\langle \frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right\rangle = \left\langle \frac{6}{\sqrt{5}}, -\frac{3}{\sqrt{5}}\right\rangle$

12. **(a)** $\frac{\overrightarrow{AB}}{|\overrightarrow{AB}|} = \frac{\langle 2, 0\rangle}{\sqrt{2^2 + 0^2}} = \frac{\langle 2, 0\rangle}{2} = \langle 1, 0\rangle$

(b) $-3 \cdot \frac{\overrightarrow{AB}}{|\overrightarrow{BA}|} = -3\langle 1, 0\rangle = \langle -3, 0\rangle$

For #13 and 14, the direction angle θ of $\langle a, b\rangle$ has $\tan\theta = \frac{b}{a}$; start with $\tan^{-1}\left(\frac{b}{a}\right)$, and add (or subtract) 180° if the angle is not in the correct quadrant. The angle between two vectors is the absolute value of the difference between their angles; if this difference is greater than 180°, subtract it from 360°.

13. **(a)** $\theta_u = \tan^{-1}\left(\frac{3}{4}\right) \approx 0.64, \theta_v = \tan^{-1}\left(\frac{5}{2}\right) \approx 1.19$

(b) $\theta_v - \theta_u \approx 0.55$

14. **(a)** $\theta_u = \pi + \tan^{-1}(-2) = \cos^{-1}\left(-\frac{1}{\sqrt{5}}\right) \approx 2.03$,
$\theta_v = \tan^{-1}\left(\frac{2}{3}\right) \approx 0.59$

(b) $\theta_u - \theta_v \approx 1.45$

15. $(-2.5\cos 25°, -2.5\sin 25°) \approx (-2.27, -1.06)$

16. $(-3.1\cos 135°, -3.1\sin 135°) = (1.55\sqrt{2}, -1.55\sqrt{2})$

17. $(2\cos(-\pi/4), 2\sin(-\pi/4)) = (\sqrt{2}, -\sqrt{2})$

18. $(3.6\cos(3\pi/4), 3.6\sin(3\pi/4)) = (-1.8\sqrt{2}, 1.8\sqrt{2})$

19. $\left(1, -\frac{2\pi}{3} + (2n + 1)\pi\right)$ and $\left(-1, -\frac{2\pi}{3} + 2n\pi\right)$, n an integer.

20. $\left(2, \frac{5\pi}{6} + (2n + 1)\pi\right)$ and $\left(-2, \frac{5\pi}{6} + 2n\pi\right)$, n an integer.

21. **(a)** $\left(-\sqrt{13}, \pi + \tan^{-1}\left(-\frac{3}{2}\right)\right) \approx (-\sqrt{13}, 2.16)$ or
$\left(\sqrt{13}, 2\pi + \tan^{-1}\left(-\frac{3}{2}\right)\right) \approx (\sqrt{13}, 5.30)$

(b) $\left(\sqrt{13}, \tan^{-1}\left(-\frac{3}{2}\right)\right) \approx (\sqrt{13}, -0.98)$ or
$\left(-\sqrt{13}, \pi + \tan^{-1}\left(-\frac{3}{2}\right)\right) \approx (-\sqrt{13}, 2.16)$

(c) The answers from **(a)**, and also
$\left(-\sqrt{13}, 3\pi + \tan^{-1}\left(-\frac{3}{2}\right)\right) \approx (-\sqrt{13}, 8.44)$ or
$\left(\sqrt{13}, 4\pi + \tan^{-1}\left(-\frac{3}{2}\right)\right) \approx (\sqrt{13}, 11.58)$

22. **(a)** $(-10, 0)$ or $(10, \pi)$ or $(-10, 2\pi)$
(b) $(10, -\pi)$ or $(-10, 0)$ or $(10, \pi)$
(c) The answers from **(a)**, and also $(10, 3\pi)$ or $(-10, 4\pi)$

23. **(a)** $(5, 0)$ or $(-5, \pi)$ or $(5, 2\pi)$
(b) $(-5, -\pi)$ or $(5, 0)$ or $(-5, \pi)$
(c) The answers from **(a)**, and also $(-5, 3\pi)$ or $(5, 4\pi)$

24. **(a)** $\left(-2, \frac{\pi}{2}\right)$ or $\left(2, \frac{3\pi}{2}\right)$

(b) $\left(2, -\frac{\pi}{2}\right)$ or $\left(-2, \frac{\pi}{2}\right)$

(c) The answers from **(a)**, and also $\left(-2, \frac{5\pi}{2}\right)$ or $\left(2, \frac{7\pi}{2}\right)$

25. $t = -\frac{1}{5}x + \frac{3}{5}$, so $y = 4 + 3\left(-\frac{1}{5}x + \frac{3}{5}\right) = -\frac{3}{5}x + \frac{29}{5}$:
Line through $\left(0, \frac{29}{5}\right)$ with slope $m = -\frac{3}{5}$

26. $t = x - 4$, so $y = -8 - 5(x - 4) = -5x + 12$, $1 \le x \le 9$: segment from $(1, 7)$ to $(9, -33)$.

27. $t = y + 1$, so $x = 2(y + 1)^2 + 3$: Parabola that opens to right with vertex at $(3, -1)$.

28. $x^2 + y^2 = (3\cos t)^2 + (3\sin t)^2 =$
$9\cos^2 t + 9\sin^2 t = 9$, so $x^2 + y^2 = 9$: Circle of radius 3 centered at $(0, 0)$.

29. $x + 1 = e^{2t}$, $t = \dfrac{\ln(x+1)}{2}$, so $y = e^{\frac{1}{2}\ln(x+1)} = e^{\ln}$
$= \sqrt{x+1}$: square root function starting at $(-1, 0)$

30. $t = \sqrt[3]{x}$, so $y = \ln\left(\sqrt[3]{x}\right) = \ln x^{1/3} = \dfrac{1}{3}\ln x$: the logarithmic function, with asymptote at $x = 0$

31. $m = \dfrac{4-(-2)}{3-(-1)} = \dfrac{6}{4} = \dfrac{3}{2}$, so $\Delta x = 2$ when $\Delta y = 3$.
One possibility for the parametrization of the line is:
$x = 2t + 3$, $y = 3t + 4$.

32. $m = \dfrac{1-3}{5-(-2)} - \dfrac{-2}{7}$, so $\Delta x = 7$ when $\Delta y = -2$.
One possibility for the parametrization of the segment is:
$x = 7t + 5$, $y = -2t + 1$, $-1 \le t \le 0$. Another possibility is $x = 7t - 2$, $y = -2t + 3$, $0 \le t \le 1$.

33. $a = -3$, $b = 4$, $|z_1| = \sqrt{3^2 + 4^2} = 5$

34. $z_1 = 5\left\{\cos\left[\cos^{-1}\left(-\dfrac{3}{5}\right)\right] + i\sin\left[\cos^{-1}\left(-\dfrac{3}{5}\right)\right]\right\}$
$\approx 5[\cos(2.21) + i\sin(2.21)]$

35. $6(\cos 30° + i\sin 30°) = 6\left(\dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i\right) = 3\sqrt{3} + 3i$

36. $3(\cos 150° + i\sin 150°) = 3\left(-\dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i\right)$
$= -1.5\sqrt{3} + 1.5i$

37. $2.5\left(\cos\dfrac{4\pi}{3} + i\sin\dfrac{4\pi}{3}\right) = 2.5\left(-\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i\right)$
$= -1.25 - 1.25\sqrt{3}\,i$

38. $4(\cos 2.5 + i\sin 2.5) \approx -3.20 + 2.39i$

39. $3 - 3i = 3\sqrt{2}\left(\cos\dfrac{7\pi}{4} + i\sin\dfrac{7\pi}{4}\right)$. Other representations would use angles $\dfrac{7\pi}{4} + 2n\pi$, n an integer.

40. $-1 + i\sqrt{2} = \sqrt{3}\left\{\cos\left[\cos^{-1}\left(-\dfrac{1}{\sqrt{3}}\right)\right]\right.$
$\left. + i\sin\left[\cos^{-1}\left(-\dfrac{1}{\sqrt{3}}\right)\right]\right\} \approx \sqrt{3}[\cos(2.19) + i\sin(2.19)]$.
Other representations would use angles $2.19 + 2n\pi$, n an integer.

41. $3 - 5i = \sqrt{34}\left\{\cos\left[\tan^{-1}\left(-\dfrac{5}{3}\right)\right]\right.$
$\left. + i\sin\left[\tan^{-1}\left(-\dfrac{5}{3}\right)\right]\right\}$
$\approx \sqrt{34}[\cos(-1.03) + i\sin(-1.03)]$
$\approx \sqrt{34}[\cos(5.25) + i\sin(5.25)]$
Other representations would use angles $\approx 5.25 + 2n\pi$, n an integer.

42. $-2 - 2i = 2\sqrt{2}\left(\cos\dfrac{5\pi}{4} + i\sin\dfrac{5\pi}{4}\right)$. Other representations would use angles $\dfrac{5\pi}{4} + 2n\pi$, n an integer.

43. $z_1 z_2 = (3)(4)[\cos(30° + 60°) + i\sin(30° + 60°)]$
$= 12(\cos 90° + i\sin 90°)$
$z_1/z_2 = \dfrac{3}{4}[\cos(30° - 60°) + i\sin(30° - 60°)]$
$= \dfrac{3}{4}[\cos(-30°) + i\sin(-30°)]$
$= \dfrac{3}{4}(\cos 330° + i\sin 330°)$

44. $z_1 z_2 = (5)(-2)[\cos(20° + 45°) + i\sin(20° + 45°)]$
$= -10(\cos 65° + i\sin 65°)$
$= 10(\cos 245° + i\sin 245°)$
$z_1/z_2 = \dfrac{5}{-2}[\cos(20° - 45°) + i\sin(20° - 45°)]$
$= -2.5[\cos(-25°) + i\sin(-25°)]$
$= 2.5(\cos 155° + i\sin 155°)$

45. **(a)** $\left[3\left(\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}\right)\right]^5 = 3^5\left(\cos\dfrac{5\pi}{4} + i\sin\dfrac{5\pi}{4}\right)$
$= 243\left(\cos\dfrac{5\pi}{4} + i\sin\dfrac{5\pi}{4}\right)$
(b) $-\dfrac{243\sqrt{2}}{2} - \dfrac{243\sqrt{2}}{2}i$

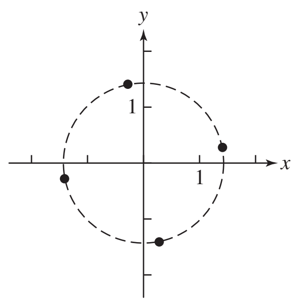
46. **(a)** $\left[2\left(\cos\dfrac{\pi}{12} + i\sin\dfrac{\pi}{12}\right)\right]^8 = 2^8\left(\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}\right)$
$= 256\left(\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}\right)$
(b) $-128 + 128\sqrt{3}\,i$

47. **(a)** $\left[5\left(\cos\dfrac{5\pi}{3} + i\sin\dfrac{5\pi}{3}\right)\right]^3 = 5^3(\cos 5\pi + i\sin 5\pi)$
$= 125(\cos\pi + i\sin\pi)$
(b) $-125 + 0i = -125$

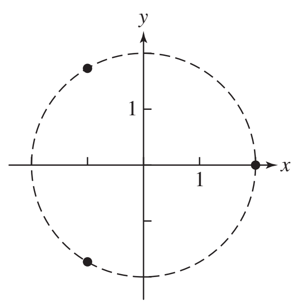
48. **(a)** $\left[7\left(\cos\dfrac{\pi}{24} + i\sin\dfrac{\pi}{24}\right)\right]^6 = 7^6\left(\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}\right)$
$= 117,649\left(\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}\right)$
(b) $\dfrac{117,649\sqrt{2}}{2} + \dfrac{117,649\sqrt{2}}{2}i$

For #49–52, the nth roots of $r(\cos\theta + i\sin\theta)$ are
$\sqrt[n]{r}\left(\cos\dfrac{2k\pi + \theta}{n} + i\sin\dfrac{2k\pi + \theta}{n}\right)$, $k = 0, 1, 2, \ldots, n - 1$.

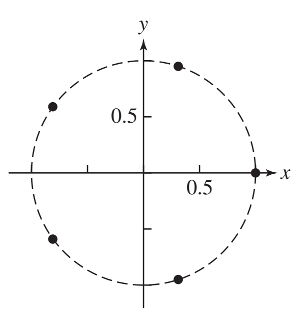
49. $3 + 3i = 3\sqrt{2}\left(\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}\right)$, so the roots are
$\sqrt[4]{3\sqrt{2}}\left(\cos\dfrac{2k\pi + \pi/4}{4} + i\sin\dfrac{2k\pi + \pi/4}{4}\right)$
$= \sqrt[8]{18}\left(\cos\dfrac{\pi(8k+1)}{16} + i\sin\dfrac{\pi(8k+1)}{16}\right)$,
$k = 0, 1, 2, 3$:
$\sqrt[8]{18}\left(\cos\dfrac{\pi}{16} + i\sin\dfrac{\pi}{16}\right)$,
$\sqrt[8]{18}\left(\cos\dfrac{9\pi}{16} + i\sin\dfrac{9\pi}{16}\right)$,
$\sqrt[8]{18}\left(\cos\dfrac{17\pi}{16} + i\sin\dfrac{17\pi}{16}\right)$,
$\sqrt[8]{18}\left(\cos\dfrac{25\pi}{16} + i\sin\dfrac{25\pi}{16}\right)$



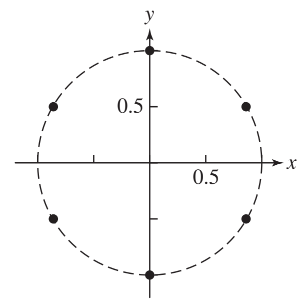
50. $8 = 8\,(\cos 0 + i \sin 0)$, so the roots are
$\sqrt[3]{8}\left(\cos \frac{2k\pi + 0}{3} + i \sin \frac{2k\pi + 0}{3}\right)$
$= 2\left(\cos \frac{2k\pi}{3} + i \sin \frac{2k\pi}{3}\right), k = 0, 1, 2:$
$2\,(\cos 0 + i \sin 0) = 2,$
$2\left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right),$
$2\left(\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}\right)$



51. $1 = \cos 0 + i \sin 0$, so the roots are
$\cos \frac{2k\pi + 0}{5} + i \sin \frac{2k\pi + 0}{5} = \cos \frac{2k\pi}{5} + i \sin \frac{2k\pi}{5},$
$k = 0, 1, 2, 3, 4:$
$\cos 0 + i \sin 0 = 1, \cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5},$
$\cos \frac{4\pi}{5} + i \sin \frac{4\pi}{5}, \cos \frac{6\pi}{5} + i \sin \frac{6\pi}{5}$
$\cos \frac{8\pi}{5} + i \sin \frac{8\pi}{5}$



52. $-1 = 1\,(\cos \pi + i \sin \pi)$, so the roots are
$\cos \frac{2k\pi + \pi}{6} + i \sin \frac{2k\pi + \pi}{6}$
$= \cos \frac{\pi(2k + 1)}{6} + i \sin \frac{\pi(2k + 1)}{6},$
$k = 0, 1, 2, 3, 4, 5:$
$\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}, \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} = i,$
$\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6}, \cos \frac{7\pi}{6} + i \sin \frac{7\pi}{6},$
$\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2} = -i, \cos \frac{11\pi}{6} + i \sin \frac{11\pi}{6}$

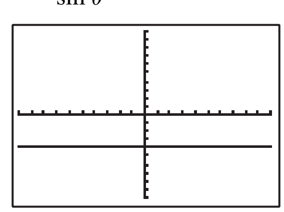


53. Graph **(b)**
54. Not shown
55. Graph **(a)**
56. Not shown
57. Not shown
58. Graph **(d)**
59. Graph **(c)**
60. Not shown
61. $x^2 + y^2 = 4$ — a circle with center (0, 0) and radius 2
62. $r^2 + 2r \sin\theta = x^2 + y^2 + 2y = 0$. Completing the square: $x^2 + (y^2 + 1)^2 = 1$ — a circle of radius 1 with center $(0, -1)$
63. $r^2 + 3r\cos\theta + 2r\sin\theta = x^2 + y^2 + 3x + 2y = 0$. Completing the square: $\left(x + \frac{3}{2}\right)^2 + (y + 1)^2 = \frac{13}{4}$

— a circle of radius $\frac{\sqrt{13}}{2}$ with center $\left(-\frac{3}{2}, -1\right)$

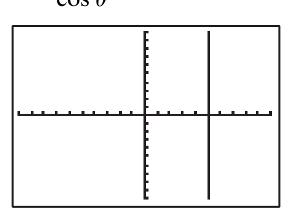
64. $1 - \frac{3}{r\cos\theta} = 1 - \frac{3}{x} = 0 \Rightarrow x - 3 = 0, x = 3$ — a vertical line through (3, 0)

65. $r = \frac{-4}{\sin\theta} = -4 \csc\theta$



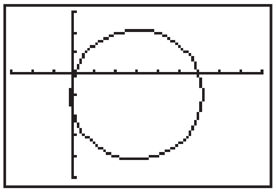
[−10, 10] by [−10, 10]

66. $r = \frac{5}{\cos\theta} = 5 \sec\theta$



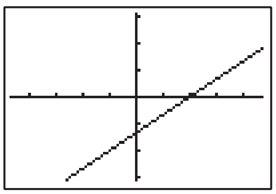
[−10, 10] by [−10, 10]

67. $(r \cos \theta - 3)^2 + (r \sin \theta + 1)^2 = 10$, so $r^2(\cos^2 \theta + \sin^2 \theta) + r(-6 \cos \theta + 2 \sin \theta) + 10 = 10$, or $r = 6 \cos \theta - 2 \sin \theta$



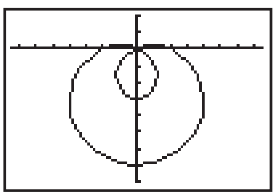
[–3, 9] by [–5, 3]

68. $2r \cos \theta - 3r \sin \theta = 4$, $r = \dfrac{4}{2 \cos \theta - 3 \sin \theta}$



[−4.7, 4.7] by [−3.1, 3.1]

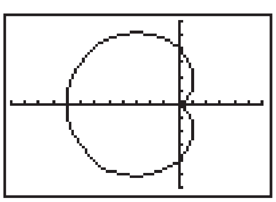
69.



[–7.5, 7.5] by [–8, 2]

Domain: $(-\infty, \infty)$
Range: $[-3, 7]$
Symmetric about the y-axis
Continuous
Bounded
Maximum $|r|$ value: 7
No asymptotes

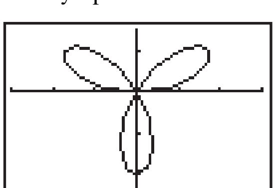
70.



[–12, 6] by [–6, 6]

Domain: $(-\infty, \infty)$
Range: $[0, 8]$
Symmetric about the x-axis
Continuous
Bounded
Maximum $|r|$ value: 8
No asymptotes

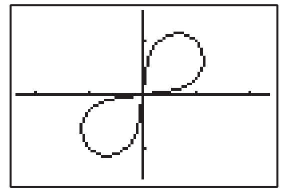
71.



[–3, 3] by [–2.5, 1.5]

Domain: $(-\infty, \infty)$
Range: $[-2, 2]$
Symmetric about the y-axis
Continuous
Bounded
Maximum $|r|$ value: 2
No asymptotes

72.



[−2.35, 2.35] by [−1.55, 1.55]

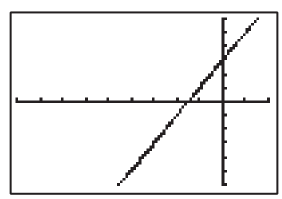
Domain: $\left[0, \dfrac{\pi}{2}\right] \cup \left[\pi, \dfrac{3\pi}{2}\right]$
Range: $[0, \sqrt{2}]$
Symmetric about the origin
Bounded
Maximum $|r|$ value: $\sqrt{2}$
No asymptotes

73. **(a)** $r = a \sec \theta \Rightarrow \dfrac{r}{\sec \theta} = a \Rightarrow r \cos \theta = a \Rightarrow x = a.$

(b) $r = b \csc \theta \Rightarrow \dfrac{r}{\csc \theta} = b \Rightarrow r \sin \theta = b \Rightarrow y = b.$

(c) $y = mx + b \Rightarrow r \sin \theta = mr \cos \theta + b \Rightarrow$
$r(\sin \theta - m \cos \theta) = b \Rightarrow r = \dfrac{b}{\sin \theta - m \cos \theta}.$
The domain of r is any value of θ for which $\sin \theta \neq m \cos \theta \Rightarrow \tan \theta \neq m \Rightarrow \theta \neq \arctan(m)$.

(d)



[−9, 2] by [−6, 6]

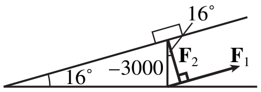
74. **(a)** $\mathbf{v} = 540 \langle \sin 80°, \cos 80° \rangle \approx \langle 531.80, 93.77 \rangle$

(b) The wind vector is $\mathbf{w} = 55 \langle \sin 100°, \cos 100° \rangle \approx \langle 54.16, -9.55 \rangle$. Actual velocity vector: $\mathbf{v} + \mathbf{w} \approx \langle 585.96, 84.22 \rangle$. Actual speed: $\|\mathbf{v} + \mathbf{w}\| \approx \sqrt{585.96^2 + 84.22^2} \approx 591.98$ mph. Actual bearing: $\tan^{-1}\left(\dfrac{585.96}{84.22}\right) \approx 81.82°$.

75. **(a)** $\mathbf{v} = 480 \langle \sin 285°, \cos 285° \rangle \approx \langle -463.64, 124.23 \rangle$

(b) The wind vector is $\mathbf{w} = 30 \langle \sin 265°, \cos 265° \rangle \approx \langle -29.89, -2.61 \rangle$. Actual velocity vector: $\mathbf{v} + \mathbf{w} \approx \langle -493.53, 121.62 \rangle$. Actual speed: $\|\mathbf{v} + \mathbf{w}\| \approx \sqrt{493.53^2 + 121.62^2} \approx 508.29$ mph. Actual bearing: $360° + \tan^{-1}\left(\dfrac{-493.53}{121.62}\right) \approx 283.84°$.

76. $\mathbf{F} = \langle 120 \cos 20°, 120 \sin 20° \rangle + \langle 300 \cos(-5°), 300 \sin(-5°) \rangle \approx \langle 411.62, 14.90 \rangle$, so $\|\mathbf{F}\| \approx \sqrt{411.62^2 + 14.90^2} \approx 411.89$ lb and $\theta = \tan^{-1}\left(\dfrac{14.90}{411.62}\right) \approx 2.07°$.

77. 

$\mathbf{F}_1$ Force to keep car from going downhill

$\mathbf{F}_2$ Force perpendicular to the street

(a) $\mathbf{F}_1 = -3000 \sin 16° \approx -826.91$, so the force required to keep the car from rolling down the hill is approximately 826.91 pounds.

(b) $\mathbf{F}_2 = -3000 \cos 16° \approx -2883.79$, so the force perpendicular to the street is approximately 2883.79 pounds.

78. $\mathbf{F} = 36 \cdot \dfrac{\langle 3, 5\rangle}{\sqrt{3^2 + 5^2}} = \dfrac{\langle 108, 180\rangle}{\sqrt{34}}$

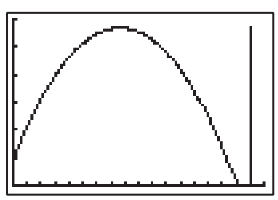
Since $\overrightarrow{AB} = \langle 10, 0\rangle$, $\mathbf{F} \cdot \overrightarrow{AB} = (10)\left(\dfrac{108}{\sqrt{34}}\right) + 0$

$= \dfrac{1080}{\sqrt{34}} \approx 185.22$ foot-pounds.

79. (a) $h = -16t^2 + v_0 t + s_0 = -16t^2 + 245t + 200$

(b) Graph and trace: $x = 17$ and $y = -16t^2 + 245t + 200$ with $0 \le t \le 16.1$ (upper limit may vary) on $[0, 18]$ by $[0, 1200]$. This graph will appear as a vertical line from about (17, 0) to about (17, 1138). Tracing shows how the arrow begins at a height of 200 ft, rises to over 1000 ft, then falls back to the ground.

(c) Graph $x = t$ and $y = -16t^2 + 245t + 200$ with $0 \le t \le 16.1$ (upper limit may vary).



[0, 18] by [0, 1200]

(d) When $t = 4$, $h = 924$ ft.

(e) When $t \approx 7.66$, the arrow is at its peak: about 1138 ft.

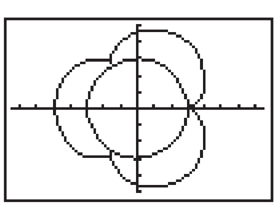
(f) The arrow hits the ground ($h = 0$) after about 16.09 sec.

80. $x = 35 \cos\left(\dfrac{\pi}{10} t\right)$, $y = 50 + 35 \sin\left(\dfrac{\pi}{10} t\right)$, assuming the wheel turns counterclockwise.

81. $x = 40 \sin\left(\dfrac{2\pi}{15} t\right)$, $y = 50 - 40 \cos\left(\dfrac{2\pi}{15} t\right)$, assuming the wheel turns counterclockwise.

82. $x = -40 \sin\left(\dfrac{\pi}{9} t\right)$, $y = 50 + 40 \cos\left(\dfrac{\pi}{9} t\right)$, assuming the wheel turns counterclockwise.

83. (a)



[−7.5, 7.5] by [−5, 5]

(b) All 4's should be changed to 5's.

84. $x = (66 \cos 5°)t$ and $y = -16t^2 + (66 \sin 5°)t + 4$. $y = 0$ when $t \approx 0.71$ sec (and also when $t \approx -0.352$, but that is not appropriate in this problem). When $t \approx 0.71$ sec, $x \approx 46.75$ ft.

85. $x = (66 \cos 12°)t$ and $y = -16t^2 + (66 \sin 12°)t + 3.5$. $y = 0$ when $t \approx 1.06$ sec (and also when $t \approx -0.206$, but that is not appropriate in this problem). When $t \approx 1.06$ sec, $x \approx 68.65$ ft.

86. $x = (70 \cos 45°)t$ and $y = -16t^2 + (70 \sin 45°)t$. The ball traveled 40 yd (120 ft) horizontally after about 2.42 sec, at which point it is about 25.96 ft above the ground, so it clears the crossbar.

87. If we assume that the initial height is 0 ft, then $x = (85 \cos 56°)t$ and $y = -16t^2 + (85 \sin 56°)t$. [If the assumed initial height is something other than 0 ft, add that amount to y.]

(a) Find graphically: The maximum y value is about 77.59 ft (after about 2.20 seconds).

(b) $y = 0$ when $t \approx 4.404$ sec

88. $x = (v_0 \cos 30°)t$ and $y = -16t^2 + (v_0 \sin 30°)t + 2.5$. v_0 must be (at least) just over 125 ft/sec. This can be found graphically (by trial and error), or algebraically: the ball is 400 ft from the plate (i.e., $x = 400$) when $t = \dfrac{400}{v_0 \cos 30°} = \dfrac{800/\sqrt{3}}{v_0}$. Substitute this value of t in the parametric equation for y. Then solve to see what value of v_0 makes y equal to 15 ft.

$$-16\left(\frac{800}{\sqrt{3}v_0}\right)^2 + v_0 \sin 30°\left(\frac{800}{\sqrt{3}v_0}\right) + 2.5 = 15$$

$$\frac{-16(640{,}000)}{3v_0^2} + \frac{400}{\sqrt{3}} = 12.5$$

$$\frac{-16(640{,}000)}{3} + \frac{400v_0^2}{\sqrt{3}} = 12.5v_0^2$$

$$\frac{-16(640{,}000)}{3} = v_0^2\left(12.5 - \frac{400}{\sqrt{3}}\right)$$

$$\frac{-16(640{,}000)}{3\left(12.5 - \frac{400}{\sqrt{3}}\right)} = v_0^2$$

$$\pm 125 \approx v_0$$

The negative root doesn't apply to this problem, so the initial velocity needed is just over 125 ft/sec.

89. Kathy's position: $x_1 = 60 \cos\left(\dfrac{\pi}{6} t\right)$ and

$y_1 = 60 + 60 \sin\left(\dfrac{\pi}{6} t\right)$

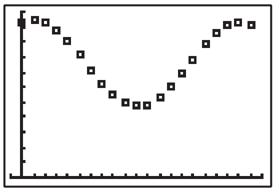
Ball's position:
$x_2 = -80 + (100 \cos 70°)t$ and
$y_2 = -16t^2 + (100 \sin 70°)t$

Find (graphically) the minimum of $d(t) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. It occurs when $t \approx 2.64$ sec; the minimum distance is about 17.65 ft.

90. $x = (20 \cos 50°)t$ and $y = -16t^2 + (20 \sin 50°)t + 5$. $y = 0$ when $t = 1.215$ sec (and also when $t = -0.257$, but that is not appropriate in this problem). When $t = 1.215$ sec, $x = 15.62$ ft. The dart falls several feet short of the target.

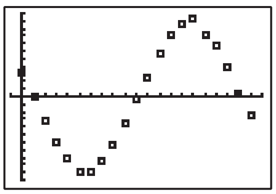
Chapter 6 Project

Answers are based on the sample data shown in the table.

1. 

[−0.1, 2.1] by [0, 1.1]

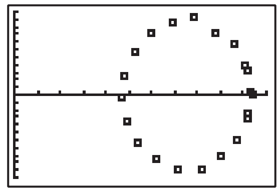
2. Sinusoidal regression produces $y = 0.28\sin(3.46x + 1.20) + 0.75$ or, with a phase shift of 2π, $y = 0.28\sin(3.46x - 5.09) + 0.75 = 0.28\sin(3.46(x - 1.47)) + 0.75$.

3. 

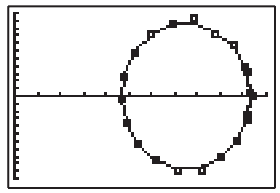
[−0.1, 2.1] by [−1.1, 1.1]

The curve $y = 0.9688\cos(3.46(x - 1.47))$ closely fits the data.

4. The distance and velocity both vary sinusoidally, with the same period but a phase shift of 90° — like the x- and y-coordinates of a point moving around a circle. A scatter plot of distance versus time should have the shape of a circle (or ellipse).



[0, 1.1] by [−1.1, 1.1]

5. 

[0, 1.1] by [−1.1, 1.1]