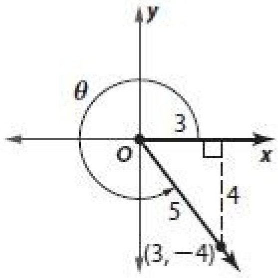
5-5 Multiple-Angle and Product-to-Sum Identities

Find the values of sin 2θ, cos 2θ, and tan 2θ for the given value and interval.

1. $\cos\theta = \frac{3}{5}$, (270°, 360°)

SOLUTION:

Since $\cos\theta = \frac{3}{5}$ on the interval (270°, 360°), one point on the terminal side of θ has x-coordinate 3 and a distance of 5 units from the origin as shown. The y-coordinate of this point is therefore $-\sqrt{5^2 - 3^2}$ or −4.



Using this point, we find that $\sin\theta = \frac{y}{r}$ or $-\frac{4}{5}$ and $\tan\theta = \frac{y}{x}$ or $-\frac{4}{3}$. Now use the double-angle identities for sine, cosine, and tangent to find sin 2θ, cos 2θ, and tan 2θ.

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta\\ &= 2\left(-\frac{4}{5}\right)\left(\frac{3}{5}\right)\\ &= -\frac{24}{25}\end{aligned}$$

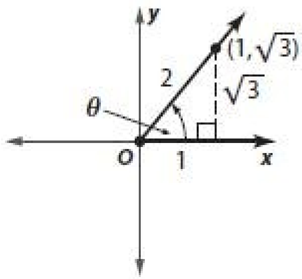
$$\begin{aligned}\cos 2\theta &= 2\cos^2\theta - 1\\ &= 2\left(\frac{3}{5}\right)^2 - 1\\ &= 2\left(\frac{9}{25}\right) - 1\\ &= \frac{18}{25} - \frac{25}{25}\\ &= -\frac{7}{25}\end{aligned}$$

$$\begin{aligned}
\tan 2\theta &= \frac{2\tan\theta}{1-\tan^2\theta} \\
&= \frac{2\left(-\frac{4}{3}\right)}{1-\left(-\frac{4}{3}\right)^2} \\
&= \frac{-\frac{8}{3}}{1-\frac{16}{9}} \\
&= \frac{-\frac{8}{3}}{-\frac{7}{9}} \\
&= \frac{24}{7}
\end{aligned}$$

6. $\tan\theta = \sqrt{3}$, $\left(0, \frac{\pi}{2}\right)$

SOLUTION:

If $\tan\theta = \sqrt{3}$, then $\tan\theta = \frac{\sqrt{3}}{1}$. Since $\tan\theta = \frac{\sqrt{3}}{1}$ on the interval $\left(0, \frac{\pi}{2}\right)$, one point on the terminal side of θ has x-coordinate 1 and y-coordinate $\sqrt{3}$ as shown. The distance from the point to the origin is $\sqrt{\left(\sqrt{3}\right)^2 + 1^2}$ or 2.



Using this point, we find that $\sin\theta = \frac{y}{r}$ or $\frac{\sqrt{3}}{2}$ and $\cos\theta = \frac{x}{r}$ or $\frac{1}{2}$. Now use the double-angle identities for sine, cosine, and tangent to find $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$.

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta \\ &= 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) \\ &= \frac{\sqrt{3}}{2}\end{aligned}$$

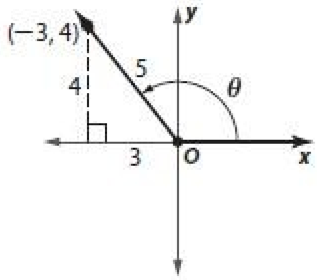
$$\begin{aligned}\cos 2\theta &= 2\cos^2\theta - 1 \\ &= 2\left(\frac{1}{2}\right)^2 - 1 \\ &= 2\left(\frac{1}{4}\right) - 1 \\ &= \frac{1}{2} - \frac{2}{2} \\ &= -\frac{1}{2}\end{aligned}$$

$$\begin{aligned}\tan 2\theta &= \frac{2\tan\theta}{1-\tan^2\theta} \\ &= \frac{2\left(\sqrt{3}\right)}{1-\left(\sqrt{3}\right)^2} \\ &= \frac{2\sqrt{3}}{-2} \\ &= -\sqrt{3}\end{aligned}$$

7. $\sin\theta = \frac{4}{5}$, $\left(\frac{\pi}{2}, \pi\right)$

SOLUTION:

Since $\sin\theta = \frac{4}{5}$ on the interval $\left(\frac{\pi}{2}, \pi\right)$, one point on the terminal side of θ has y-coordinate 4 and a distance of 5 units from the origin as shown. The x-coordinate of this point is therefore $-\sqrt{5^2 - 4^2}$ or −3.



Using this point, we find that $\cos\theta = \frac{x}{r}$ or $-\frac{3}{5}$ and $\tan\theta = \frac{y}{x}$ or $-\frac{4}{3}$. Now use the double-angle identities for sine and cosine to find sin 2θ and cos 2θ.

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta \\ &= 2\left(\frac{4}{5}\right)\left(-\frac{3}{5}\right) \\ &= -\frac{24}{25}\end{aligned}$$

$$\begin{aligned}\cos 2\theta &= 2\cos^2\theta - 1 \\ &= 2\left(-\frac{3}{5}\right)^2 - 1 \\ &= 2\left(\frac{9}{25}\right) - 1 \\ &= \frac{18}{25} - \frac{25}{25} \\ &= -\frac{7}{25}\end{aligned}$$

Use the definition of tangent to find tan 2θ.

$$\begin{aligned}\tan 2\theta &= \frac{2\tan\theta}{1-\tan^2\theta}\\ &= \frac{2\left(-\frac{4}{3}\right)}{1-\left(-\frac{4}{3}\right)^2}\\ &= \frac{-\frac{8}{3}}{1-\left(\frac{16}{9}\right)}\\ &= \frac{-\frac{8}{3}}{-\frac{7}{9}}\\ &= \frac{24}{7}\end{aligned}$$

Solve each equation on the interval [0, 2π].

9. sin 2θ = cos θ

SOLUTION:

$$\begin{aligned}\sin 2\theta &= \cos\theta\\ 2\sin\theta\cos\theta &= \cos\theta\\ 2\sin\theta\cos\theta-\cos\theta &= 0\\ \cos\theta(2\sin\theta-1) &= 0\end{aligned}$$

$\cos\theta = 0$ or $2\sin\theta - 1 = 0$, $\sin\theta = \frac{1}{2}$

On the interval [0, 2π), cos θ = 0 when $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$ and sin $\theta = \frac{1}{2}$ when $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$.

10. cos 2θ = cos θ

SOLUTION:

$$\begin{aligned}\cos 2\theta &= \cos\theta\\ 2\cos^2\theta-1 &= \cos\theta\\ 2\cos^2\theta-\cos\theta-1 &= 0\\ (2\cos\theta+1)(\cos\theta-1) &= 0\end{aligned}$$

$2\cos\theta + 1 = 0$, $\cos\theta = -\frac{1}{2}$ or $\cos\theta - 1 = 0$, $\cos\theta = 1$

On the interval [0, 2π), $\cos\theta = -\frac{1}{2}$ when $\theta = \frac{2\pi}{3}$ and $\theta = \frac{4\pi}{3}$ and cos θ = 1 when θ = 0.

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11. $\cos 2\theta - \sin \theta = 0$

SOLUTION:

$$\begin{aligned} \cos 2\theta - \sin\theta &= 0 \\ 1 - 2\sin^2\theta - \sin\theta &= 0 \\ 2\sin^2\theta + \sin\theta - 1 &= 0 \\ (\sin\theta + 1)(2\sin\theta - 1) &= 0 \end{aligned}$$

$$\begin{aligned} \sin\theta + 1 &= 0 \\ \sin\theta &= -1 \end{aligned} \quad \text{or} \quad \begin{aligned} 2\sin\theta - 1 &= 0 \\ \sin\theta &= \frac{1}{2} \end{aligned}$$

On the interval $[0, 2\pi)$, $\sin \theta = -1$ when $\theta = \frac{3\pi}{2}$ and $\sin \theta = \frac{1}{2}$ when $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$.

12. $\tan 2\theta - \tan 2\theta \tan^2 \theta = 2$

SOLUTION:

$$\begin{aligned} \tan 2\theta - \tan 2\theta \tan^2\theta &= 2 \\ \tan 2\theta(1 - \tan^2\theta) &= 2 \\ \frac{2\tan\theta}{1 - \tan^2\theta}(1 - \tan^2\theta) &= 2 \\ 2\tan\theta &= 2 \\ \tan\theta &= 1 \end{aligned}$$

On the interval $[0, 2\pi)$, $\tan \theta = 1$ when $\theta = \frac{\pi}{4}$ and $\theta = \frac{5\pi}{4}$.

Both of these are extraneous solutions since $1 - \tan^2 \frac{\pi}{4} = 0$ and $1 - \tan^2 \frac{5\pi}{4} = 0$.

Thus, there are no values on the interval $[0, 2\pi)$ for which the equation will be true. So, the solution is Ø.

13. $\sin 2\theta \csc \theta = 1$

SOLUTION:

$$\begin{aligned} \sin 2\theta \csc\theta &= 1 \\ 2\sin\theta\cos\theta\csc\theta &= 1 \\ 2\sin\theta\cos\theta \cdot \frac{1}{\sin\theta} &= 1 \\ 2\cos\theta &= 1 \\ \cos\theta &= \frac{1}{2} \end{aligned}$$

On the interval $[0, 2\pi)$, $\cos \theta = \frac{1}{2}$ when $\theta = \frac{\pi}{3}$ and $\theta = \frac{5\pi}{3}$.