Chapter 10
Angular Momentum

Conceptual Problems

1 • True or false:
(a) If two vectors are exactly opposite in direction, their cross product must be zero.
(b) The magnitude of the cross product of 2 vectors is at a minimum when the two vectors are perpendicular.
(c) Knowing the magnitude of the cross product of two nonzero vectors and their individual magnitudes uniquely determines the angle between them.

Determine the Concept The cross product of vectors $\vec{A}$ and $\vec{B}$ is defined to be $\vec{A}\times\vec{B} = AB\sin\phi\,\hat{n}$ where $\hat{n}$ is a unit vector normal to the plane defined by $\vec{A}$ and $\vec{B}$.

(a) True. If $\vec{A}$ and $\vec{B}$ are in opposite direction, then $\sin\phi = \sin(180°) = 0$.

(b) False. If $\vec{A}$ and $\vec{B}$ are perpendicular, then $\sin\phi = \sin(90°) = 1$ and the cross product of $\vec{A}$ and $\vec{B}$ is a maximum.

(c) False. $\phi = \sin^{-1}\left(\frac{\left|\vec{A}\times\vec{B}\right|}{AB}\right)$, because of the magnitude of $\vec{A}\times\vec{B}$, gives the reference angle associated with $\vec{A}\times\vec{B}$.

2 • Consider two nonzero vectors $\vec{A}$ and $\vec{B}$. Their cross product has the greatest magnitude if $\vec{A}$ and $\vec{B}$ are (a) parallel, (b) perpendicular, (c) antiparallel, (d) at an angle of 45° to each other.

Determine the Concept The cross product of the vectors $\vec{A}$ and $\vec{B}$ is defined to be $\vec{A}\times\vec{B} = AB\sin\phi\,\hat{n}$ where $\hat{n}$ is a unit vector normal to the plane defined by $\vec{A}$ and $\vec{B}$. Hence, the cross product is a maximum when $\sin\phi = 1$. This condition is satisfied provided $\vec{A}$ and $\vec{B}$ are *perpendicular*. $\boxed{(b)}$ is correct.

3 • What is the angle between a force $\vec{F}$ and a torque vector $\vec{\tau}$ produced by $\vec{F}$?

Determine the Concept Because $\vec{\tau} = \vec{r}\times\vec{F} = rF\sin\phi\,\hat{n}$, where $\hat{n}$ is a unit vector normal to the plane defined by $\vec{r}$ and $\vec{F}$, the angle between $\vec{F}$ and $\vec{\tau}$ is $\boxed{90°.}$

4 • A particle of mass m is moving with a constant speed v along a straight line that passes through point P. What can you say about the angular momentum of the particle relative to point P? (a) Its magnitude is mv. (b) Its magnitude is zero. (c) Its magnitude changes sign as the particle passes through point P. (d) It varies in magnitude as the particle approaches point P.

Determine the Concept $\vec{\boldsymbol{L}}$ and $\vec{\boldsymbol{p}}$ are related according to $\vec{\boldsymbol{L}} = \vec{\boldsymbol{r}} \times \vec{\boldsymbol{p}}$. Because the motion is along a line that passes through point P, $r = 0$ and so is L. $\boxed{(\boldsymbol{b})}$ is correct.

5 • **[SSM]** A particle travels in a circular path and point P is at the center of the circle. (a) If the particle's linear momentum $\vec{\boldsymbol{p}}$ is doubled without changing the radius of the circle, how is the magnitude of its angular momentum about P affected? (b) If the radius of the circle is doubled but the speed of the particle is unchanged, how is the magnitude of its angular momentum about P affected?

Determine the Concept $\vec{\boldsymbol{L}}$ and $\vec{\boldsymbol{p}}$ are related according to $\vec{\boldsymbol{L}} = \vec{\boldsymbol{r}} \times \vec{\boldsymbol{p}}$.

(a) Because $\vec{\boldsymbol{L}}$ is directly proportional to $\vec{\boldsymbol{p}}$, L is doubled.

(b) Because $\vec{\boldsymbol{L}}$ is directly proportional to $\vec{\boldsymbol{r}}$, L is doubled.

6 • A particle moves along a straight line at constant speed. How does its angular momentum about any fixed point vary with time?

Determine the Concept We can determine how the angular momentum of the particle about any fixed point varies with time by examining the derivative of the cross product of $\vec{r}$ and $\vec{p}$.

The angular momentum of the particle is given by:

$$\vec{L} = \vec{r} \times \vec{p}$$

Differentiate $\vec{L}$ with respect to time to obtain:

$$\frac{d\vec{L}}{dt} = \left(\vec{r} \times \frac{d\vec{p}}{dt}\right) + \left(\frac{d\vec{r}}{dt} \times \vec{p}\right) \qquad (1)$$

Because $\vec{p} = m\vec{v}$, $\frac{d\vec{p}}{dt} = \vec{F}_{\text{net}}$, and $\frac{d\vec{r}}{dt} = \vec{v}$:

$$\frac{d\vec{L}}{dt} = \left(\vec{r} \times \vec{F}_{\text{net}}\right) + \left(\vec{v} \times \vec{p}\right)$$

Because the particle moves along a straight line at constant speed: $F_{\text{net}} = 0 \Rightarrow \vec{r} \times \vec{F}_{\text{net}} = 0$

Because $\vec{v}$ and $\vec{p}(= m\vec{v})$ are parallel: $\vec{v} \times \vec{p} = 0$

Substitute in equation (1) to obtain: $\frac{d\vec{L}}{dt} = 0 \Rightarrow \vec{L}$ does not change in time.

7 •• True or false: If the net torque on a rotating system is zero, the angular velocity of the system cannot change. If your answer is false, give an example of such a situation.

False. The net torque acting on a rotating system equals the change in the system's angular momentum; that is, $\tau_{\text{net}} = dL/dt$ where $L = I\omega$. Hence, if τ_{net} is zero, all we can say for sure is that the angular momentum (the product of I and ω) is constant. If I changes, so must ω. An example is a high diver going from a tucked to a layout position.

8 •• You are standing on the edge of a frictionless turntable that is initially rotating When you catch a ball that was thrown in the same direction that you are moving, and on a line tangent to the edge of the turntable. Assume you do not move relative to the turntable. (*a*) Does the angular speed of the turntable increase, decrease, or remain the same during the catch? (*b*) Does the magnitude of your angular momentum (about the rotation axis of the table) increase, decrease, or remain the same after the catch? (*c*) How does the ball's angular momentum (relative to the center of the table) change after the catch? (*d*) How does the total angular momentum of the system you-table-ball (about the rotation axis of the table) change after the catch?

Determine the Concept You can apply conservation of angular momentum to the you-table-ball system to answer each of these questions.

(*a*) Because the ball is moving in the same direction that you are moving, your angular speed will increase when you catch it.

(*b*) The ball has angular momentum relative to the rotation axis of the table before you catch it and so catching it increases your angular momentum relative to the rotation axis of the table.

(*c*) The ball will slow down as a result of your catch and so its angular momentum relative to the center of the table will decrease.

(*d*) Because there is zero net torque on the you-table-ball system, its angular momentum **remains the same.**

9 •• If the angular momentum of a system about a fixed point *P* is constant, which one of the following statements must be true?

(*a*) No torque about *P* acts on any part of the system.
(*b*) A constant torque about *P* acts on each part of the system.
(*c*) Zero net torque about *P* acts on each part of the system.
(*d*) A constant external torque about *P* acts on the system.
(*e*) Zero net external torque about *P* acts on the system.

Determine the Concept If *L* is constant, we know that the *net* torque acting on the system is zero. There may be multiple constant or time-dependent torques acting on the system as long as the net torque is zero. **(*e*)** is correct.

10 •• A block sliding on a frictionless table is attached to a string that passes through a narrow hole through the tabletop. Initially, the block is sliding with speed v_0 in a circle of radius r_0. A student under the table pulls slowly on the string. What happens as the block spirals inward? Give supporting arguments for your choice. (The term angular momentum refers to the angular momentum about a vertical axis through the hole.) (*a*) Its energy and angular momentum are conserved. (*b*) Its angular momentum is conserved and its energy increases. (*c*) Its angular momentum is conserved and its energy decreases. (*d*) Its energy is conserved and its angular momentum increases. (*e*) Its energy is conserved and its angular momentum decreases.

Determine the Concept The pull that the student exerts on the block is at right angles to its motion and exerts no torque (recall that $\vec{\tau} = \vec{r} \times \vec{F}$ and $\tau = rF\sin\phi$). Therefore, we can conclude that the angular momentum of the block is conserved. The student does, however, do work in displacing the block in the direction of the radial force and so the block's energy increases. **(*b*)** is correct.

11 •• **[SSM]** One way to tell if an egg is hardboiled or uncooked without breaking the egg is to lay the egg flat on a hard surface and try to spin it. A hardboiled egg will spin easily, while an uncooked egg will not. However, once spinning, the uncooked egg will do something unusual; if you stop it with your finger, it may start spinning again. Explain the difference in the behavior of the two types of eggs.

Determine the Concept The hardboiled egg is solid inside, so everything rotates with a uniform angular speed. By contrast, when you start an uncooked egg spinning, the yolk will not immediately spin with the shell, and when you stop it from spinning the yolk will initially continue to spin.

12 •• Explain why a helicopter with just one main rotor has a second smaller rotor mounted on a horizontal axis at the rear as in Figure 10-40. Describe the resultant motion of the helicopter if this rear rotor fails during flight.

Determine the Concept The purpose of the second smaller rotor is to prevent the body of the helicopter from rotating. If the rear rotor fails, the body of the helicopter will tend to rotate on the main axis due to angular momentum being conserved.

13 •• The spin angular momentum vector for a spinning wheel is parallel with its axle and is pointed east. To cause this vector to rotate toward the south, it is necessary to exert a force on the east end of the axle in which direction? (*a*) up, (*b*) down, (*c*) north, (*d*) south, (*e*) east.

Determine the Concept The vector $\Delta\vec{L} = \vec{L}_\mathrm{f} - \vec{L}_\mathrm{i}$ (and the torque that is responsible for this change in the direction of the angular momentum vector) is initially points to the south and eventually points south-west. One can use a right-hand rule to determine the direction of this torque, and hence the force exerted on the east end of the axle, required to turn the angular momentum vector from east to south. Letting the fingers of your right hand point east, rotate your wrist until your thumb points south. Note that fingers, which point in the direction of the force that must be exerted on the east end of the axle, points upward. $\boxed{(\boldsymbol{a})}$ is correct.

14 •• You are walking toward the north and with your left hand you are carrying a suitcase that contains a massive spinning wheel mounted on an axle attached to the front and back of the case. The angular velocity of the gyroscope points north. You now begin to turn to walk toward the south. As a result, the front end of the suitcase will (*a*) resist your attempt to turn it and will try to maintain its original orientation, (*b*) resist your attempt to turn and will pull to the west, (*c*) rise upward, (*d*) dip downward, (*e*) show no effect whatsoever.

Determine the Concept In turning toward the south, you redirect the angular momentum vector from north to south by exerting a torque on the spinning wheel. The force that you must exert to produce this torque (use a right-hand rule with your thumb pointing either east of north or west of north and note that your fingers point upward) is upward. That is, the force you exert on the front end of the suitcase is upward and the force the suitcase exerts on you is downward.

Consequently, the front end of the suitcase will dip downward. $\boxed{(\boldsymbol{d})}$ is correct.

15 •• **[SSM]** The angular momentum of the propeller of a small single-engine airplane points forward. The propeller rotates clockwise if viewed from behind. (*a*) Just after liftoff, as the nose lifts and the airplane tends to veer to one side. To which side does it veer and why? (*b*) If the plane is flying horizontally and suddenly turns to the right, does the nose of the plane tend to move up or down? Why?

(*a*) The plane tends to veer to the right. The change in angular momentum $\Delta\vec{\boldsymbol{L}}_{\text{prop}}$ for the propeller is up, so the net torque $\vec{\boldsymbol{\tau}}$ on the propeller is up as well. The propeller must exert an equal but opposite torque on the plane. This downward torque exerted on the plane by the propeller tends to cause a downward change in the angular momentum of the plane. This means the plane tends to rotate clockwise as viewed from above.

(*b*) The plane tends to veer downward. The change in angular momentum $\Delta\vec{\boldsymbol{L}}_{\text{prop}}$ for the propeller is to the right, so the net torque $\vec{\boldsymbol{\tau}}$ on the propeller is toward the right as well. The propeller must exert an equal but opposite torque on the plane. This leftward directed torque exerted by the propeller on the plane tends to cause a leftward-directed change in angular momentum for the plane. This means the plane tends to rotate clockwise as viewed from the right.

16 •• You have designed a car that is powered by the energy stored in a single flywheel with a spin angular momentum $\vec{L}$. In the morning, you plug the car into an electrical outlet and a motor spins the flywheel up to speed, adding a huge amount of rotational kinetic energy to it—energy that will be changed into translational kinetic energy of the car during the day. Having taken a physics course involving angular momentum and torques, you realize that problems would arise during various maneuvers of the car. Discuss some of these problems. For example, suppose the flywheel is mounted so $\vec{L}$ points vertically upward when the car is on a horizontal road. What would happen as the car travels over a hilltop? Through a valley? Suppose the flywheel is mounted so $\vec{L}$ points forward, or to one side, when the car is on a horizontal road. Then what would happen as the car attempts to turn to the left or right? In each case that you examine, consider the direction of the torque exerted on the car by the road.

Determine the Concept If $\vec{L}$ points up and the car travels over a hill or through a valley, the force the road exerts on the wheels on one side (or the other) will increase and car will tend to tip. If $\vec{L}$ points forward and the car turns left or right, the front (or rear) of the car will tend to lift. These problems can be averted by having two identical flywheels that rotate on the same shaft in opposite directions.

17 •• **[SSM]** You are sitting on a spinning piano stool with your arms folded. (*a*) When you extend your arms out to the side, what happens to your kinetic energy? What is the cause of this change? (*b*) Explain what happens to your moment of inertia, angular speed and angular momentum as you extend your arms.

Determine the Concept The rotational kinetic energy of the you-stool system is given by $K_{\text{rot}} = \frac{1}{2}I\omega^2 = \frac{L^2}{2I}$. Because the net torque acting on the you-stool system is zero, its angular momentum $\vec{L}$ is conserved.

(*a*) Your kinetic energy decreases. Increasing your moment of inertia I while conserving your angular momentum L decreases your kinetic energy $K = L^2/(2I)$.

(*b*) Extending your arms out to the side increases your moment of inertia I s and decreases your angular speed. The angular momentum of the system is unchanged.

18 •• A uniform rod of mass M and length L rests on a horizontal frictionless table. A blob of putty of mass $m = M/4$ moves along a line perpendicular to the rod, strikes the rod near its end, and sticks to the rod. Describe qualitatively the subsequent motion of the rod and putty.

Determine the Concept The center of mass of the rod-and-putty system moves in a straight line, and the system rotates about its center of mass.

Estimation and Approximation

19 •• **[SSM]** An ice skater starts her pirouette with arms outstretched, rotating at 1.5 rev/s. Estimate her rotational speed (in revolutions per second) when she brings her arms tight against her body.

Picture the Problem Because we have no information regarding the mass of the skater, we'll assume that her body mass (not including her arms) is 50 kg and that each arm has a mass of 4.0 kg. Let's also assume that her arms are 1.0 m long and that her body is cylindrical with a radius of 20 cm. Because the net external torque acting on her is zero, her angular momentum will remain constant during her pirouette.

Because the net external torque acting on her is zero:

$$\Delta L = L_f - L_i = 0$$

or

$$I_{\text{armsin}}\omega_{\text{armsin}} - I_{\text{armsout}}\omega_{\text{armsout}} = 0 \quad (1)$$

Express her total moment of inertia with her arms out:

$$I_{\text{arms out}} = I_{\text{body}} + I_{\text{arms}}$$

Treating her body as though it is cylindrical, calculate the moment of inertia of her body, minus her arms:

$$\boldsymbol{I}_{\text{body}} = \tfrac{1}{2}\boldsymbol{mr}^2 = \tfrac{1}{2}(50\,\text{kg})(0.20\,\text{m})^2 = 1.00\,\text{kg}\cdot\text{m}^2$$

Modeling her arms as though they are rods, calculate their moment of inertia when she has them out:

$$\boldsymbol{I}_{\text{arms}} = 2\left[\tfrac{1}{3}(4\,\text{kg})(1.0\,\text{m})^2\right] = 2.67\,\text{kg}\cdot\text{m}^2$$

Substitute to determine her total moment of inertia with her arms out:

$$I_{\text{arms out}} = 1.00\,\text{kg}\cdot\text{m}^2 + 2.67\,\text{kg}\cdot\text{m}^2 = 3.67\,\text{kg}\cdot\text{m}^2$$

Express her total moment of inertia with her arms in:

$$\boldsymbol{I}_{\text{arms in}} = \boldsymbol{I}_{\text{body}} + \boldsymbol{I}_{\text{arms}} = 1.00\,\text{kg}\cdot\text{m}^2 + 2\left[(4.0\,\text{kg})(0.20\,\text{m})^2\right] = 1.32\,\text{kg}\cdot\text{m}^2$$

Solve equation (1) for $\omega_{\text{arms in}}$ to obtain:

$$\omega_{\text{arms in}} = \frac{I_{\text{arms out}}}{I_{\text{arms in}}}\omega_{\text{arms out}}$$

Substitute numerical values and evaluate $\omega_{\text{arms in}}$:

$$\boldsymbol{\omega}_{\text{arms in}} = \frac{3.67\,\text{kg}\cdot\text{m}^2}{1.32\,\text{kg}\cdot\text{m}^2}(1.5\,\text{rev/s}) \approx \boxed{4\,\text{rev/s}}$$

20 •• Estimate the ratio of angular velocities for the rotation of a diver between the full tuck position and the full-layout position.

Picture the Problem Because the net external torque acting on the diver is zero, the diver's angular momentum will remain constant as she rotates from the full tuck to the full layout position. Assume that, in layout position, the diver is a thin rod of length 2.5 m and that, in the full tuck position, the diver is a sphere of radius 0.50 m.

Because the net external torque acting on the diver is zero:

$$\Delta\boldsymbol{L} = \boldsymbol{L}_{\text{layout}} - \boldsymbol{L}_{\text{tuck}} = 0$$

or

$$\boldsymbol{I}_{\text{layout}}\boldsymbol{\omega}_{\text{layout}} - \boldsymbol{I}_{\text{tuck}}\boldsymbol{\omega}_{\text{tuck}} = 0$$

Solving for the ratio of the angular velocities yields:

$$\frac{\omega_{\text{tuck}}}{\omega_{\text{layout}}}=\frac{I_{\text{layout}}}{I_{\text{tuck}}}$$

Substituting for the moment of inertia of a thin rod relative to an axis through its center of mass and the moment of inertia of a sphere relative to its center of mass and simplifying yields:

$$\frac{\omega_{\text{tuck}}}{\omega_{\text{layout}}}=\frac{\frac{1}{12}m\ell^2}{\frac{2}{5}mr^2}=\frac{5\ell^2}{24r^2}$$

Substitute numerical values and evaluate $\omega_{\text{tuck}}/\omega_{\text{layout}}$:

$$\frac{\omega_{\text{tuck}}}{\omega_{\text{layout}}}=\frac{5(2.5\text{ m})^2}{24(0.50\text{ m})^2}\approx\boxed{5}$$

21 •• Mars and Earth have nearly identical lengths of days. Earth's mass is 9.35 times Mars' mass, its radius is 1.88 times Mars' radius, and Mars' orbital radius is, on average, 1.52 times greater than Earth's orbital radius. The Martian year is 1.88 times longer than Earth's year. Assume they are both uniform spheres and their orbits about the Sun are circles. Estimate the ratio (Earth to Mars) of (*a*) their spin angular momenta, (*b*) their spin kinetic energies, (*c*) their orbital angular momenta, and (*d*) their orbital kinetic energies.

Picture the Problem We can use the definitions of spin angular momentum, spin kinetic energy, orbital angular momentum, and orbital kinetic energy to evaluate these ratios.

(*a*) The ratio of the spin angular momenta of Earth and Mars is:

$$\left(\frac{L_{\text{E}}}{L_{\text{M}}}\right)_{\text{spin}}=\frac{I_{\text{E}}\omega_{\text{E}}}{I_{\text{M}}\omega_{\text{M}}}$$

Because Mars and Earth have nearly identical lengths of days, $\omega_{\text{E}}\approx\omega_{\text{M}}$:

$$\left(\frac{L_{\text{E}}}{L_{\text{M}}}\right)_{\text{spin}}\approx\frac{I_{\text{E}}}{I_{\text{M}}}$$

Substituting for the moments of inertia and simplifying yields:

$$\left(\frac{L_{\text{E}}}{L_{\text{M}}}\right)_{\text{spin}}\approx\frac{\frac{2}{5}M_{\text{E}}R_{\text{E}}^2}{\frac{2}{5}M_{\text{M}}R_{\text{M}}^2}=\frac{M_{\text{E}}}{M_{\text{M}}}\left(\frac{R_{\text{E}}}{R_{\text{M}}}\right)^2$$

Substitute numerical values for the ratios and evaluate $\left(\frac{L_{\text{E}}}{L_{\text{M}}}\right)_{\text{spin}}$:

$$\left(\frac{L_{\text{E}}}{L_{\text{M}}}\right)_{\text{spin}}\approx 9.35(1.88)^2\approx\boxed{33}$$

(*b*) The ratio of the spin kinetic energies of Earth and Mars is:

$$\left(\frac{K_{\text{E}}}{K_{\text{M}}}\right)_{\text{spin}}=\frac{\frac{1}{2}I_{\text{E}}\omega_{\text{E}}^2}{\frac{1}{2}I_{\text{M}}\omega_{\text{M}}^2}=\frac{I_{\text{E}}\omega_{\text{E}}^2}{I_{\text{M}}\omega_{\text{M}}^2}$$

Because Mars and Earth have nearly identical lengths of days, $\omega_E \approx \omega_M$:

$$\left(\frac{K_E}{K_M}\right)_{spin} \approx \frac{I_E}{I_M}$$

Substituting for the moments of inertia and simplifying yields:

$$\left(\frac{K_E}{K_M}\right)_{spin} \approx \frac{\frac{2}{5}M_E R_E^2}{\frac{2}{5}M_M R_M^2} = \frac{M_E}{M_M}\left(\frac{R_E}{R_M}\right)^2$$

Substitute numerical values for the ratios and evaluate $\left(\frac{K_E}{K_M}\right)_{spin}$:

$$\left(\frac{K_E}{K_M}\right)_{spin} \approx 9.35(1.88)^2 \approx \boxed{33}$$

(c) Treating Earth and Mars as point objects, the ratio of their orbital angular momenta is:

$$\left(\frac{L_E}{L_M}\right)_{orb} = \frac{I_E\omega_E}{I_M\omega_M}$$

Substituting for the moments of inertia and angular speeds yields:

$$\left(\frac{L_E}{L_M}\right)_{orb} = \frac{M_E r_E^2\left(\frac{2\pi}{T_E}\right)}{M_M r_M^2\left(\frac{2\pi}{T_M}\right)}$$

where r_E and r_M are the radii of the orbits of Earth and Mars, respectively.

Simplify to obtain:

$$\left(\frac{L_E}{L_M}\right)_{orb} = \left(\frac{M_E}{M_M}\right)\left(\frac{r_E}{r_M}\right)^2\left(\frac{T_M}{T_E}\right)$$

Substitute numerical values for the three ratios and evaluate $\left(\frac{L_E}{L_M}\right)_{orb}$:

$$\left(\frac{L_E}{L_M}\right)_{orb} = (9.35)\left(\frac{1}{1.52}\right)^2(1.88) \approx \boxed{8}$$

(d) The ratio of the orbital kinetic energies of Earth and Mars is:

$$\left(\frac{K_E}{K_M}\right)_{orb} = \frac{\frac{1}{2}I_E\omega_E^2}{\frac{1}{2}I_M\omega_M^2}$$

Substituting for the moments of inertia and angular speeds and simplifying yields:

$$\left(\frac{K_E}{K_M}\right)_{orb} = \frac{M_E r_E^2\left(\frac{2\pi}{T_E}\right)^2}{M_M r_M^2\left(\frac{2\pi}{T_M}\right)^2} = \left(\frac{M_E}{M_M}\right)\left(\frac{r_E}{r_M}\right)^2\left(\frac{T_M}{T_E}\right)^2$$

Substitute numerical values for the ratios and evaluate $\left(\frac{K_E}{K_M}\right)_{orb}$:

$$\left(\frac{K_E}{K_M}\right)_{orb} = (9.35)\left(\frac{1}{1.52}\right)^2 (1.88)^2 \approx \boxed{14}$$

22 •• The polar ice caps contain about 2.3×10^{19} kg of ice. This mass contributes negligibly to the moment of inertia of Earth because it is located at the poles, close to the axis of rotation. Estimate the change in the length of the day that would be expected if the polar ice caps were to melt and the water were distributed uniformly over the surface of Earth.

Picture the Problem The change in the length of the day is the difference between its length when the ice caps have melted and the water has been distributed over the surface of the Earth and the length of the day before the ice caps melt. Because the net torque acting on the Earth during this process is zero, angular momentum is conserved and we can relate the angular speed (which are related to the length of the day) of the Earth before and after the ice caps melt to the moments of inertia of the Earth-plus-spherical shell the ice caps melt.

Express the change in the length of a day as:

$$\Delta \boldsymbol{T} = \boldsymbol{T}_{after} - \boldsymbol{T}_{before} \qquad (1)$$

Because the net torque acting on the Earth during this process is zero, angular momentum is conserved:

$$\Delta \boldsymbol{L} = \boldsymbol{L}_{after} - \boldsymbol{L}_{before} = 0$$

Substituting for L_{after} and L_{before} yields:

$$(\boldsymbol{I}_{sphere} + \boldsymbol{I}_{shell})\boldsymbol{\omega}_{after} - \boldsymbol{I}_{sphere}\boldsymbol{\omega}_{before} = 0$$

Because $\boldsymbol{\omega} = 2\boldsymbol{\pi}/\boldsymbol{T}$:

$$(\boldsymbol{I}_{sphere} + \boldsymbol{I}_{shell})\frac{2\boldsymbol{\pi}}{\boldsymbol{T}_{after}} - \boldsymbol{I}_{sphere}\frac{2\boldsymbol{\pi}}{\boldsymbol{T}_{before}} = 0$$

or, simplifying,

$$\frac{\boldsymbol{I}_{sphere} + \boldsymbol{I}_{shell}}{\boldsymbol{T}_{after}} - \frac{\boldsymbol{I}_{sphere}}{\boldsymbol{T}_{before}} = 0$$

Solve for T_{after} to obtain:

$$T_{after} = \left(1 + \frac{I_{shell}}{I_{sphere}}\right)T_{before}$$

Substituting for T_{after} in equation (1) and simplifying yields:

$$\Delta T = \left(1 + \frac{I_{\text{shell}}}{I_{\text{sphere}}}\right) T_{\text{before}} - T_{\text{before}} = \frac{I_{\text{shell}}}{I_{\text{sphere}}} T_{\text{before}}$$

Substitute for I_{shell} and I_{sphere} and simplify to obtain:

$$\Delta T = \frac{\frac{2}{3} mr^2}{\frac{2}{5} M_{\text{E}} R_{\text{E}}^2} T_{\text{before}} = \frac{5m}{3M_{\text{E}}} T_{\text{before}}$$

Substitute numerical values and evaluate ΔT:

$$\Delta \boldsymbol{T} = \frac{5\left(2.3\times10^{19}\ \text{kg}\right)}{3\left(5.98\times10^{24}\ \text{kg}\right)}\left(1\,\text{d}\times\frac{24\,\text{h}}{\text{d}}\times\frac{3600\,\text{s}}{\text{h}}\right) = \boxed{0.55\,\text{s}}$$

23 •• **[SSM]** A 2.0-g particle moves at a constant speed of 3.0 mm/s around a circle of radius 4.0 mm. (*a*) Find the magnitude of the angular momentum of the particle. (*b*) If $L = \sqrt{\ell(\ell+1)}\hbar$, where ℓ is an integer, find the value of $\ell(\ell+1)$ and the approximate value of ℓ. (*c*) By how much does ℓ change if the particle's speed increases by one-millionth of a percent, nothing else changing? Use your result to explain why the quantization of angular momentum is not noticed in macroscopic physics.

Picture the Problem We can use $L = mvr$ to find the angular momentum of the particle. In (*b*) we can solve the equation $L = \sqrt{\ell(\ell+1)}\hbar$ for $\ell(\ell+1)$ and the approximate value of ℓ.

(*a*) Use the definition of angular momentum to obtain:

$$L = mvr = \left(2.0\times10^{-3}\ \text{kg}\right)\left(3.0\times10^{-3}\ \text{m/s}\right)\left(4.0\times10^{-3}\ \text{m}\right) = 2.40\times10^{-8}\ \text{kg}\cdot\text{m}^2/\text{s}$$
$$= \boxed{2.4\times10^{-8}\ \text{kg}\cdot\text{m}^2/\text{s}}$$

(*b*) Solve the equation $L = \sqrt{\ell(\ell+1)}\hbar$ for $\ell(\ell+1)$:

$$\ell(\ell+1) = \frac{L^2}{\hbar^2} \qquad (1)$$

Substitute numerical values and evaluate $\ell(\ell+1)$:

$$\ell(\ell+1) = \left(\frac{2.40\times10^{-8}\ \text{kg}\cdot\text{m}^2/\text{s}}{1.05\times01^{-34}\ \text{J}\cdot\text{s}}\right)^2 = \boxed{5.2\times10^{52}}$$

Because $\ell >> 1$, approximate its value with the square root of $\ell(\ell+1)$:

$$\ell \approx \boxed{2.3\times10^{26}}$$

(c) The change in ℓ is:

$$\Delta\ell = \ell_{\text{new}} - \ell \qquad (2)$$

If the particle's speed increases by one-millionth of a percent while nothing else changes:

$$\boldsymbol{v} \to \boldsymbol{v} + 10^{-8}\boldsymbol{v} = \left(1+10^{-8}\right)\boldsymbol{v}$$

and

$$\boldsymbol{L} \to \boldsymbol{L} + 10^{-8}\boldsymbol{L} = \left(1+10^{-8}\right)\boldsymbol{L}$$

Equation (1) becomes:

$$\ell_{\text{new}}\left(\ell_{\text{new}}+1\right) = \frac{\left[\left(1+10^{-8}\right)\boldsymbol{L}\right]^2}{\hbar^2}$$

and

$$\ell_{\text{new}} \approx \frac{\left(1+10^{-8}\right)\boldsymbol{L}}{\hbar}$$

Substituting in equation (2) yields:

$$\Delta\ell = \ell_{\text{new}} - \ell \approx \frac{\left(1+10^{-8}\right)\boldsymbol{L}}{\hbar} - \frac{\boldsymbol{L}}{\hbar} = 10^{-8}\frac{\boldsymbol{L}}{\hbar}$$

Substitute numerical values and evaluate $\Delta\ell$:

$$\Delta\ell = 10^{-8}\left(\frac{2.40\times10^{-8}\ \text{kg}\cdot\text{m}^2/\text{s}}{1.05\times01^{-34}\ \text{J}\cdot\text{s}}\right) = \boxed{2.3\times10^{18}}$$

and

$$\frac{\Delta\ell}{\ell} = \frac{2.3\times10^{18}}{2.3\times10^{26}} \approx 10^{-6}\%$$

The quantization of angular momentum is not noticed in macroscopic physics because no experiment can detect a fractional change in ℓ of $10^{-6}\%$.

24 ••• Astrophysicists in the 1960s tried to explain the existence and structure of *pulsars*—extremely regular astronomical sources of radio pulses whose periods ranged from seconds to milliseconds. At one point, these radio sources were given the acronym LGM, standing for "Little Green Men," a reference to the idea that they might be signals of extraterrestrial civilizations. The explanation given today is no less interesting. Consider the following. Our Sun, which is a fairly typical star, has a mass of 1.99×10^{30} kg and a radius of 6.96×10^{8} m. Although it does not rotate uniformly, because it isn't a solid body, its average rate of rotation is about 1 rev/25 d. Stars larger than the Sun can end their life in spectacular explosions called supernovae, leaving behind a collapsed remnant of the star called a *neutron star*. Neutron stars have masses comparable to the original masses of the stars, but radii of only a few kilometers! The high rotation rates are due to the conservation of angular momentum during the collapse. These stars emit beams of radio waves. Because of the rapid angular speed of the stars, the beam sweeps past Earth at regular, very short, intervals. To produce the observed radio-wave pulses, the star has to rotate at rates from about 1 rev/s to 1000 rev/s. (*a*) Using data from the textbook, estimate the rotation rate of the Sun if it were to collapse into a neutron star of radius 10 km. The Sun is not a uniform sphere of gas and its moment of inertia is given by $I = 0.059MR^2$. Assume that the neutron star is spherical and has a uniform mass distribution. (*b*) Is the rotational kinetic energy of our Sun greater or smaller after the collapse? By what factor does it change, and where does the energy go to or come from?

Picture the Problem We can use conservation of angular momentum in Part (*a*) to relate the before-and-after collapse rotation rates of the sun. In Part (*b*), we can express the fractional change in the rotational kinetic energy of the Sun as it collapses into a neutron star to decide whether its rotational kinetic energy is greater initially or after the collapse.

(*a*) Use conservation of angular momentum to relate the angular momenta of the Sun before and after its collapse:

$$I_b\omega_b = I_a\omega_a \Rightarrow \omega_a = \frac{I_b}{I_a}\omega_b \qquad (1)$$

Using the given formula, approximate the moment of inertia I_b of the Sun before collapse:

$$\boldsymbol{I}_b = 0.059\boldsymbol{MR}_{sun}^2 = 0.059\left(1.99\times10^{30}\ \text{kg}\right)\left(6.96\times10^5\ \text{km}\right)^2 = 5.69\times10^{46}\ \text{kg}\cdot\text{m}^2$$

Find the moment of inertia I_a of the Sun when it has collapsed into a spherical neutron star of radius 10 km and uniform mass distribution:

$$I_a = \tfrac{2}{5}MR^2 = \tfrac{2}{5}\left(1.99\times10^{30}\ \text{kg}\right)\left(10\,\text{km}\right)^2 = 7.96\times10^{37}\ \text{kg}\cdot\text{m}^2$$

Substitute numerical values in equation (1) and simplify to obtain:

$$\omega_a = \frac{5.69\times10^{46}\ \text{kg}\cdot\text{m}^2}{7.96\times10^{37}\ \text{kg}\cdot\text{m}^2}\omega_b = 7.15\times10^8\ \omega_b$$

Given that ω_b = 1 rev/25 d, evaluate ω_a:

$$\boldsymbol{\omega}_a = 7.15\times10^8\left(\frac{1\,\text{rev}}{25\,\text{d}}\right) = 2.86\ \text{rev/d} = \boxed{2.9\times10^7\ \text{rev/d}}$$

Note that the rotational period decreases by the same factor of I_b/I_a and becomes:

$$\boldsymbol{T}_a=\frac{2\pi}{\boldsymbol{\omega}_a}=\frac{2\pi}{2.86\times10^7\,\frac{\text{rev}}{\text{d}}\times\frac{2\pi\,\text{rad}}{\text{rev}}\times\frac{1\,\text{d}}{24\,\text{h}}\times\frac{1\,\text{h}}{3600\,\text{s}}}=3.0\times10^{-3}\,\text{s}$$

(*b*) Express the fractional change in the Sun's rotational kinetic energy as a consequence of its collapse:

$$\frac{\Delta K}{K_b}=\frac{K_a-K_b}{K_b}=\frac{K_a}{K_b}-1$$

Substituting for the kinetic energies and simplifying yields:

$$\frac{\Delta K}{K_b}=\frac{\frac{1}{2}I_a\omega_a^2}{\frac{1}{2}I_b\omega_b^2}-1=\frac{I_a\omega_a^2}{I_b\omega_b^2}-1$$

Substitute numerical values and evaluate $\Delta K/K_b$:

$$\frac{\Delta K}{K_b}=\left(\frac{1}{7.15\times10^8}\right)\left(\frac{2.86\times10^7\,\text{rev/d}}{1\,\text{rev/25}\,\text{d}}\right)^2-1=\boxed{7.1\times10^8}$$

That is, the rotational kinetic energy *increases* by a factor of approximately 7×10^8. The additional rotational kinetic energy comes at the expense of gravitational potential energy, which decreases as the Sun gets smaller.

25 •• The moment of inertia of Earth about its spin axis is approximately 8.03×10^{37} kg·m². (*a*) Because Earth is nearly spherical, assume that the moment of inertia can be written as $I = CMR^2$, where C is a dimensionless constant, $M = 5.98\times10^{24}$ kg is the mass of Earth, and $R = 6370$ km is its radius. Determine C. (*b*) If the earth's mass were distributed uniformly, C would equal 2/5. From the value of C calculated in Part (*a*), is Earth's density greater near its center or near its surface? Explain your reasoning.

Picture the Problem We can solve $I = CMR^2$ for C and substitute numerical values in order to determine an experimental value of C for the earth. We can then compare this value to those for a spherical shell and a sphere in which the mass is uniformly distributed to decide whether the earth's mass density is greatest near its core or near its crust.

(*a*) Express the moment of inertia of Earth in terms of the constant C:

$$I=CMR^2\Rightarrow C=\frac{I}{MR^2}$$

Substitute numerical values and evaluate C:

$$C=\frac{8.03\times10^{37}\,\text{kg}\cdot\text{m}^2}{\left(5.98\times10^{24}\,\text{kg}\right)\left(6370\,\text{km}\right)^2}=\boxed{0.331}$$

(*b*) If all of the mass were in the crust, the moment of inertia of Earth would be that of a thin spherical shell:

$$I_{\text{spherical shell}} = \tfrac{2}{3}MR^2$$

If the mass of Earth were uniformly distributed throughout its volume, its moment of inertia would be:

$$I_{\text{solid sphere}} = \tfrac{2}{5}MR^2$$

Because experimentally $C < 0.4$, the mass density must be greater near the center of Earth..

26 ••• Estimate Timothy Goebel's initial takeoff speed, rotational velocity, and angular momentum when he performs a quadruple Lutz (Figure 10-41). Make any assumptions you think reasonable, but justify them. Goebel's mass is about 60 kg and the height of the jump is about 0.60 m. Note that his angular speed will change quite a bit during the jump, as he begins with arms outstretched and pulls them in. Your answer should be accurate to within a factor of 2, if you're careful.

Picture the Problem We'll assume that he launches himself at an angle of 45° with the horizontal with his arms spread wide, and then pulls them in to increase his rotational speed during the jump. We'll also assume that we can model him as a 2.0-m long cylinder with an average radius of 0.15 m and a mass of 60 kg. We can then find his take-off speed and ″air time″ using constant-acceleration equations, and use the latter, together with the definition of rotational velocity, to find his initial rotational velocity. Finally, we can apply conservation of angular momentum to find his initial angular momentum.

Using a constant-acceleration equation, relate his takeoff speed v_0 to his maximum elevation Δy:

$$v^2 = v_{0y}^2 + 2a_y\Delta y$$

or, because $v_{0y} = v_0\sin(45°)$, $v = 0$, and $a_y = -g$,

$$0 = v_0^2\sin^2 45° - 2g\Delta y$$

Solving for v_0 and simplifying yields:

$$v_0 = \sqrt{\frac{2g\Delta y}{\sin^2 45°}} = \frac{\sqrt{2g\Delta y}}{\sin 45°}$$

Substitute numerical values and evaluate v_0:

$$\boldsymbol{v}_0 = \frac{\sqrt{2(9.81\,\text{m/s}^2)(0.60\,\text{m})}}{\sin 45°} = \boxed{4.9\,\text{m/s}}$$

Use its definition to express Goebel's angular velocity:

$$\omega = \frac{\Delta\theta}{\Delta t}$$

Use a constant-acceleration equation to express Goebel's "air time" Δt:

$$\Delta t = 2\Delta t_{\text{rise }0.6\text{ m}} = 2\sqrt{\frac{2\Delta y}{g}}$$

Substitute numerical values and evaluate Δt:

$$\Delta \boldsymbol{t} = 2\sqrt{\frac{2(0.60\text{ m})}{9.81\text{ m/s}^2}} = 0.699\text{ s}$$

Substitute numerical values and evaluate ω:

$$\boldsymbol{\omega} = \frac{4\text{ rev}}{0.699\text{ s}} \times \frac{2\pi\text{ rad}}{\text{rev}} = \boxed{36\text{ rad/s}}$$

Use conservation of angular momentum to relate his take-off angular velocity ω_0 to his average angular velocity ω as he performs a quadruple Lutz:

$$I_0\omega_0 = I\omega$$

Assuming that he can change his moment of inertia by a factor of 2 by pulling his arms in, solve for and evaluate ω_0:

$$\boldsymbol{\omega}_0 = \frac{\boldsymbol{I}}{\boldsymbol{I}_0}\boldsymbol{\omega} = \frac{1}{2}(36\text{ rad/s}) = \boxed{18\text{ rad/s}}$$

Express his take-off angular momentum:

$$L_0 = I_0\omega_0$$

Assuming that we can model him as a solid cylinder of length ℓ with an average radius r and mass m, express his moment of inertia with arms drawn in (his take-off configuration):

$$I_0 = 2\left(\tfrac{1}{2}mr^2\right) = mr^2$$

where the factor of 2 represents our assumption that he can double his moment of inertia by extending his arms.

Substitute for I_0 to obtain:

$$L_0 = mr^2\omega_0$$

Substitute numerical values and evaluate L_0:

$$\boldsymbol{L}_0 = (60\text{ kg})(0.15\text{ m})^2(18\text{ rad/s}) = \boxed{24\text{ kg}\cdot\text{m}^2/\text{s}}$$

The Cross Product and the Vector Nature of Torque and Rotation

27 • **[SSM]** A force of magnitude F is applied horizontally in the negative x direction to the rim of a disk of radius R as shown in Figure 10-42. Write $\vec{\boldsymbol{F}}$ and $\vec{\boldsymbol{r}}$ in terms of the unit vectors $\hat{\boldsymbol{i}}$, $\hat{\boldsymbol{j}}$, and $\hat{\boldsymbol{k}}$, and compute the torque produced by this force about the origin at the center of the disk.

Picture the Problem We can express $\vec{F}$ and $\vec{r}$ in terms of the unit vectors $\hat{i}$ and $\hat{j}$ and then use the definition of the cross product to find $\vec{\tau}$.

Express $\vec{F}$ in terms of F and the unit vector $\hat{i}$:

$$\vec{F} = -F\hat{i}$$

Express $\vec{r}$ in terms of R and the unit vector $\hat{j}$:

$$\vec{r} = R\hat{j}$$

Calculate the cross product of $\vec{r}$ and $\vec{F}$:

$$\vec{\tau} = \vec{r}\times\vec{F} = FR\left(\hat{j}\times-\hat{i}\right) = FR\left(\hat{i}\times\hat{j}\right) = \boxed{FR\hat{k}}$$

28 • Compute the torque about the origin of the gravitational force $\vec{F} = -mg\hat{j}$ acting on a particle of mass m located at $\vec{r} = x\hat{i} + y\hat{j}$ and show that this torque is independent of the y coordinate.

Picture the Problem We can find the torque from the cross product of $\vec{r}$ and $\vec{F}$.

Compute the cross product of $\vec{r}$ and $\vec{F}$:

$$\begin{aligned}\vec{\tau} &= \vec{r}\times\vec{F} = \left(x\hat{i} + y\hat{j}\right)\left(-mg\hat{j}\right)\\ &= -mgx\left(\hat{i}\times\hat{j}\right) - mgy\left(\hat{j}\times\hat{j}\right)\\ &= \boxed{-mgx\hat{k}}\end{aligned}$$

29 • Find $\vec{A}\times\vec{B}$ for the following choices: (a) $\vec{A} = 4\hat{i}$ and $\vec{B} = 6\hat{i} + 6\hat{j}$, ($b$) $\vec{A} = 4\hat{i}$ and $\vec{B} = 6\hat{i} + 6\hat{k}$, and ($c$) $\vec{A} = 2\hat{i} + 3\hat{j}$ and $\vec{B} = 3\hat{i} + 2\hat{j}$.

Picture the Problem We can use the definitions of the cross products of the unit vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$ to evaluate $\vec{A}\times\vec{B}$ in each case.

(a) Evaluate $\vec{A}\times\vec{B}$ for $\vec{A} = 4\hat{i}$ and $\vec{B} = 6\hat{i} + 6\hat{j}$:

$$\begin{aligned}\vec{A}\times\vec{B} &= 4\hat{i}\times\left(6\hat{i} + 6\hat{j}\right)\\ &= 24\left(\hat{i}\times\hat{i}\right) + 24\left(\hat{i}\times\hat{j}\right)\\ &= 24(0) + 24\hat{k}\\ &= \boxed{24\hat{k}}\end{aligned}$$

(*b*) Evaluate $\vec{A}\times\vec{B}$ for $\vec{A} = 4\hat{i}$ and $\vec{B} = 6\hat{i} + 6\hat{k}$:

$$\begin{aligned}\vec{A}\times\vec{B} &= 4\hat{i}\times\left(6\hat{i}+6\hat{k}\right)\\ &= 24\left(\hat{i}\times\hat{i}\right)+24\left(\hat{i}\times\hat{k}\right)\\ &= 24(0)+24\left(-\hat{j}\right)\\ &= \boxed{-24\hat{j}}\end{aligned}$$

(*c*) Evaluate $\vec{A}\times\vec{B}$ for $\vec{A} = 2\hat{i} + 3\hat{j}$ and $\vec{B} = 3\hat{i} + 2\hat{j}$:

$$\begin{aligned}\vec{A}\times\vec{B} &= \left(2\hat{i}+3\hat{j}\right)\times\left(3\hat{i}+2\hat{j}\right)\\ &= 6\left(\hat{i}\times\hat{i}\right)+4\left(\hat{i}\times\hat{j}\right)+9\left(\hat{j}\times\hat{i}\right)\\ &\quad+6\left(\hat{j}\times\hat{j}\right)\\ &= 6(0)+4\left(\hat{k}\right)+9\left(-\hat{k}\right)+6(0)\\ &= \boxed{-5\hat{k}}\end{aligned}$$

30 •• For each case in Problem 31, compute $\left|\vec{A}\times\vec{B}\right|$. Compare it to $\left|\vec{A}\right|\left|\vec{B}\right|$ to estimate which of the pairs of vectors are closest to being perpendicular. Verify your answers by calculating the angle using the dot product.

Picture the Problem Because $\left|\vec{A}\times\vec{B}\right| = \left|\vec{A}\right|\left|\vec{B}\right|\sin\phi$, if vectors $\vec{A}$ and $\vec{B}$ are perpendicular, then $\left|\vec{A}\times\vec{B}\right| = \left|\vec{A}\right|\left|\vec{B}\right|$ or $\dfrac{\left|\vec{A}\times\vec{B}\right|}{\left|\vec{A}\right|\left|\vec{B}\right|} = 1$. The dot product of vectors $\vec{A}$ and $\vec{B}$ is $\vec{A}\cdot\vec{B} = \left|\vec{A}\right|\left|\vec{B}\right|\cos\phi$. We can verify our estimations using this definition to calculate ϕ for each pair of vectors.

(*a*) For $\vec{A} = 4\hat{i}$ and $\vec{B} = 6\hat{i} + 6\hat{j}$:

$$\frac{\left|\vec{A}\times\vec{B}\right|}{\left|\vec{A}\right|\left|\vec{B}\right|} = \frac{\left|4\hat{i}\times\left(6\hat{i}+6\hat{j}\right)\right|}{(4)\left(6\sqrt{2}\right)} = \frac{\left|24\hat{k}\right|}{24\sqrt{2}} = \frac{1}{\sqrt{2}} \approx 0.707$$

and the vectors $\vec{A}$ and $\vec{B}$ are not perpendicular.

The angle between $\vec{A}$ and $\vec{B}$ is:

$$\phi = \cos^{-1}\frac{\vec{A}\cdot\vec{B}}{\left|\vec{A}\right|\left|\vec{B}\right|} = \cos^{-1}\left(\frac{4\hat{i}\cdot\left(6\hat{i}+6\hat{j}\right)}{24\sqrt{2}}\right)$$

$$= \cos^{-1}\frac{24}{24\sqrt{2}} = 45°,$$

a result confirming that obtained above.

(*b*) For $\vec{\boldsymbol{A}} = 4\hat{\boldsymbol{i}}$ and $\vec{\boldsymbol{B}} = 6\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{k}}$:

$$\frac{\left|\vec{A}\times\vec{B}\right|}{\left|\vec{A}\right|\left|\vec{B}\right|} = \frac{\left|4\hat{i}\times\left(6\hat{i}+6\hat{k}\right)\right|}{(4)\left(6\sqrt{2}\right)} = \frac{\left|-24\hat{j}\right|}{24\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\approx 0.707$$

and the vectors $\vec{A}$ and $\vec{B}$ are not perpendicular.

The angle between $\vec{A}$ and $\vec{B}$ is:

$$\phi = \cos^{-1}\frac{\vec{A}\cdot\vec{B}}{\left|\vec{A}\right|\left|\vec{B}\right|} = \cos^{-1}\left(\frac{4\hat{\boldsymbol{i}}\cdot\left(6\hat{\boldsymbol{i}}+6\hat{\boldsymbol{k}}\right)}{24\sqrt{2}}\right)$$

$$= \cos^{-1}\frac{24}{24\sqrt{2}} = 45°,$$

a result confirming that obtained above.

(*c*) For $\vec{\boldsymbol{A}} = 2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$ and $\vec{\boldsymbol{B}} = 3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$:

$$\frac{\left|\vec{A}\times\vec{B}\right|}{\left|\vec{A}\right|\left|\vec{B}\right|} = \frac{\left|\left(2\hat{\boldsymbol{i}}+3\hat{\boldsymbol{j}}\right)\times\left(3\hat{\boldsymbol{i}}+2\hat{\boldsymbol{j}}\right)\right|}{\sqrt{13}\sqrt{13}} = \frac{\left|-5\hat{\boldsymbol{k}}\right|}{13}$$

$$= \frac{5}{13} \approx 0.385$$

and the vectors $\vec{A}$ and $\vec{B}$ are not perpendicular.

The angle between $\vec{A}$ and $\vec{B}$ is:

$$\phi = \cos^{-1}\frac{\vec{A}\cdot\vec{B}}{\left|\vec{A}\right|\left|\vec{B}\right|}$$

$$= \cos^{-1}\left(\frac{\left(2\hat{\boldsymbol{i}}+3\hat{\boldsymbol{j}}\right)\cdot\left(3\hat{\boldsymbol{i}}+2\hat{\boldsymbol{j}}\right)}{\sqrt{13}\sqrt{13}}\right)$$

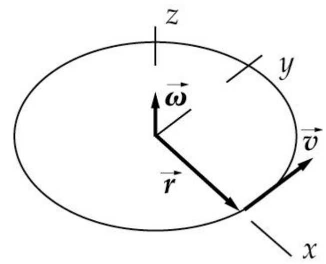
$$= \cos^{-1}\left(\frac{12}{13}\right) = 23°,$$

a result confirming that obtained above.

While none of these sets of vectors are perpendicular, those in (*a*) and (*b*) are the closest, with $\phi = 45°$, to being perpendicular.

31 •• A particle moves in a circle that is centered at the origin. The particle has position $\vec{\boldsymbol{r}}$ and angular velocity $\vec{\boldsymbol{\omega}}$. (*a*) Show that its velocity is given by $\vec{\boldsymbol{v}} = \vec{\omega}\times\vec{\boldsymbol{r}}$. (*b*) Show that its centripetal acceleration is given by $\vec{\boldsymbol{a}}_c = \vec{\omega}\times\vec{\boldsymbol{v}} = \vec{\omega}\times\left(\vec{\omega}\times\vec{\boldsymbol{r}}\right)$.

Picture the Problem Let $\vec{r}$ be in the xy plane and point in the $+x$ direction. Then $\vec{\omega}$ points in the $+z$ direction. We can establish the results called for in this problem by forming the appropriate cross products and by differentiating $\vec{v}$.



(a) Express $\vec{\omega}$ using unit vector notation:

$$\vec{\omega} = \omega\hat{k}$$

Express $\vec{r}$ using unit vector notation:

$$\vec{r} = r\hat{i}$$

Form the cross product of $\vec{\omega}$ and $\vec{r}$:

$$\vec{\omega}\times\vec{r} = \omega\hat{k}\times r\hat{i} = r\omega\left(\hat{k}\times\hat{i}\right) = r\omega\,\hat{j}$$
$$= v\hat{j}$$

and $\boxed{\vec{v} = \vec{\omega}\times\vec{r}}$

(b) Differentiate $\vec{v}$ with respect to t to express $\vec{a}$:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\vec{\omega}\times\vec{r}\right) = \frac{d\vec{\omega}}{dt}\times\vec{r} + \vec{\omega}\times\frac{d\vec{r}}{dt} = \frac{d\vec{\omega}}{dt}\times\vec{r} + \vec{\omega}\times\vec{v} = \vec{a}_{\rm t} + \vec{\omega}\times\left(\vec{\omega}\times\vec{r}\right)$$
$$= \vec{a}_{\rm t} + \vec{a}_{\rm c}$$

where $\vec{a}_{\rm c} = \boxed{\vec{\omega}\times\left(\vec{\omega}\times\vec{r}\right)}$ and $\vec{a}_{\rm t}$ and $\vec{a}_{\rm c}$ are the tangential and centripetal accelerations, respectively.

32 •• You are given three vectors and their components in the form: $\vec{A} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$, $\vec{B} = b_x\hat{i} + b_y\hat{j} + b_z\hat{k}$, and $\vec{C} = c_x\hat{i} + c_y\hat{j} + c_z\hat{k}$. Show that the following equalities hold: $\vec{A}\cdot\left(\vec{B}\times\vec{C}\right) = \vec{C}\cdot\left(\vec{A}\times\vec{B}\right) = \vec{B}\cdot\left(\vec{C}\times\vec{A}\right)$

Picture the Problem We can establish these equalities by carrying out the details of the cross- and dot-products and comparing the results of these operations.

Evaluate the cross product of $\vec{B}$ and $\vec{C}$ to obtain:

$$\vec{B}\times\vec{C} = \left(b_yc_z - b_zc_y\right)\hat{i} + \left(b_zc_x - b_xc_z\right)\hat{j} + \left(b_xc_y - b_yc_x\right)\hat{k}$$

Form the dot product of $\vec{A}$ with $\vec{B}\times\vec{C}$ to obtain:

$$\vec{A}\cdot\left(\vec{B}\times\vec{C}\right) = a_xb_yc_z - a_xb_zc_y + a_yb_zc_x - a_yb_xc_z + a_zb_xc_y - a_zb_yc_x \qquad (1)$$

Evaluate the cross product of $\vec{A}$ and $\vec{B}$ to obtain:

$$\vec{A}\times\vec{B}=\left(a_y b_z - a_z b_y\right)\hat{i}+\left(a_z b_x - a_x b_z\right)\hat{j}+\left(a_x b_y - a_y b_x\right)\hat{k}$$

Form the dot product of $\vec{C}$ with $\vec{A}\times\vec{B}$ to obtain:

$$\vec{C}\cdot\left(A\times\vec{B}\right)=c_x a_y b_z - c_x a_z b_y + c_y a_z b_x - c_y a_x b_z + c_z a_x b_y - c_z a_y b_x \qquad (2)$$

Evaluate the cross product of $\vec{C}$ and $\vec{A}$ to obtain:

$$\vec{C}\times\vec{A}=\left(c_y a_z - a_z a_y\right)\hat{i}+\left(c_z a_x - a_x a_z\right)\hat{j}+\left(c_x a_y - a_y a_x\right)\hat{k}$$

Form the dot product of $\vec{B}$ with $\vec{C}\times\vec{A}$ to obtain:

$$\vec{B}\cdot\left(\vec{C}\times\vec{A}\right)=b_x c_y a_z - b_x c_z a_y + b_y c_z a_x - b_y c_x a_z + b_z c_x a_y - b_z c_y a_x \qquad (3)$$

The equality of equations (1), (2), and (3) establishes the equalities.

33 •• If $\vec{A}=3\hat{j}$, $\vec{A}\times\vec{B}=9\hat{i}$, and $\vec{A}\cdot\vec{B}$ = 12, find $\vec{B}$.

Picture the Problem We can write $\vec{B}$ in the form $\vec{B}=B_x\hat{i}+B_y\hat{j}+B_z\hat{k}$ and use the dot product of $\vec{A}$ and $\vec{B}$ to find B_y and their cross product to find B_x and B_z.

Express $\vec{B}$ in terms of its components:

$$\vec{B}=B_x\hat{i}+B_y\hat{j}+B_z\hat{k} \qquad (1)$$

Evaluate $\vec{A}\cdot\vec{B}$:

$$\vec{A}\cdot\vec{B}=3B_y=12\Rightarrow B_y=4$$

Evaluate $\vec{A}\times\vec{B}$:

$$\vec{A}\times\vec{B}=3\hat{j}\times\left(B_x\hat{i}+4\hat{j}+B_z\hat{k}\right)$$
$$=-3B_x\hat{k}+3B_z\hat{i}$$

Because $\vec{A}\times\vec{B}=9\hat{i}$:

$B_x = 0$ and $B_z = 3$.

Substitute for B_y and B_z in equation (1) to obtain:

$$\vec{B}=\boxed{4\hat{j}+3\hat{k}}$$

34 •• If $\vec{A}=4\hat{i}$, $B_z = 0$, $\left|\vec{B}\right|=5$, and $\vec{A}\times\vec{B}=12\hat{k}$, determine $\vec{B}$.

Picture the Problem Because $B_z = 0$, we can express $\vec{B}$ as $\vec{B}=B_x\hat{i}+B_y\hat{j}$ and form its cross product with $\vec{A}$ to determine B_x and B_y.

Express $\vec{B}$ in terms of its components:

$$\vec{B} = B_x\hat{i} + B_y\hat{j} \qquad (1)$$

Express $\vec{A} \times \vec{B}$:

$$\vec{A} \times \vec{B} = 4\hat{i} \times \left(B_x\hat{i} + B_y\hat{j}\right) = 4B_y\hat{k} = 12\hat{k}$$

Solving for B_y yields:

$$B_y = 3$$

Relate B to B_x and B_y:

$$B^2 = B_x^2 + B_y^2$$

Solve for and evaluate B_x:

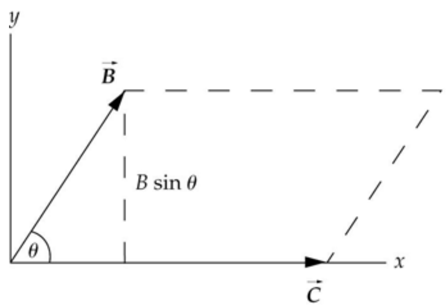
$$B_x = \sqrt{B^2 - B_y^2} = \sqrt{5^2 - 3^2} = 4$$

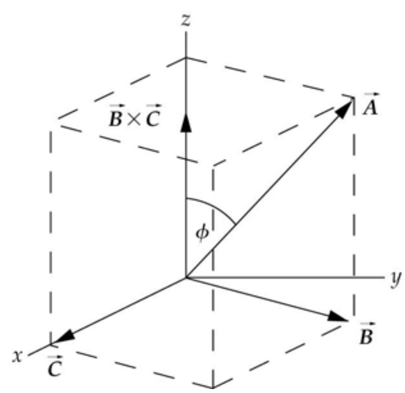
Substitute for B_x and B_y in equation (1) to obtain:

$$\vec{B} = \boxed{4\hat{i} + 3\hat{j}}$$

35 ••• Given three noncoplanar vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$, show that $\vec{A} \cdot \left(\vec{B} \times \vec{C}\right)$ is the volume of the parallelepiped formed by the three vectors.

Picture the Problem Let, without loss of generality, the vector $\vec{C}$ lie along the x axis and the vector $\vec{B}$ lie in the xy plane as shown below to the left. The diagram to the right shows the parallelepiped spanned by the three vectors. We can apply the definitions of the cross- and dot-products to show that $\vec{A} \cdot \left(\vec{B} \times \vec{C}\right)$ is the volume of the parallelepiped.





Express the cross-product of $\vec{B}$ and $\vec{C}$:

$$\vec{B} \times \vec{C} = \left(BC\sin\theta\right)\left(-\hat{k}\right)$$

and

$$\left|\vec{B} \times \vec{C}\right| = \left(B\sin\theta\right)C$$

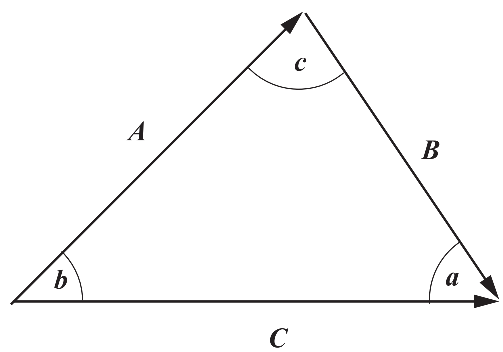
$$= \text{area of the parallelogram}$$

Form the dot-product of $\vec{A}$ with the cross-product of $\vec{B}$ and $\vec{C}$ to obtain:

$$\vec{A}\cdot\left(\vec{B}\times\vec{C}\right)=A(B\sin\theta)C\cos\phi$$
$$=(BC\sin\theta)(A\cos\phi)$$
$$=(\text{area of base})(\text{height})$$
$$=\boxed{V_{\text{parallelepiped}}}$$

36 ••• Using the cross product, prove the *law of sines* for the triangle shown in Figure 10-43. That is, if *A*, *B*, and *C* are the lengths of each side of the triangle, show that *A*/sin *a* = *B*/sin *b* = *C*/sin *c*.

Picture the Problem Draw the triangle using the three vectors as shown below. Note that $\vec{A}+\vec{B}=\vec{C}$. We can find the magnitude of the cross product of $\vec{A}$ and $\vec{B}$ and of $\vec{A}$ and $\vec{C}$ and then use the cross product of $\vec{A}$ and $\vec{C}$, using $\vec{A}+\vec{B}=\vec{C}$, to show that $\boldsymbol{AC}\sin\boldsymbol{b}=\boldsymbol{AB}\sin\boldsymbol{c}$ or $\dfrac{\boldsymbol{B}}{\sin\boldsymbol{b}}=\dfrac{\boldsymbol{C}}{\sin\boldsymbol{c}}$. Proceeding similarly, we can extend the law of sines to the third side of the triangle and the angle opposite it.



Express the magnitude of the cross product of $\vec{A}$ and $\vec{B}$:

$$\left|\vec{A}\times\vec{B}\right|=\boldsymbol{AB}\sin(180°-\boldsymbol{c})=\boldsymbol{AB}\sin\boldsymbol{c}$$

Express the magnitude of the cross product of $\vec{A}$ and $\vec{C}$:

$$\left|\vec{A}\times\vec{C}\right|=AC\sin b$$

Form the cross product of $\vec{A}$ with $\vec{C}$ to obtain:

$$\vec{A}\times\vec{C}=\vec{A}\times\left(\vec{A}+\vec{B}\right)$$
$$=\vec{A}\times\vec{A}+\vec{A}\times\vec{B}$$
$$=\vec{A}\times\vec{B}$$

because $\vec{A}\times\vec{A}=0$.

Because $\vec{A}\times\vec{C}=\vec{A}\times\vec{B}$:

$$\left|\vec{A}\times\vec{C}\right|=\left|\vec{A}\times\vec{B}\right|$$

and

$$AC\sin b=AB\sin c$$

Simplify and rewrite this expression to obtain:

$$\boxed{\frac{B}{\sin b}=\frac{C}{\sin c}}$$

Proceed similarly to extend this result to the law of sines:

$$\boxed{\frac{A}{\sin a}=\frac{B}{\sin b}=\frac{C}{\sin c}}$$

Torque and Angular Momentum

37 • **[SSM]** A 2.0-kg particle moves directly eastward at a constant speed of 4.5 m/s along an east-west line. (*a*) What is its angular momentum (including direction) about a point that lies 6.0 m north of the line? (*b*) What is its angular momentum (including direction) about a point that lies 6.0 m south of the line? (*c*) What is its angular momentum (including direction) about a point that lies 6.0 m directly east of the particle?

Picture the Problem The angular momentum of the particle is $\vec{L}=\vec{r}\times\vec{p}$ where $\vec{r}$ is the vector locating the particle relative to the reference point and $\vec{p}$ is the particle's linear momentum.

(*a*) The magnitude of the particle's angular momentum is given by:

$$L = rp\sin\phi = rmv\sin\phi = mv(r\sin\phi)$$

Substitute numerical values and evaluate L:

$$\boldsymbol{L} = (2.0\,\text{kg})(4.5\,\text{m/s})(6.0\,\text{m}) = 54\,\text{kg}\cdot\text{m}^2/\text{s}$$

Use a right-hand rule to establish the direction of $\vec{L}$:

$$\boldsymbol{L} = \boxed{54\,\text{kg}\cdot\text{m}^2/\text{s, upward}}$$

(*b*) Because the distance to the line along which the particle is moving is the same, only the direction of $\vec{L}$ differs:

$$\boldsymbol{L} = \boxed{54\,\text{kg}\cdot\text{m}^2/\text{s, downward}}$$

(*c*) Because $\vec{r}\times\vec{p}=0$ for a point on the line along which the particle is moving:

$$\vec{L} = \boxed{0}$$

38 • You observe a 2.0-kg particle moving at a constant speed of 3.5 m/s in a clockwise direction around a circle of radius 4.0 m. (*a*) What is its angular momentum (including direction) about the center of the circle? (*b*) What is its moment of inertia about an axis through the center of the circle and perpendicular to the plane of the motion? (*c*) What is the angular velocity of the particle?

Picture the Problem The angular momentum of the particle is $\vec{L} = \vec{r} \times \vec{p}$ where $\vec{r}$ is the vector locating the particle relative to the reference point and $\vec{p}$ is the particle's linear momentum.

(*a*) The magnitude of the particle's angular momentum is given by:

$$L = rp\sin\phi = rmv\sin\phi = mv(r\sin\phi)$$

Substitute numerical values and evaluate the magnitude of *L*:

$$\boldsymbol{L} = (2.0\,\text{kg})(3.5\,\text{m/s})(4.0\,\text{m}) = 28\,\text{kg}\cdot\text{m}^2/\text{s}$$

Use a right-hand rule to establish the direction of $\vec{L}$:

$$\boldsymbol{L} = \boxed{28\,\text{kg}\cdot\text{m}^2/\text{s, away from you}}$$

(*b*) Treat the 2.0-kg particle as a point particle to obtain:

$$I = mr^2$$

Substitute numerical values and evaluate *I*:

$$\boldsymbol{I} = (2.0\,\text{kg})(4.0\,\text{m})^2 = \boxed{32\,\text{kg}\cdot\text{m}^2}$$

(*c*) Because $L = I\omega$, the angular speed of the particle is the ratio of its angular momentum and its moment of inertia:

$$\boldsymbol{\omega} = \frac{\boldsymbol{L}}{\boldsymbol{I}}$$

Substitute numerical values and evaluate ω:

$$\boldsymbol{\omega} = \frac{28\,\text{kg}\cdot\text{m}^2/\text{s}}{32\,\text{kg}\cdot\text{m}^2} = \boxed{0.88\,\text{rad/s}^2}$$

39 •• (*a*) A particle moving at constant velocity has zero angular momentum about a particular point. Use the definition of angular momentum to show that under this condition the particle is moving either directly toward or directly away from the point. (*b*) You are a right-handed batter and let a waist-high fastball go past you without swinging. What is the direction of its angular momentum relative to your navel? (Assume the ball travels in a straight horizontal line as it passes you.)

Picture the Problem $\vec{L}$ and $\vec{p}$ are related according to $\vec{L} = \vec{r} \times \vec{p}$. If $\vec{L} = 0$, then examination of the magnitude of $\vec{r} \times \vec{p}$ will allow us to conclude that $\sin\phi = 0$ and that the particle is moving either directly toward the point, directly away from the point, or through the point.

(*a*) Because $\vec{L} = 0$:	$\vec{r} \times \vec{p} = \vec{r} \times m\vec{v} = m\vec{r} \times \vec{v} = 0$ or $\vec{r} \times \vec{v} = 0$
Express the magnitude of $\vec{r} \times \vec{v}$:	$\left\|\vec{r} \times \vec{v}\right\| = rv\sin\phi = 0$
Because neither *r* nor *v* is zero:	$\sin\phi = 0$ where ϕ is the angle between $\vec{r}$ and $\vec{v}$.
Solving for ϕ yields:	$\phi = \sin^{-1}(0) = \boxed{0° \text{ or } 180°}$

(*b*) Use the right-hand rule to establish that the ball's angular momentum is downward.

40 •• A particle that has a mass *m* is traveling with a constant velocity $\vec{v}$ along a straight line that is a distance *b* from the origin *O* (Figure 10-44). Let *dA* be the area swept out by the position vector from *O* to the particle during a time interval *dt*. Show that *dA*/*dt* is constant and is equal to $L/2m$, where *L* is the magnitude of the angular momentum of the particle about the origin.

Picture the Problem We can use the formula for the area of a triangle to find the area swept out at $t = t_1$, add this area to the area swept out in time *dt*, and then differentiate this expression with respect to time to obtain the given expression for *dA*/*dt*.

Express the area swept out at $t = t_1$:	$A_1 = \frac{1}{2}br_1\cos\theta_1 = \frac{1}{2}bx_1$ where θ_1 is the angle between $\vec{r}_1$ and $\vec{v}$ and x_1 is the component of $\vec{r}_1$ in the direction of $\vec{v}$.
The area swept out at $t = t_1 + dt$ is:	$A = A_1 + dA$
Substitute for A_1 to obtain:	$A = A_1 + dA = \frac{1}{2}b(x_1 + dx)$
Because $dx = vdt$:	$A = \frac{1}{2}b(x_1 + vdt)$
Differentiate *A* with respect to *t* to obtain:	$\frac{dA}{dt} = \frac{1}{2}b\frac{dx}{dt} = \frac{1}{2}bv = \text{constant}$

Because $r\sin\theta = b$:

$$\tfrac{1}{2}bv = \tfrac{1}{2}(r\sin\theta)v = \frac{1}{2m}(rp\sin\theta)$$

$$= \boxed{\frac{L}{2m}}$$

41 •• A 15-g coin that has a diameter of 1.5 cm is spinning at 10 rev/s about a fixed vertical axis. The coin is spinning on edge with its center directly above the point of contact with the tabletop. As you look down on the tabletop, the coin spins clockwise. (*a*) What is the angular momentum (including direction) of the coin about its center of mass? Model the coin as a thin disk with a radius *R*. (To find the moment of inertia about the axis, see Table 9-1.) (*b*) What is its angular momentum (including direction) about a point on the tabletop 10 cm from the axis? (*c*) Now the coin's center of mass travels in a straight line east across the tabletop at 5.0 cm/s, in addition to spinning the same way as in part (*a*). What is the angular momentum (including direction) of the coin about a point on the line of motion of the center of mass? (*d*) When it is both spinning and sliding, what is the angular momentum of the coin (including direction) about a point 10 cm north of the line of motion of the center of mass?

Picture the Problem We can find the total angular momentum of the coin from the sum of its spin and orbital angular momenta.

(*a*) The spin angular momentum of the coin is:

$$\boldsymbol{L}_{\text{spin}} = \boldsymbol{I\omega}_{\text{spin}}$$

From Table 9-1, for *L* negligible compared to *R*:

$$I = \tfrac{1}{4}MR^2$$

Substitute for *I* to obtain:

$$L_{\text{spin}} = \tfrac{1}{4}MR^2\omega_{\text{spin}}$$

Substitute numerical values and evaluate L_{spin}:

$$\boldsymbol{L}_{\text{spin}} = \tfrac{1}{4}(0.015\,\text{kg})(0.0075\,\text{m})^2\left(10\frac{\text{rev}}{\text{s}}\times\frac{2\boldsymbol{\pi}\,\text{rad}}{\text{rev}}\right) = 1.33\times10^{-5}\,\text{kg}\cdot\text{m}^2/\text{s}$$

Use a right-hand rule to establish the direction of $\vec{\boldsymbol{L}}_{\text{spin}}$:

$$\vec{\boldsymbol{L}}_{\text{spin}} = \boxed{1.3\times10^{-5}\,\text{kg}\cdot\text{m}^2/\text{s, away from you}}$$

(*b*)The total angular momentum of the coin is the sum of its orbital and spin angular momenta:

$$\boldsymbol{L}_{\text{total}} = \boldsymbol{L}_{\text{orbital}} + \boldsymbol{L}_{\text{spin}}$$

Substitute numerical values and evaluate L_{total}:

$$\boldsymbol{L}_{\text{total}} = 0 + \boldsymbol{L}_{\text{spin}} = 1.3\times10^{-5}\ \text{kg}\cdot\text{m}^2/\text{s}$$

Use a right-hand rule to establish the direction of $\vec{\boldsymbol{L}}_{\text{total}}$:

$$\vec{\boldsymbol{L}}_{\text{total}} = \boxed{1.3\times10^{-5}\ \text{kg}\cdot\text{m}^2/\text{s, away from you}}$$

(*c*) Because $\boldsymbol{L}_{\text{orbital}} = 0$:

$$\vec{\boldsymbol{L}}_{\text{total}} = \boxed{1.3\times10^{-5}\ \text{kg}\cdot\text{m}^2/\text{s, away from you}}$$

(*d*) When it is both spinning and sliding, the total angular momentum of the coin is:

$$\boldsymbol{L}_{\text{total}} = \boldsymbol{L}_{\text{orbital}} + \boldsymbol{L}_{\text{spin}}$$

The orbital angular momentum of the coin is:

$$L_{\text{orbital}} = MvR$$

The spin angular momentum of the coin is:

$$L_{\text{spin}} = I_{\text{spin}}\omega_{\text{spin}} = \tfrac{1}{4}MR^2\omega_{\text{spin}}$$

Substituting for L_{orbital} and L_{spin} yields:

$$L_{\text{total}} = MvR + \tfrac{1}{4}MR^2\omega_{\text{spin}}$$

Substitute numerical values and evaluate L_{total}:

$$L_{\text{total}} = (0.015\,\text{kg})(0.050\,\text{m/s})(0.10\,\text{m}) + \tfrac{1}{4}(0.015\,\text{kg})(0.0075\,\text{m})^2\left(10\,\frac{\text{rev}}{\text{s}}\times\frac{2\pi\ \text{rad}}{\text{rev}}\right)$$

$$= \boxed{8.8\times10^{-5}\ \text{kg}\cdot\text{m}^2/\text{s, pointing toward you}}$$

42 •• (*a*) Two stars of masses m_1 and m_2 are located at $\vec{r}_1$ and $\vec{r}_2$ relative to some origin *O,* as shown in Figure 10-45. They exert equal and opposite attractive gravitational forces on each other. For this two-star system, calculate the net torque exerted by these internal forces about the origin *O* and show that it is zero only if both forces lie along the line joining the particles. (*b*)The fact that the Newton's third-law pair of forces are not only equal and oppositely directed but also lie along the line connecting the two objects is sometimes called the strong form of Newton's third law. Why is it important to add that last phrase? *Hint: Consider what would happen to these two objects if the forces were offset from each other.*

Picture the Problem Both the forces acting on the particles exert torques with respect to an axis perpendicular to the page and through point O and the net torque about this axis is their vector sum.

(*a*) The net torque about an axis perpendicular to the page and through point O is given by:

$$\vec{\tau}_{net} = \sum_i \vec{\tau}_i = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2$$

or, because $\vec{F}_2 = -\vec{F}_1$,

$$\vec{\tau}_{net} = (\vec{r}_1 - \vec{r}_2) \times \vec{F}_1$$

Because $\vec{r}_1 - \vec{r}_2$ points along $-\vec{F}_1$:

$$\vec{\tau}_{net} = (\vec{r}_1 - \vec{r}_2) \times \vec{F}_1 = \boxed{0}$$

(*b*) If the forces are not along the same line, there will be a net torque (but still no net force) acting on the system. This net torque would cause the system to accelerate angularly, contrary to observation, and hence makes no sense physically.

43 •• A 1.8-kg particle moves in a circle of radius 3.4 m. As you look down on the plane of its orbit, it is initially moving clockwise. If we call the clockwise direction positive, its angular momentum relative to the center of the circle varies with time according to $\boldsymbol{L}(\boldsymbol{t}) = 10\ \text{N}\cdot\text{m}\cdot\text{s} - (4.0\ \text{N}\cdot\text{m})\boldsymbol{t}$. (*a*) Find the magnitude and direction of the torque acting on the particle. (*b*) Find the angular velocity of the particle as a function of time.

Picture the Problem The angular momentum of the particle changes because a *net* torque acts on it. Because we know how the angular momentum depends on time, we can find the net torque acting on the particle by differentiating its angular momentum. We can use a constant-acceleration equation and Newton's 2nd law to relate the angular speed of the particle to its angular acceleration.

(*a*) The magnitude of the torque acting on the particle is the rate at which its angular momentum changes:

$$\tau_{net} = \frac{dL}{dt}$$

Evaluate *dL*/*dt* to obtain:

$$\tau_{net} = \frac{d}{dt}\left[10\ \text{N}\cdot\text{m}\cdot\text{s} - (4.0\ \text{N}\cdot\text{m})t\right] = \boxed{-4.0\ \text{N}\cdot\text{m}}$$

Note that, because *L* decreases as the particle rotates clockwise, the angular acceleration and the net torque are both upward.

(*b*) The angular speed of the particle is given by:

$$\omega_{\text{orbital}} = \frac{L_{\text{orbital}}}{I_{\text{orbital}}}$$

Treating the 1.8-kg particle as a point particle, express its moment of inertia relative to an axis through the center of the circle and normal to it:

$$I_{\text{orbital}} = MR^2$$

Substitute for I_{orbital} and L_{orbital} to obtain:

$$\boldsymbol{\omega}_{\text{orbital}} = \frac{10\ \text{N}\cdot\text{m}\cdot\text{s} - (4.0\ \text{N}\cdot\text{m})\boldsymbol{t}}{\boldsymbol{MR}^2}$$

Substitute numerical values and evaluate ω_{orbital}:

$$\boldsymbol{\omega}_{\text{orbital}} = \frac{10\ \text{N}\cdot\text{m}\cdot\text{s} - (4.0\ \text{N}\cdot\text{m})\boldsymbol{t}}{(1.8\ \text{kg})(3.4\ \text{m})^2} = \boxed{0.48\ \text{rad/s} - (0.19\ \text{rad/s}^2)\boldsymbol{t}}$$

Note that the direction of the angular velocity is downward.

44 •• You are designing a lathe motor and part of it consists of a uniform cylinder whose mass is 90 kg and radius is 0.40 m that is mounted so that it turns without friction on its axis, which is fixed. The cylinder is driven by a belt that wraps around its perimeter and exerts a constant torque. At $t = 0$, the cylinder's angular velocity is zero. At $t = 25$ s, its angular speed is 500 rev/min. (*a*) What is the magnitude of its angular momentum at $t = 25$ s? (*b*) At what rate is the angular momentum increasing? (*c*) What is the magnitude of the torque acting on the cylinder? (*d*) What is the magnitude of the frictional force acting on the rim of the cylinder?

Picture the Problem The angular momentum of the cylinder changes because a *net* torque acts on it. We can find the angular momentum at $t = 25$ s from its definition and the magnitude of the *net* torque acting on the cylinder from the rate at which the angular momentum is changing. The magnitude of the frictional force acting on the rim can be found using the definition of torque.

(*a*) The angular momentum of the cylinder is given by:

$$L = I\omega = \tfrac{1}{2}mr^2\omega$$

Substitute numerical values and evaluate L:

$$\boldsymbol{L} = \tfrac{1}{2}(90\ \text{kg})(0.40\ \text{m})^2\left(500\frac{\text{rev}}{\text{min}}\times\frac{2\boldsymbol{\pi}\ \text{rad}}{\text{rev}}\times\frac{1\ \text{min}}{60\ \text{s}}\right) = 377\ \text{kg}\cdot\text{m}^2/\text{s}$$

$$= \boxed{3.8\times10^2\ \text{kg}\cdot\text{m}^2/\text{s}}$$

(*b*) The rate at which the angular momentum of the cylinder is increasing is given by:

$$\frac{dL}{dt}=\frac{\left(377\,\text{kg}\cdot\text{m}^2/\text{s}\right)}{25\,\text{s}}=15\,\text{kg}\cdot\text{m}^2/\text{s}^2$$
$$=\boxed{15\,\text{kg}\cdot\text{m}^2/\text{s}^2}$$

(*c*) Because the torque acting on the uniform cylinder is constant, the rate of change of the angular momentum is constant and hence the instantaneous rate of change of the angular momentum at any instant is equal to the average rate of change over the time during which the torque acts:

$$\tau=\frac{dL}{dt}=\boxed{15\,\text{kg}\cdot\text{m}^2/\text{s}^2}$$

(*d*) The magnitude of the frictional force *f* acting on the rim is:

$$f=\frac{\tau}{\ell}=\frac{15.1\,\text{kg}\cdot\text{m}^2/\text{s}^2}{0.40\,\text{m}}=\boxed{38\,\text{N}}$$

45 •• **[SSM]** In Figure 10-46, the incline is frictionless and the string passes through the center of mass of each block. The pulley has a moment of inertia *I* and radius *R*. (*a*) Find the net torque acting on the system (the two masses, string, and pulley) about the center of the pulley. (*b*)Write an expression for the total angular momentum of the system about the center of the pulley. Assume the masses are moving with a speed *v*. (*c*) Find the acceleration of the masses by using your results for Parts (*a*) and (*b*) and by setting the net torque equal to the rate of change of the system's angular momentum.

Picture the Problem Let the system include the pulley, string, and the blocks and assume that the mass of the string is negligible. The angular momentum of this system changes because a *net* torque acts on it.

(*a*) Express the net torque about the center of mass of the pulley:

$$\tau_{\text{net}}=Rm_2g\sin\theta-Rm_1g$$
$$=\boxed{Rg\left(m_2\sin\theta-m_1\right)}$$

where we have taken clockwise to be positive to be consistent with a positive upward velocity of the block whose mass is m_1 as indicated in the figure.

(*b*) Express the total angular momentum of the system about an axis through the center of the pulley:

$$L=I\omega+m_1vR+m_2vR$$
$$=\boxed{vR\left(\frac{I}{R^2}+m_1+m_2\right)}$$

(*c*) Express τ as the time derivative of the angular momentum:

$$\tau = \frac{dL}{dt} = \frac{d}{dt}\left[vR\left(\frac{I}{R^2} + m_1 + m_2\right)\right]$$
$$= aR\left(\frac{I}{R^2} + m_1 + m_2\right)$$

Equate this result to that of Part (*a*) and solve for *a* to obtain:

$$\boxed{a = \frac{g(m_2 \sin\theta - m_1)}{\frac{I}{R^2} + m_1 + m_2}}$$

46 •• Figure 10-47 shows the rear view of a space capsule that was left rotating rapidly about its longitudinal axis at 30 rev/min after a collision with another capsule. You are the flight controller and have just moments to tell the crew how to stop this rotation before they become ill from the rotation and the situation becomes dangerous. You know that they have access to two small jets mounted tangentially at a distance of 3.0 m from the axis, as indicated in the figure. These jets can each eject 10 g/s of gas with a nozzle speed of 800 m/s. Determine the length of time these jets must run to stop the rotation. In flight, the moment of inertia of the ship about its axis (assumed constant) is known to be 4000 kg·m^2.

Picture the Problem The forces resulting from the release of gas from the jets will exert a torque on the spaceship that will slow and eventually stop its rotation. We can relate this net torque to the angular momentum of the spaceship and to the time the jets must fire.

Relate the firing time of the jets to the desired change in angular momentum:

$$\Delta t = \frac{\Delta L}{\tau_{\text{net}}} = \frac{I\Delta\omega}{\tau_{\text{net}}} \qquad (1)$$

Express the magnitude of the net torque exerted by the jets:

$$\tau_{\text{net}} = 2FR$$

Letting $\Delta m/\Delta t'$ represent the mass of gas per unit time exhausted from the jets, relate the force exerted by the gas on the spaceship to the rate at which the gas escapes:

$$F = \frac{\Delta m}{\Delta t'}v$$

Substituting for *F* yields:

$$\tau_{\text{net}} = 2vR\frac{\Delta m}{\Delta t'}$$

Substitute for τ_{net} in equation (1) to obtain:

$$\Delta t = \frac{I\Delta\omega}{2vR\dfrac{\Delta m}{\Delta t'}}$$

Substitute numerical values and evaluate Δt:

$$\Delta \boldsymbol{t} = \frac{\left(4000\,\text{kg}\cdot\text{m}^2\right)\left(30\,\dfrac{\text{rev}}{\text{min}}\times\dfrac{2\boldsymbol{\pi}\,\text{rad}}{\text{rev}}\times\dfrac{1\,\text{min}}{60\,\text{s}}\right)}{2\left(10^{-2}\,\text{kg/s}\right)\left(800\,\text{m/s}\right)\left(3.0\,\text{m}\right)} = \boxed{2.6\times10^{2}\,\text{s}}$$

47 •• A projectile (mass M) is launched at an angle θ with an initial speed v_0. Considering the torque and angular momentum about the launch point, explicitly show that $dL/dt = \tau$. Ignore the effects of air resistance. (The equations for projectile motion are found in Chapter 3.)

Picture the Problem We can use constant-acceleration equations to express the projectile's position and velocity coordinates as functions of time. We can use these coordinates to express the particle's position and velocity vectors $\vec{r}$ and $\vec{v}$. Using its definition, we can express the projectile's angular momentum $\vec{L}$ as a function of time and then differentiate this expression to obtain $d\vec{L}/dt$. Finally, we can use the definition of the torque, relative to an origin located at the launch position, the gravitational force exerts on the projectile to express $\vec{\tau}$ and complete the demonstration that $d\vec{L}/dt = \vec{\tau}$.

Using its definition, express the angular momentum vector $\vec{L}$ of the projectile:

$$\vec{L} = \vec{r}\times m\vec{v} \qquad (1)$$

Using constant-acceleration equations, express the position coordinates of the projectile as a function of time:

$$x = v_{0x}t = \left(v_0\cos\theta\right)t$$

and

$$y = y_0 + v_{0y}t + \tfrac{1}{2}a_y t^2 = \left(v_0\sin\theta\right)t - \tfrac{1}{2}gt^2$$

Express the projectile's position vector $\vec{r}$:

$$\vec{r} = \left[\left(v_0\cos\theta\right)t\right]\hat{\boldsymbol{i}} + \left[\left(v_0\sin\theta\right)t - \tfrac{1}{2}gt^2\right]\hat{\boldsymbol{j}}$$

Using constant-acceleration equations, express the velocity of the projectile as a function of time:

$$v_x = v_{0x} = v_0\cos\theta$$

and

$$v_y = v_{0y} + a_y t = v_0\sin\theta - gt$$

Express the projectile's velocity vector $\vec{v}$:

$$\vec{v} = \left[v_0\cos\theta\right]\hat{\boldsymbol{i}} + \left[v_0\sin\theta - gt\right]\hat{\boldsymbol{j}}$$

Substituting in equation (1) and simplifying yields:

$$\vec{L} = \left\{\left[(V\cos\theta)t\right]\hat{i} + \left[(V\sin\theta)t - \tfrac{1}{2}gt^2\right]\hat{j}\right\} \times m\left\{\left[V\cos\theta\right]\hat{i} + \left[V\sin\theta - gt\right]\hat{j}\right\}$$
$$= \left(-\tfrac{1}{2}mgt^2 V\cos\theta\right)\hat{k}$$

Differentiate $\vec{L}$ with respect to t to obtain:

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}\left(-\tfrac{1}{2}mgt^2 V\cos\theta\right)\hat{k} = \left(-mgtV\cos\theta\right)\hat{k} \quad (2)$$

Using its definition, express the torque acting on the projectile:

$$\vec{\tau} = \vec{r} \times (-mg)\hat{j} = \left[(v_0\cos\theta)t\right]\hat{i} + \left[(v_0\sin\theta)t - \tfrac{1}{2}gt^2\right]\hat{j} \times (-mg)\hat{j} = \left(-mgtV\cos\theta\right)\hat{k} \quad (3)$$

Comparing equations (2) and (3) we see that:

$$\boxed{\frac{d\vec{L}}{dt} = \vec{\tau}}$$

Conservation of Angular Momentum

48 • A planet moves in an elliptical orbit about the sun with the sun at one focus of the ellipse as in Figure 10-48. (*a*) What is the torque about the center of the Sun due to the gravitational force of attraction of the Sun on the planet? (*b*) At position A, the planet has an orbital radius r_1 and is moving with a speed v_1 perpendicular to the line from the sun to the planet. At position B, the planet has an orbital radius r_2 and is moving with speed v_2, again perpendicular to the line from the sun to the planet. What is the ratio of v_1 to v_2 in terms of r_1 and r_2?

Picture the Problem Let m represent the mass of the planet and apply the definition of torque to find the torque produced by the gravitational force of attraction. We can use Newton's 2nd law of motion in the form $\vec{\tau} = d\vec{L}/dt$ to show that $\vec{L}$ is constant and apply conservation of angular momentum to the motion of the planet at points *A* and *B*.

(*a*) Express the torque produced by the gravitational force of attraction of the sun for the planet:

$\vec{\tau} = \vec{r} \times \vec{F} = \boxed{0}$ because $\vec{F}$ acts along the direction of $\vec{r}$.

(*b*) Because $\vec{\tau} = 0$:

$$\frac{d\vec{L}}{dt} = 0 \Rightarrow \vec{L} = \vec{r} \times m\vec{v} = \text{constant}$$

Noting that at points A and B $|\vec{r}\times\vec{v}| = rv$, express the relationship between the distances from the sun and the speeds of the planets:

$$r_1 v_1 = r_2 v_2 \Rightarrow \frac{v_1}{v_2} = \boxed{\frac{r_2}{r_1}}$$

49 •• **[SSM]** You stand on a frictionless platform that is rotating at an angular speed of 1.5 rev/s. Your arms are outstretched, and you hold a heavy weight in each hand. The moment of inertia of you, the extended weights, and the platform is 6.0 kg·m^2. When you pull the weights in toward your body, the moment of inertia decreases to 1.8 kg·m^2. (*a*) What is the resulting angular speed of the platform? (*b*) What is the change in kinetic energy of the system? (*c*) Where did this increase in energy come from?

Picture the Problem Let the system consist of you, the extended weights, and the platform. Because the net external torque acting on this system is zero, its angular momentum remains constant during the pulling in of the weights.

(*a*) Using conservation of angular momentum, relate the initial and final angular speeds of the system to its initial and final moments of inertia:

$$\boldsymbol{I}_\text{i}\boldsymbol{\omega}_\text{i} - \boldsymbol{I}_\text{f}\boldsymbol{\omega}_\text{f} = 0 \Rightarrow \omega_\text{f} = \frac{I_\text{i}}{I_\text{f}}\omega_\text{i}$$

Substitute numerical values and evaluate ω_f:

$$\boldsymbol{\omega}_\text{f} = \frac{6.0\,\text{kg}\cdot\text{m}^2}{1.8\,\text{kg}\cdot\text{m}^2}(1.5\,\text{rev/s}) = \boxed{5.0\,\text{rev/s}}$$

(*b*) Express the change in the kinetic energy of the system:

$$\Delta K = K_\text{f} - K_\text{i} = \tfrac{1}{2}I_\text{f}\omega_\text{f}^2 - \tfrac{1}{2}I_\text{i}\omega_\text{i}^2$$

Substitute numerical values and evaluate ΔK:

$$\Delta \boldsymbol{K} = \tfrac{1}{2}(1.8\,\text{kg}\cdot\text{m}^2)\left(5.0\frac{\text{rev}}{\text{s}}\times\frac{2\boldsymbol{\pi}\,\text{rad}}{\text{rev}}\right)^2 - \tfrac{1}{2}(6.0\,\text{kg}\cdot\text{m}^2)\left(1.5\frac{\text{rev}}{\text{s}}\times\frac{2\boldsymbol{\pi}\,\text{rad}}{\text{rev}}\right)^2$$
$$= \boxed{0.62\,\text{kJ}}$$

(*c*) Because no external agent does work on the system, the energy comes from your internal energy.

50 •• A small blob of putty of mass m falls from the ceiling and lands on the outer rim of a turntable of radius R and moment of inertia I_0 that is rotating freely with angular speed ω_0 about its vertical fixed-symmetry axis. (*a*) What is the post-collision angular speed of the turntable-putty system? (*b*) After several turns, the blob flies off the edge of the turntable. What is the angular speed of the turntable after the blob's departure?

Picture the Problem Let the system consist of the blob of putty and the turntable. Because the net external torque acting on this system is zero, its angular momentum remains constant when the blob of putty falls onto the turntable.

(*a*) Using conservation of angular momentum, relate the initial and final angular speeds of the turntable to its initial and final moments of inertia and solve for ω_f:

$$I_0\omega_0 - I_f\omega_f = 0 \Rightarrow \omega_f = \frac{I_0}{I_f}\omega_0 \quad (1)$$

Express the final rotational inertia of the turntable-plus-blob:

$$I_f = I_0 + I_{blob} = I_0 + mR^2$$

Substitute for I_f in equation (1) and simplify to obtain:

$$\omega_f = \frac{I_0}{I_0 + mR^2}\omega_0 = \boxed{\frac{1}{1+\dfrac{mR^2}{I_0}}\omega_0}$$

(*b*) If the blob flies off tangentially to the turntable, its angular momentum doesn't change (with respect to an axis through the center of turntable). Because there is no external torque acting on the blob-turntable system, the total angular momentum of the system will remain constant and the angular momentum of the turntable will not change. The turntable will continue to spin at $\boxed{\omega' = \omega_f}$.

51 •• **[SSM]** A Lazy Susan consists of a heavy plastic cylinder mounted on a frictionless bearing resting on a vertical shaft. The cylinder has a radius R = 15 cm and mass M = 0.25 kg. A cockroach (mass m = 0.015 kg) is on the Lazy Susan, at a distance of 8.0 cm from the center. Both the cockroach and the Lazy Susan are initially at rest. The cockroach then walks along a circular path concentric with the center of the Lazy Susan at a constant distance of 8.0 cm from the axis of the shaft. If the speed of the cockroach with respect to the Lazy Susan is 0.010 m/s, what is the speed of the cockroach with respect to the room?

Picture the Problem Because the net external torque acting on the Lazy Susan-cockroach system is zero, the net angular momentum of the system is constant (equal to zero because the Lazy Susan is initially at rest) and we can use conservation of angular momentum to find the angular velocity ω of the Lazy Susan. The speed of the cockroach relative to the floor v_f is the difference between its speed with respect to the Lazy Susan and the speed of the Lazy Susan at the location of the cockroach with respect to the floor.

Relate the speed of the cockroach with respect to the floor v_f to the speed of the Lazy Susan at the location of the cockroach:

$$v_f = v - \omega r \qquad (1)$$

Use conservation of angular momentum to obtain:

$$L_{LS} - L_C = 0 \qquad (2)$$

Express the angular momentum of the Lazy Susan:

$$L_{LS} = I_{LS}\omega = \tfrac{1}{2}MR^2\omega$$

Express the angular momentum of the cockroach:

$$L_C = I_C\omega_C = mr^2\left(\frac{v}{r} - \omega\right)$$

Substitute for L_{LS} and L_C in equation (2) to obtain:

$$\tfrac{1}{2}MR^2\omega - mr^2\left(\frac{v}{r} - \omega\right) = 0$$

Solving for ω yields:

$$\omega = \frac{2mrv}{MR^2 + 2mr^2}$$

Substitute for ω in equation (1) to obtain:

$$v_f = v - \frac{2mr^2v}{MR^2 + 2mr^2}$$

Substitute numerical values and evaluate v_f:

$$\boldsymbol{v}_f = 0.010\,\text{m/s} - \frac{2(0.015\,\text{kg})(0.080\,\text{m})^2(0.010\,\text{m/s})}{(0.25\,\text{m})(0.15\,\text{m})^2 + 2(0.015\,\text{kg})(0.080\,\text{m})^2} = \boxed{10\,\text{mm/s}}$$

52 •• Two disks of identical mass but different radii (r and $2r$) are spinning on frictionless bearings at the same angular speed ω_0 but in opposite directions (Figure 10-49). The two disks are brought slowly together. The resulting frictional force between the surfaces eventually brings them to a common angular velocity. (*a*) What is the magnitude of that final angular velocity in terms of ω_0? (*b*) What is the change in rotational kinetic energy of the system? Explain.

Picture the Problem The net external torque acting on this system is zero and so we know that angular momentum is conserved as these disks are brought together. Let the numeral 1 refer to the disk to the left and the numeral 2 to the disk to the right. Let the angular momentum of the disk with the larger radius be positive.

(*a*) Using conservation of angular momentum, relate the initial angular speeds of the disks to their common final speed and to their moments of inertia:

$$I_i\omega_i = I_f\omega_f$$

or

$$I_1\omega_0 - I_2\omega_0 = (I_1 + I_2)\omega_f$$

Solving for ω_f yields:

$$\omega_f = \frac{I_1 - I_2}{I_1 + I_2}\omega_0 \qquad (1)$$

Express I_1 and I_2:

$$I_1 = \tfrac{1}{2}m(2r)^2 = 2mr^2$$

and

$$I_2 = \tfrac{1}{2}mr^2$$

Substitute for I_1 and I_2 in equation (1) and simplify to obtain:

$$\boldsymbol{\omega}_f = \frac{2\boldsymbol{mr}^2 - \frac{1}{2}\boldsymbol{mr}^2}{2\boldsymbol{mr}^2 + \frac{1}{2}\boldsymbol{mr}^2}\boldsymbol{\omega}_0 = \boxed{\tfrac{3}{5}\boldsymbol{\omega}_0}$$

(*b*) The change in kinetic energy of the system is given by:

$$\Delta \boldsymbol{K} = \boldsymbol{K}_f - \boldsymbol{K}_i \qquad (2)$$

The initial kinetic energy of the system is the sum of the kinetic energies of the two disks:

$$\begin{aligned}\boldsymbol{K}_i &= \boldsymbol{K}_1 + \boldsymbol{K}_2 \\ &= \tfrac{1}{2}\boldsymbol{I}_1\boldsymbol{\omega}_0^2 + \tfrac{1}{2}\boldsymbol{I}_2\boldsymbol{\omega}_0^2 \\ &= \tfrac{1}{2}(\boldsymbol{I}_1 + \boldsymbol{I}_2)\boldsymbol{\omega}_0^2\end{aligned}$$

Substituting for K_f and K_i in equation (2) yields:

$$\Delta \boldsymbol{K} = \tfrac{1}{2}(\boldsymbol{I}_1 + \boldsymbol{I}_2)\boldsymbol{\omega}_f^2 - \tfrac{1}{2}(\boldsymbol{I}_1 + \boldsymbol{I}_2)\boldsymbol{\omega}_0^2$$

Substitute for ω_f from part (*a*) and simplify to obtain:

$$\begin{aligned}\Delta \boldsymbol{K} &= \tfrac{1}{2}(\boldsymbol{I}_1 + \boldsymbol{I}_2)\left(\tfrac{3}{5}\boldsymbol{\omega}_0\right)^2 - \tfrac{1}{2}(\boldsymbol{I}_1 + \boldsymbol{I}_2)\boldsymbol{\omega}_0^2 \\ &= -\tfrac{16}{25}\left[\tfrac{1}{2}(\boldsymbol{I}_1 + \boldsymbol{I}_2)\boldsymbol{\omega}_0^2\right]\end{aligned}$$

Noting that the quantity in brackets is K_i, substitute to obtain:

$$\Delta \boldsymbol{K} = \boxed{-\tfrac{16}{25}\boldsymbol{K}_i}$$

The frictional force between the surfaces is responsible for some of the initial kinetic energy being converted to thermal energy as the two disks come together.

53 •• A block of mass *m* sliding on a frictionless table is attached to a string that passes through a narrow hole through the center of the table. The block is sliding with speed v_0 in a circle of radius r_0. Find (*a*) the angular momentum of the block, (*b*) the kinetic energy of the block, and (*c*) the tension in the string.

(*d*) A student under the table now slowly pulls the string downward. How much work is required to reduce the radius of the circle from r_0 to $r_0/2$?

Picture the Problem (*a*) and (*b*) We can express the angular momentum and kinetic energy of the block directly from their definitions. (*c*) The tension in the string provides the centripetal force required for the uniform circular motion and can be expressed using Newton's 2nd law. (*d*) Finally, we can use the work-kinetic energy theorem to express the work required to reduce the radius of the circle by a factor of two.

(*a*) Express the initial angular momentum of the block:

$$L_0 = \boxed{r_0 m v_0}$$

(*b*) Express the initial kinetic energy of the block:

$$K_0 = \boxed{\tfrac{1}{2} m v_0^2}$$

(*c*) Using Newton's 2nd law, relate the tension in the string to the centripetal force required for the circular motion:

$$T = F_c = \boxed{m\frac{v_0^2}{r_0}}$$

(*d*) Use the work-kinetic energy theorem to relate the required work to the change in the kinetic energy of the block:

$$W = \Delta K = K_f - K_0 = \frac{L_f^2}{2I_f} - \frac{L_0^2}{2I_0}$$

$$= \frac{L_0^2}{2I_f} - \frac{L_0^2}{2I_0} = \frac{L_0^2}{2}\left(\frac{1}{I_f - I_0}\right)$$

$$= \frac{L_0^2}{2}\left(\frac{1}{m\left(\tfrac{1}{2}r_0\right)^2 - mr_0^2}\right) = -\frac{2}{3}\frac{L_0^2}{mr_0^2}$$

Substitute the result from Part (*a*) and simplify to obtain:

$$W = \boxed{-\tfrac{2}{3} m v_0^2}$$

54 ••• A 0.20-kg point mass moving on a frictionless horizontal surface is attached to a rubber band whose other end is fixed at point *P*. The rubber band exerts a force whose magnitude is $F = bx$, where x is the length of the rubber band and b is an unknown constant. The rubber band force points inward towards *P*. The mass moves along the dotted line in Figure 10-50. When it passes point *A*, its velocity is 4.0 m/s, directed as shown. The distance *AP* is 0.60 m and *BP* is 1.0 m. (*a*) Find the speed of the mass at points *B* and *C*. (*b*) Find *b*.

Picture the Problem Because the force exerted by the rubber band is parallel to the position vector of the point mass, the net external torque acting on it is zero and we can use the conservation of angular momentum to determine the speeds of the ball at points *B* and *C*. We'll use mechanical energy conservation to find *b* by relating the kinetic and elastic potential energies at *A* and *B*.

(*a*) Use conservation of momentum to relate the angular momenta at points *A*, *B* and *C*:

$$L_A = L_B = L_C$$

or

$$mv_A r_A = mv_B r_B = mv_C r_C \qquad (1)$$

Solve for v_B in terms of v_A:

$$v_B = v_A \frac{r_A}{r_B}$$

Substitute numerical values and evaluate v_B:

$$\boldsymbol{v_B} = (4.0\,\text{m/s})\frac{0.60\,\text{m}}{1.0\,\text{m}} = \boxed{2.4\,\text{m/s}}$$

Solve equation (1) for v_C in terms of v_A:

$$v_C = v_A \frac{r_A}{r_C}$$

Substitute numerical values and evaluate v_C:

$$\boldsymbol{v_C} = (4.0\,\text{m/s})\frac{0.60\,\text{m}}{0.60\,\text{m}} = \boxed{4.0\,\text{m/s}}$$

(*b*) Use conservation of mechanical energy between points *A* and *B* to relate the kinetic energy of the point mass and the energy stored in the stretched rubber band:

$$\Delta E = E_A - E_B = 0$$

or

$$\tfrac{1}{2}mv_A^2 + \tfrac{1}{2}br_A^2 - \tfrac{1}{2}mv_B^2 - \tfrac{1}{2}br_B^2 = 0$$

Solving for *b* yields:

$$b = \frac{m\left(v_B^2 - v_A^2\right)}{r_A^2 - r_B^2}$$

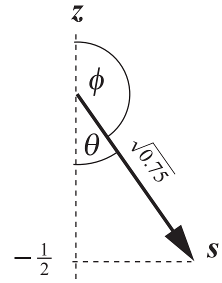
Substitute numerical values and evaluate *b*:

$$b = \frac{(0.20\,\text{kg})\left[(2.4\,\text{m/s})^2 - (4.0\,\text{m/s})^2\right]}{(0.60\,\text{m})^2 - (1.0\,\text{m})^2} = \boxed{3\,\text{N/m}}$$

*Quantization of Angular Momentum

55 •• **[SSM]** The *z* component of the spin of an electron is $-\frac{1}{2}\hbar$, but the magnitude of the spin vector is $\sqrt{0.75}\hbar$. What is the angle between the electron's spin angular momentum vector and the positive *z*-axis?

Picture the Problem The electron's spin angular momentum vector is related to its z component as shown in the diagram. The angle between $\vec{s}$ and the positive z-axis is ϕ.



Express ϕ in terms of θ to obtain:

$$\phi = 180° - \theta$$

Using trigonometry, relate the magnitude of $\vec{s}$ to its $-z$ component:

$$\theta = \cos^{-1}\left(\frac{\frac{1}{2}\hbar}{\sqrt{0.75}\hbar}\right)$$

Substitute for θ in the expression for ϕ to obtain:

$$\theta = 180° - \cos^{-1}\left(\frac{\frac{1}{2}\hbar}{\sqrt{0.75}\hbar}\right) = \boxed{125°}$$

56 •• Show that the energy difference between one rotational state of a molecule and the next higher state is proportional to $\ell + 1$.

Picture the Problem Equation 10-29*a* describes the quantization of rotational energy. We can show that the energy difference between a given state and the next higher state is proportional to $\ell + 1$ by using Equation 10-27*a* to express the energy difference.

From Equation 10-29*a* we have:

$$K_\ell = \ell(\ell + 1)E_{0r}$$

Using this equation, express the difference between one rotational state and the next higher state:

$$\Delta E = (\ell + 1)(\ell + 2)E_{0r} - \ell(\ell + 1)E_{0r} = \boxed{2(\ell + 1)E_{0r}}$$

57 •• **[SSM]** You work in a bio-chemical research lab, where you are investigating the rotational energy levels of the HBr molecule. After consulting the periodic chart, you know that the mass of the bromine atom is 80 times that of the hydrogen atom. Consequently, in calculating the rotational motion of the molecule, you assume, to a good approximation, that the Br nucleus remains stationary as the H atom (mass 1.67×10^{-27} kg) revolves around it. You also know that the separation between the H atom and bromine nucleus is 0.144 nm. Calculate (*a*) the moment of inertia of the HBr molecule about the bromine nucleus, and (*b*) the rotational energies for the bromine nucleus's *ground state* (lowest energy) $\ell = 0$, and the next two states of higher energy (called the first and second *excited states*) described by $\ell = 1$, and $\ell = 2$.

Picture the Problem The rotational energies of HBr molecule are related to ℓ and E_{0r} according to $K_\ell = \ell(\ell+1)E_{0r}$ where $E_{0r} = \hbar^2/2I$.

(*a*) Neglecting the motion of the bromine molecule:

$$\boldsymbol{I}_{HBr} \approx \boldsymbol{m}_p\boldsymbol{r}^2 = \boldsymbol{m}_H\boldsymbol{r}^2$$

Substitute numerical values and evaluate I_{HBr}:

$$\boldsymbol{I}_{HBr} \approx (1.67\times10^{-27}\text{ kg})(0.144\times10^{-9}\text{ m})^2 = 3.463\times10^{-47}\text{ kg}\cdot\text{m}^2 = \boxed{3.46\times10^{-47}\text{ kg}\cdot\text{m}^2}$$

(*b*) Relate the rotational energies to ℓ and E_{0r}:

$$K_\ell = \ell(\ell+1)E_{0r} \text{ where } \boldsymbol{E}_{0r} = \frac{\hbar^2}{2\boldsymbol{I}_{HBr}}$$

Substitute numerical values and evaluate E_{0r}:

$$\boldsymbol{E}_{0r} = \frac{\hbar^2}{2\boldsymbol{I}} = \frac{(1.055\times10^{-34}\text{ J}\cdot\text{s})^2}{2(3.463\times10^{-47}\text{ kg}\cdot\text{m}^2)} = 1.607\times10^{-22}\text{ J}\times\frac{1\text{ eV}}{1.602\times10^{-19}\text{ J}} = 1.003\text{ meV}$$

Evaluate E_0 to obtain:

$$\boldsymbol{E}_0 = \boldsymbol{K}_0 = \boxed{1.00\text{ meV}}$$

Evaluate E_1 to obtain:

$$\boldsymbol{E}_1 = \boldsymbol{K}_1 = (1+1)(1.003\text{ meV}) = \boxed{2.01\text{ meV}}$$

Evaluate E_2 to obtain:

$$\boldsymbol{E}_2 = \boldsymbol{K}_2 = 2(2+1)(1.003\text{ meV}) = \boxed{6.02\text{ meV}}$$

58 ••• The equilibrium separation between the nuclei of the nitrogen molecule (N_2, consisting of two nitrogen atoms) is 0.110 nm and the mass of each nitrogen nucleus is 14.0 u, where u = 1.66×10^{-27} kg. For rotational energies, the total energy is due to rotational kinetic energy. (*a*) Approximate the nitrogen molecule as a rigid dumbbell of two equal point masses and calculate the moment of inertia about its center of mass. (*b*) Find the energy E_ℓ of the lowest three energy levels using $E_\ell = K_\ell = \ell(\ell+1)\hbar^2/(2I)$. (*c*) Molecules emit a particle (or *quantum*) of light called a *photon* when they make a *transition* from a higher energy state to a lower one. Determine the energy of a photon emitted when a nitrogen molecule drops from the $\ell = 2$ to the $\ell = 1$ state. Visible light photons each have between about 2 and 3 eV of energy. Are these photons in the visible region?

Picture the Problem We can use the definition of the moment of inertia of point particles to calculate the rotational inertia of the nitrogen molecule. The rotational energies of nitrogen molecule are related to ℓ and E_{0r} according to $E_\ell = K_\ell = \ell(\ell+1)E_{0r}$ where $E_{0r} = \hbar^2/2I$.

(*a*) Using a rigid dumbbell model, express and evaluate the moment of inertia of the nitrogen molecule about its center of mass:

$$I_{N_2} = \sum_i m_i r_i^2 = m_N r^2 + m_N r^2 = 2m_N r^2$$

Substitute numerical values and evaluate *I*:

$$I_{N_2} = 2(14)(1.66\times10^{-27}\text{ kg})\left(\frac{0.110\text{ nm}}{2}\right)^2 = 1.406\times10^{-46}\text{ kg}\cdot\text{m}^2 = \boxed{1.41\times10^{-46}\text{ kg}\cdot\text{m}^2}$$

(*b*) Relate the rotational energies to ℓ and E_{0r}:

$$E_\ell = K_\ell = \ell(\ell+1)E_{0r}$$

where

$$E_{0r} = \frac{\hbar^2}{2I_{N_2}}$$

Substitute numerical values and evaluate E_{0r}:

$$E_{0r} = \frac{(1.055\times10^{-34}\text{ J}\cdot\text{s})^2}{2(1.406\times10^{-46}\text{ kg}\cdot\text{m}^2)} = 3.958\times10^{-23}\text{ J}\times\frac{1\text{eV}}{1.60\times10^{-19}\text{ J}} = 0.2474\text{ meV}$$

Evaluate E_0 to obtain:

$$E_0 = \boxed{0.247\text{ meV}}$$

Evaluate E_1 to obtain:

$$E_1 = (1+1)(0.2474\text{ meV}) = \boxed{0.495\text{ meV}}$$

Evaluate E_2 to obtain:

$$E_2 = 2(2+1)(0.2474\text{ meV}) = \boxed{1.48\text{ meV}}$$

(*c*) The energy of a photon emitted when a nitrogen molecule drops from the $\ell = 2$ to the $\ell = 1$ state is:

$$\Delta E_{\ell=2\to\ell=1} = E_2 - E_1 = 1.48\ \text{meV} - 0.495\ \text{meV} = \boxed{0.99\ \text{meV}}$$

No. This energy is too low to produce radiation in the visible portion of the spectrum.

59 ••• Consider a *transition* from a lower energy state to a higher one. That is, the absorption of a quantum of energy resulting in an increase in the rotational energy of an N_2 molecule (see Problem 64). Suppose such a molecule, initially in its ground rotational state, was exposed to photons each with energy equal to the three times the energy of its first excited state. (*a*) Would the molecule be able to absorb this photon energy? Explain why or why not and if it can, determine the energy level to which it goes. (*b*) To make a transition from its ground state to its second excited state requires how many times the energy of the first excited state?

Picture the Problem The rotational energies of a nitrogen molecule depend on the quantum number ℓ according to $E_\ell = L^2/2I = \ell(\ell+1)\hbar^2/2I$.

(*a*) No. None of the allowed values of E_ℓ are equal to $3E_{0\mathrm{r}}$.

(*b*) The upward transition from the ground state to the second excited state requires energy given by:

$$\Delta E_{\ell=0\to\ell=2} = E_2 - E_0$$

Set this energy difference equal to a constant *n* times the energy of the 1st excited state:

$$E_2 - E_0 = nE_1 \Rightarrow n = \frac{E_2 - E_0}{E_1}$$

Substitute numerical values and evaluate *n*:

$$n = \frac{2(2+1)E_{0\mathrm{r}} - E_{0\mathrm{r}}}{(1+1)E_{0\mathrm{r}}} = \boxed{2.5}$$

Collisions with Rotations

60 •• A 16.0-kg, 2.40-m-long rod is supported on a knife edge at its midpoint. A 3.20-kg ball of clay is dropped from rest from a height of 1.20 m and makes a perfectly inelastic collision with the rod 0.90 m from the point of support (Figure 10-51). Find the angular momentum of the rod and clay system about the point of support immediately after the inelastic collision.

Picture the Problem Let the zero of gravitational potential energy be at the elevation of the rod. Because the net external torque acting on this system is zero, we know that angular momentum is conserved in the collision. We'll use the definition of angular momentum to express the angular momentum just after the collision and conservation of mechanical energy to determine the speed of the ball just before it makes its perfectly inelastic collision with the rod.

Use conservation of angular momentum to relate the angular momentum before the collision to the angular momentum just after the perfectly inelastic collision:

$$L_f = L_i = mvr \quad (1)$$

Use conservation of mechanical energy to relate the kinetic energy of the ball just before impact to its initial potential energy:

$$K_f - K_i + U_f - U_i = 0$$

or, because $K_i = U_f = 0$,

$$K_f - U_i = 0$$

Letting h represent the distance the ball falls, substitute for K_f and U_i to obtain:

$$\tfrac{1}{2}mv^2 - mgh = 0 \Rightarrow v = \sqrt{2gh}$$

Substituting for v in equation (1) yields:

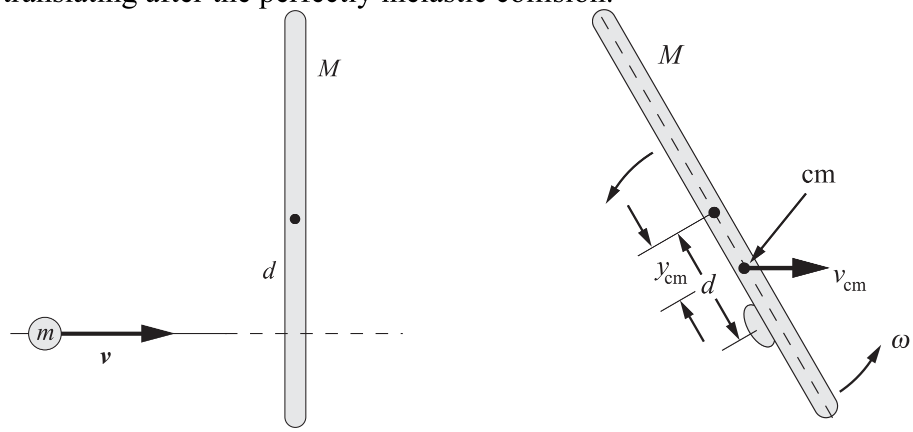
$$L_f = mr\sqrt{2gh}$$

Substitute numerical values and evaluate L_f:

$$\boldsymbol{L}_f = (3.20\,\text{kg})(0.90\,\text{m})\sqrt{2(9.81\,\text{m/s}^2)(1.20\,\text{m})} = \boxed{14\,\text{J}\cdot\text{s}}$$

61 •• **[SSM]** Figure 10-52 shows a thin uniform bar of length L and mass M and a small blob of putty of mass m. The system is supported by a frictionless horizontal surface. The putty moves to the right with velocity $\vec{v}$, strikes the bar at a distance d from the center of the bar, and sticks to the bar at the point of contact. Obtain expressions for the velocity of the system's center of mass and for the angular speed following the collision.

Picture the Problem The velocity of the center of mass of the bar-blob system does not change during the collision and so we can calculate it before the collision using its definition. Because there are no external forces or torques acting on the bar-blob system, both linear and angular momentum are conserved in the collision. Let the direction the blob of putty is moving initially be the $+x$ direction. Let lower-case letters refer to the blob of putty and upper-case letters refer to the bar. The diagram to the left shows the blob of putty approaching the bar and the diagram to the right shows the bar-blob system rotating about its center of mass and translating after the perfectly inelastic collision.



The velocity of the center of mass before the collision is given by:

$$(M+m)\vec{v}_{cm} = m\vec{v} + M\vec{V}$$

or, because $\vec{V} = 0$,

$$\vec{v}_{cm} = \boxed{\frac{m}{M+m}\vec{v}}$$

Using its definition, express the location of the center of mass relative to the center of the bar:

$$(M+m)y_{cm} = md \Rightarrow y_{cm} = \frac{md}{M+m}$$

below the center of the bar.

Express the angular momentum, relative to the center of mass, of the bar-blob system:

$$L_{cm} = I_{cm}\omega \Rightarrow \omega = \frac{L_{cm}}{I_{cm}} \qquad (1)$$

Express the angular momentum about the center of mass:

$$L_{cm} = mv(d - y_{cm}) = mv\left(d - \frac{md}{M+m}\right) = \frac{mMvd}{M+m}$$

Using the parallel axis theorem, express the moment of inertia of the system relative to its center of mass:

$$I_{cm} = \tfrac{1}{12}ML^2 + My_{cm}^2 + m(d - y_{cm})^2$$

Substitute for y_{cm} and simplify to obtain:

$$I_{cm} = \tfrac{1}{12}ML^2 + M\left(\frac{md}{M+m}\right)^2 + m\left(d - \frac{md}{M+m}\right)^2 = \tfrac{1}{12}ML^2 + \frac{mMd^2}{M+m}$$

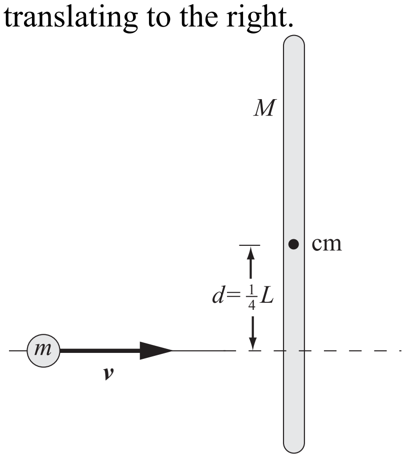
Substitute for I_{cm} and L_{cm} in equation (1) and simplify to obtain:

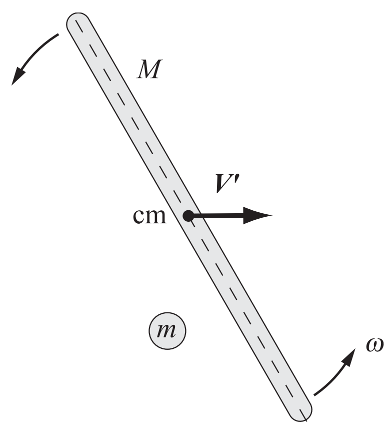
$$\omega = \boxed{\frac{mMvd}{\frac{1}{12}ML^2(M+m)+Mmd^2}}$$

Remarks: You can verify the expression for I_{cm} by letting $m \to 0$ to obtain $I_{cm} = \frac{1}{12}ML^2$ and letting $M \to 0$ to obtain $I_{cm} = 0$.

62 •• Figure 10-52 shows a thin uniform bar whose length is L and mass is M and a compact hard sphere whose mass is m. The system is supported by a frictionless horizontal surface. The sphere moves to the right with velocity $\vec{v}$, strikes the bar at a distance $\frac{1}{4}L$ from the center of the bar. The collision is elastic, and following the collision the sphere is at rest. Find the value of the ratio m/M.

Picture the Problem Because there are no external forces or torques acting on the bar-sphere system, both linear and angular momentum are conserved in the collision. Kinetic energy is also conserved in the elastic collision of the hard sphere with the bar. Let the direction the sphere is moving initially be the +x direction Let lower-case letters refer to the compact hard sphere and upper-case characters refer to the bar. Let unprimed characters refer to before the collision and primed characters to after the collision. The diagram to the left shows the path of the sphere before its collision with the bar and the diagram to the right shows the sphere at rest after the collision and the bar rotating about its center of mass and translating to the right.





Apply conservation of linear momentum to the collision to obtain:

$$mv = 0 + MV' \Rightarrow V' = \frac{m}{M}v \qquad (1)$$

Apply conservation of angular momentum to the collision to obtain:

$$mvd = 0 + I_{cm}\omega \qquad (2)$$

Apply conservation of mechanical energy to the elastic collision to obtain:

$$\tfrac{1}{2}mv^2 = 0 + \tfrac{1}{2}MV'^2 + \tfrac{1}{2}I_{\text{cm}}\omega^2 \quad (3)$$

Use Table 9-1 to find the moment of inertia of a thin bar about an axis through its center:

$$I_{\text{cm}} = \tfrac{1}{12}ML^2$$

Substitute for I_{cm} in equation (2) and simplify to obtain:

$$mvd = \tfrac{1}{12}ML^2\omega \Rightarrow \omega = \left(\frac{12vd}{L^2}\right)\frac{m}{M}$$

Substitute for I_{cm} and V' in equation (3) and simplify to obtain:

$$mv^2 = M\left(\frac{m}{M}\right)^2 v^2 + \tfrac{1}{12}ML^2\omega^2$$

Substituting for ω yields:

$$mv^2 = M\left(\frac{m}{M}\right)^2 v^2 + \tfrac{1}{12}ML^2\left[\left(\frac{12vd}{L^2}\right)\frac{m}{M}\right]^2$$

Solve this equation for $\frac{m}{M}$ to obtain:

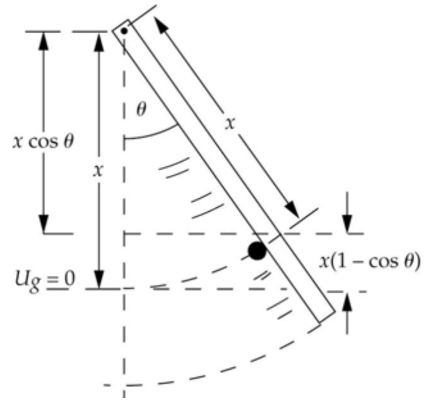
$$\frac{m}{M} = \frac{1}{1+12\left(\dfrac{d}{L}\right)^2}$$

Because $d = L/4$:

$$\frac{m}{M} = \frac{1}{1+12\left(\dfrac{1}{4}\right)^2} = \boxed{\frac{4}{7}}$$

63 •• Figure 10-53 shows a uniform rod whose length is L and whose mass is M pivoted at the top. The rod, which is initially at rest, is struck by a particle whose mass is m at a point $x = 0.8L$ below the pivot. Assume that the particle sticks to the rod. What must be the speed v of the particle so that following the collision the maximum angle between the rod and the vertical is 90°?

Picture the Problem Let the zero of gravitational potential energy be a distance x below the pivot and ignore friction between the rod and the pivot. Because the net external torque acting on the system is zero, angular momentum is conserved in this perfectly inelastic collision. We can also use conservation of mechanical energy to relate the initial kinetic energy of the system after the collision to its potential energy at the top of its swing.



Using conservation of mechanical energy, relate the rotational kinetic energy of the system just after the collision to its gravitational potential energy when it has swung through an angle θ:	$\Delta K + \Delta U = 0$ or, because $K_f = U_i = 0$, $-K_i + U_f = 0$
Substitute for K_i and U_f to obtain:	$-\frac{1}{2}I\omega^2 + \left(Mg\frac{L}{2} + mgx\right)(1-\cos\theta) = 0$ (1)
Apply conservation of momentum to the collision:	$\Delta L = L_f - L_i = 0$ or $\left[\frac{1}{3}ML^2 + (0.8L)^2 m\right]\omega - 0.8Lmv = 0$
Solving for ω yields:	$\omega = \frac{0.8Lmv}{\frac{1}{3}ML^2 + 0.64mL^2}$ (2)
Express the moment of inertia of the system about the pivot:	$I = m(0.8L)^2 + \frac{1}{3}ML^2 = 0.64mL^2 + \frac{1}{3}ML^2$ (3)

Substitute equations (2) and (3) in equation (1) and simplify to obtain:

$$\frac{0.32(Lmv)^2}{\frac{1}{3}ML^2 + 0.64mL^2} - \left(Mg\frac{L}{2} + mg(0.8L)\right)(1-\cos\theta) = 0$$

Solving for v yields:

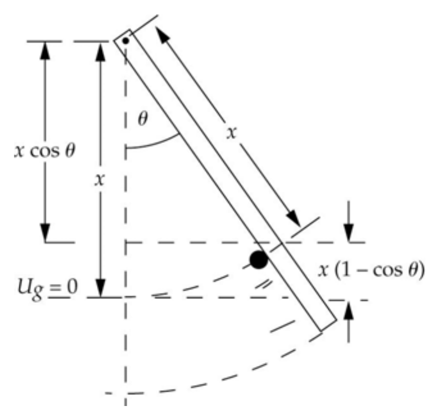
$$v=\sqrt{\frac{(0.5M+0.8m)\left(\frac{1}{3}ML^2+0.64mL^2\right)g(1-\cos\theta)}{0.32Lm^2}}$$

Evaluate v for $\theta = 90°$ to obtain:

$$\boxed{v=\sqrt{\frac{(0.5M+0.8m)\left(\frac{1}{3}ML^2+0.64mL^2\right)g}{0.32Lm^2}}}$$

64 •• If, for the system of Problem 69, L = 1.2 m, M = 0.80 kg, m = 0.30 kg, and the maximum angle between the rod and the vertical following the collision is 60°, find the speed of the particle before impact.

Picture the Problem Let the zero of gravitational potential energy be a distance x below the pivot and ignore friction between the rod and pivot. Because the net external torque acting on the system is zero, angular momentum is conserved in this perfectly inelastic collision. We can also use conservation of mechanical energy to relate the initial kinetic energy of the system after the collision to its potential energy at the top of its swing.



Using conservation of mechanical energy, relate the rotational kinetic energy of the system just after the collision to its gravitational potential energy when it has swung through an angle θ:

$$K_f - K_i + U_f - U_i = 0$$

or, because $K_f = U_i = 0$,

$$-K_i + U_f = 0$$

Substitute for K_i and U_f to obtain:

$$-\tfrac{1}{2}I\omega^2+\left(Mg\frac{L}{2}+mgx\right)(1-\cos\theta)=0 \qquad (1)$$

Apply conservation of momentum to the collision:

$$\Delta L = L_f - L_i = 0$$

or

$$\left(\tfrac{1}{3}ML^2 + (0.80L)^2 m\right)\omega - 0.80Lmv = 0$$

Solving for ω yields:

$$\omega = \frac{0.80Lmv}{\tfrac{1}{2}ML^2 + 0.64mL^2} \qquad (2)$$

The moment of inertia of the system about the pivot is:

$$I = m(0.80L)^2 + \tfrac{1}{3}ML^2 = \left(0.64m + \tfrac{1}{3}M\right)L^2$$

Substitute numerical values and evaluate I:

$$I = \left[0.64(0.30\,\text{kg}) + \tfrac{1}{3}(0.80\,\text{kg})\right](1.2\,\text{m})^2 = 0.660\,\text{kg}\cdot\text{m}^2$$

Substitute equation (2) in equation (1) and simplify to obtain:

$$-\frac{0.32(Lmv)^2}{I} + \left(Mg\frac{L}{2} + 0.80Lmg\right)(1-\cos\theta) = 0$$

Solving for v yields:

$$v = \sqrt{\frac{g(0.50M + 0.80m)(1-\cos\theta)I}{0.32Lm^2}}$$

Substitute numerical values and evaluate v for $\theta = 60°$ to obtain:

$$v = \sqrt{\frac{(9.81\,\text{m/s}^2)\left[0.50(0.80\,\text{kg}) + (0.80)(0.30\,\text{kg})\right](0.50)(0.660\,\text{kg}\cdot\text{m}^2)}{0.32(1.2\,\text{m})(0.30\,\text{kg})^2}} = \boxed{7.7\,\text{m/s}}$$

65 •• A uniform rod is resting on a frictionless table when it is suddenly struck at one end by a sharp horizontal blow in a direction perpendicular to the rod. The mass of the rod is M and the magnitude of the impulse applied by the blow is J. Immediately after the rod is struck, (*a*) what is the velocity of the center of mass of the rod, (*b*) what is the velocity of the end that is struck, (*c*) and what is the velocity of the other end of the rod? (*d*) Is there a point on the rod that remains motionless?

Picture the Problem Let the length of the uniform stick be ℓ. We can use the impulse-change in momentum theorem to express the velocity of the center of mass of the stick. By expressing the velocity V of the end of the stick in terms of the velocity of the center of mass and applying the angular impulse-change in angular momentum theorem we can find the angular velocity of the stick and, hence, the velocity of the end of the stick.

(*a*) Apply the impulse-change in momentum theorem to obtain:

$$J = \Delta p = p - p_0 = p$$

or, because $p_0 = 0$ and $p = Mv_{cm}$,

$$J = Mv_{cm} \Rightarrow v_{cm} = \boxed{\frac{J}{M}}$$

(*b*) Relate the velocity V of the end of the stick to the velocity of the center of mass v_{cm}:

$$V = v_{cm} + v_{rel\ to\ cm} = v_{cm} + \omega\left(\tfrac{1}{2}\ell\right) \quad (1)$$

Relate the angular impulse to the change in the angular momentum of the stick:

$$J\left(\tfrac{1}{2}\ell\right) = \Delta L = L - L_0 = I_{cm}\omega$$

or, because $L_0 = 0$,

$$J\left(\tfrac{1}{2}\ell\right) = I_{cm}\omega \quad (2)$$

Refer to Table 9-1 to find the moment of inertia of the stick with respect to its center of mass:

$$I_{cm} = \tfrac{1}{12}M\ell^2$$

Substitute for I_{cm} in equation (2) to obtain:

$$J\left(\tfrac{1}{2}\ell\right) = \tfrac{1}{12}M\ell^2\omega \Rightarrow \omega = \frac{6J}{M\ell}$$

Substituting for ω in equation (1) yields:

$$V = \frac{J}{M} + \left(\frac{6J}{M\ell}\right)\frac{\ell}{2} = \boxed{\frac{4J}{M}}$$

(*c*) Relate the velocity V' of the other end of the stick to the velocity of the center of mass v_{cm}:

$$V = v_{cm} - v_{rel\ to\ cm} = v_{cm} - \omega\left(\tfrac{1}{2}\ell\right)$$

$$= \frac{J}{M} - \left(\frac{6J}{M\ell}\right)\frac{\ell}{2} = \boxed{-\frac{2J}{M}}$$

(*d*) Yes, one point remains motionless, but only for a very brief time.

66 •• A projectile of mass m_p is traveling at a constant velocity $\vec{v}_0$ toward a stationary disk of mass M and radius R that is free to rotate about its axis O (Figure 10-54). Before impact, the projectile is traveling along a line displaced a distance b below the axis. The projectile strikes the disk and sticks to point B. Model the projectile as a point mass. (*a*) Before impact, what is the total angular momentum L_0 of the disk-projectile system about the axis? Answer the following questions in terms of the symbols given at the start of this problem. (*b*) What is the angular speed ω of the disk-projectile system just after the impact? (*c*) What is the kinetic energy of the disk-projectile system after impact? (*d*) How much mechanical energy is lost in this collision?

Picture the Problem Because the net external torque acting on the system is zero, angular momentum is conserved in this perfectly inelastic collision.

(*a*) Use its definition to express the total angular momentum of the disk and projectile just before impact:

$$L_0 = \boxed{m_p v_0 b}$$

(*b*) Use conservation of angular momentum to relate the angular momenta just before and just after the collision:

$$L_0 = L = I\omega \Rightarrow \omega = \frac{L_0}{I}$$

The moment of inertia of the disk-projectile after the impact is:

$$I = \tfrac{1}{2}MR^2 + m_p R^2 = \tfrac{1}{2}(M + 2m_p)R^2$$

Substitute for I in the expression for ω to obtain:

$$\omega = \boxed{\frac{2m_p v_0 b}{(M + 2m_p)R^2}}$$

(*c*) Express the kinetic energy of the system after impact in terms of its angular momentum:

$$K_f = \frac{L^2}{2I} = \frac{(m_p v_0 b)^2}{2\left[\tfrac{1}{2}(M + 2m_p)R^2\right]}$$

$$= \boxed{\frac{(m_p v_0 b)^2}{(M + 2m_p)R^2}}$$

(*d*) Express the difference between the initial and final kinetic energies, substitute, and simplify to obtain:

$$\Delta \boldsymbol{E} = \boldsymbol{K}_i - \boldsymbol{K}_f$$

$$= \tfrac{1}{2}\boldsymbol{m}_p \boldsymbol{v}_0^2 - \frac{(\boldsymbol{m}_p \boldsymbol{v}_0 \boldsymbol{b})^2}{(\boldsymbol{M} + 2\boldsymbol{m}_p)\boldsymbol{R}^2}$$

$$= \boxed{\tfrac{1}{2}\boldsymbol{m}_p \boldsymbol{v}_0^2\left[1 - \frac{2\boldsymbol{m}_p \boldsymbol{b}^2}{(\boldsymbol{M} + 2\boldsymbol{m}_p)\boldsymbol{R}^2}\right]}$$

67 •• **[SSM]** A uniform rod of length L_1 and mass M equal to 0.75 kg is supported by a hinge of negligible mass at one end and is free to rotate in the vertical plane (Figure 10-55). The rod is released from rest in the position shown. A particle of mass m = 0.50 kg is supported by a thin string of length L_2 from the hinge. The particle sticks to the rod on contact. What should be the ratio L_2/L_1 so that θ_{max} = 60° after the collision?

Picture the Problem Assume that there is no friction between the rod and the hinge. Because the net external torque acting on the system is zero, angular momentum is conserved in this perfectly inelastic collision. The rod, on its downward swing, acquires rotational kinetic energy. Angular momentum is conserved in the perfectly inelastic collision with the particle and the rotational kinetic of the after-collision system is then transformed into gravitational potential energy as the rod-plus-particle swing upward. Let the zero of gravitational potential energy be at a distance L_1 below the pivot and use both angular momentum and mechanical energy conservation to relate the distances L_1 and L_2 and the masses M and m.

Use conservation of energy to relate the initial and final potential energy of the rod to its rotational kinetic energy just before it collides with the particle:

$$K_f - K_i + U_f - U_i = 0$$

or, because $K_i = 0$,

$$K_f + U_f - U_i = 0$$

Substitute for K_f, U_f, and U_i to obtain:

$$\tfrac{1}{2}\left(\tfrac{1}{3}ML_1^2\right)\omega^2 + Mg\frac{L_1}{2} - MgL_1 = 0$$

Solving for ω yields:

$$\omega = \sqrt{\frac{3g}{L_1}}$$

Letting ω' represent the angular speed of the rod-and-particle system just after impact, use conservation of angular momentum to relate the angular momenta before and after the collision:

$$\Delta \boldsymbol{L} = \boldsymbol{L}_f - \boldsymbol{L}_i = 0$$

or

$$\left(\tfrac{1}{3}\boldsymbol{ML_1^2} + \boldsymbol{mL_2^2}\right)\boldsymbol{\omega}' - \left(\tfrac{1}{3}\boldsymbol{ML_1^2}\right)\boldsymbol{\omega} = 0$$

Solve for ω' to obtain:

$$\omega' = \frac{\tfrac{1}{3}ML_1^2}{\tfrac{1}{3}ML_1^2 + mL_2^2}\,\omega$$

Use conservation of energy to relate the rotational kinetic energy of the rod-plus-particle just after their collision to their potential energy when they have swung through an angle θ_{max}:

$$K_f - K_i + U_f - U_i = 0$$

Because $K_f = 0$:

$$-\tfrac{1}{2}I\omega'^2 + Mg\left(\tfrac{1}{2}L_1\right)\left(1-\cos\theta_{max}\right) + mgL_2\left(1-\cos\theta_{max}\right) = 0 \qquad (1)$$

Express the moment of inertia of the system with respect to the pivot:

$$I = \tfrac{1}{3}ML_1^2 + mL_2^2$$

Substitute for θ_{max}, I and ω' in equation (1):

$$\frac{3\frac{g}{L_1}\left(\frac{1}{3}ML_1^2\right)^2}{\frac{1}{3}ML_1^2 + mL_2^2} = Mg\left(\tfrac{1}{2}L_1\right) + mgL_2$$

Simplify to obtain:

$$L_1^3 = 2\frac{m}{M}L_1^2L_2 + 3L_2^2L_1 + 6\frac{m}{M}L_2^3$$

Let $\alpha = m/M$ and $\beta = L_2/L_1$ to obtain:

$$6\alpha^2\beta^3 + 3\beta^2 + 2\alpha\beta - 1 = 0$$

Substitute for α and simplify to obtain the cubic equation in β:

$$8\boldsymbol{\beta}^3 + 9\boldsymbol{\beta}^2 + 4\boldsymbol{\beta} - 3 = 0$$

Use the solver function* of your calculator to find the only real value of β:

$$\boldsymbol{\beta} = \boxed{0.36}$$

Remarks: **Most graphing calculators have a ″solver″ feature. One can solve the cubic equation using either the ″graph″ and ″trace″ capabilities or the ″solver″ feature. The root given above was found using SOLVER on a TI-85.**

68 •• A uniform rod that has a length L_1 equal to 1.2 m and a mass M equal to 2.0 kg is supported by a hinge at one end and is free to rotate in the vertical plane (Figure 10-55). The rod is released from rest in the position shown. A particle whose mass is m is supported by a thin string that has a length L_2 equal to 0.80 m from the hinge. The particle sticks to the rod on contact, and after the collision the rod continues to rotate until $\theta_{max} = 37°$. (*a*) Find m. (*b*) How much energy is dissipated during the collision?

Picture the Problem Because the net external torque acting on the system is zero, angular momentum is conserved in this perfectly inelastic collision. The rod, on its downward swing, acquires rotational kinetic energy. Angular momentum is conserved in the perfectly inelastic collision with the particle and the rotational kinetic energy of the after-collision system is then transformed into gravitational potential energy as the rod-plus-particle swing upward. Let the zero of gravitational potential energy be at a distance L_1 below the pivot and use both angular momentum and mechanical energy conservation to relate the distances L_1 and L_2 and the mass M to m.

(*a*) Use conservation of energy to relate the initial and final potential energy of the rod to its rotational kinetic energy just before it collides with the particle:

$$K_f - K_i + U_f - U_i = 0$$

or, because $K_i = 0$,

$$K_f + U_f - U_i = 0$$

Substitute for K_f, U_f, and U_i to obtain:

$$\tfrac{1}{2}\left(\tfrac{1}{3}ML_1^2\right)\omega^2 + Mg\frac{L_1}{2} - MgL_1 = 0$$

Solving for ω yields:

$$\omega = \sqrt{\frac{3g}{L_1}}$$

Letting ω' represent the angular speed of the system after impact, use conservation of angular momentum to relate the angular momenta before and after the collision:

$$\Delta L = L_f - L_i = 0$$

or

$$\left(\tfrac{1}{3}ML_1^2 + mL_2^2\right)\omega' - \left(\tfrac{1}{3}ML_1^2\right)\omega = 0 \quad (1)$$

Solving for ω' and simplifying yields:

$$\omega' = \frac{\tfrac{1}{3}ML_1^2}{\tfrac{1}{3}ML_1^2 + mL_2^2}\omega = \frac{\tfrac{1}{3}ML_1^2}{\tfrac{1}{3}ML_1^2 + mL_2^2}\sqrt{\frac{3g}{L_1}}$$

Substitute numerical values and simplify to obtain:

$$\boldsymbol{\omega}' = \frac{\tfrac{1}{3}(2.0\,\text{kg})(1.2\,\text{m})^2}{\tfrac{1}{3}(2.0\,\text{kg})(1.2\,\text{m})^2 + \boldsymbol{m}(0.80\,\text{m})^2}\sqrt{\frac{3(9.81\,\text{m/s}^2)}{1.2\,\text{m}}} = \frac{4.75\,\text{kg/s}}{0.960\,\text{kg} + 0.64\boldsymbol{m}}$$

Use conservation of energy to relate the rotational kinetic energy of the rod-plus-particle just after their collision to their potential energy when they have swung through an angle θ_{max}:

$$K_f - K_i + U_f - U_i = 0$$

or, because $K_f = 0$,

$$-K_i + U_f - U_i = 0$$

Substitute for K_i, U_f, and U_i to obtain:

$$-\tfrac{1}{2}I\omega'^2 + Mg\left(\tfrac{1}{2}L_1\right)\left(1-\cos\theta_{max}\right) + mgL_2\left(1-\cos\theta_{max}\right) = 0$$

Express the moment of inertia of the system with respect to the pivot:

$$I = \tfrac{1}{3}ML_1^2 + mL_2^2$$

Substitute for θ_{max}, I and ω' in equation (1) and simplify to obtain:

$$\frac{\tfrac{1}{2}(4.75\,\text{kg/s})^2}{0.960\,\text{kg} + 0.64m} = 0.2g\left(ML_1 + mL_2\right)$$

Substitute for M, L_1 and L_2 and simplify to obtain:

$$\frac{\tfrac{1}{2}(4.75\,\text{kg/s})^2}{0.960\,\text{kg} + 0.64\boldsymbol{m}} = 0.2\boldsymbol{g}\left(2.4\,\text{kg}\cdot\text{m} + (0.80\,\text{m})\boldsymbol{m}\right)$$

Solve for m to obtain:

$$\boldsymbol{m} = 1.18\,\text{kg} = \boxed{1.2\,\text{kg}}$$

(*b*) The energy dissipated in the inelastic collision is:

$$\Delta E = U_i - U_f \qquad (2)$$

Express U_i:

$$U_i = Mg\frac{L_1}{2}$$

Express U_f:

$$U_f = \left(1-\cos\theta_{max}\right)g\left(M\frac{L_1}{2} + mL_2\right)$$

Substitute for U_i and U_f in equation (2) to obtain:

$$\Delta E = Mg\frac{L_1}{2} - \left(1-\cos\theta_{max}\right)g\left(M\frac{L_1}{2} + mL_2\right)$$

Substitute numerical values and evaluate ΔE:

$$\Delta \boldsymbol{E} = \frac{(2.0\,\text{kg})(9.81\,\text{m/s}^2)(1.2\,\text{m})}{2} - (1-\cos 37°)(9.81\,\text{m/s}^2)\left(\frac{(2.0\,\text{kg})(1.2\,\text{m})}{2} + (1.18\,\text{kg})(0.80\,\text{m})\right) = \boxed{7.5\,\text{J}}$$

Precession

69 •• **[SSM]** A bicycle wheel that has a radius equal to 28 cm is mounted at the middle of an axle 50 cm long. The tire and rim weigh 30 N. The wheel is spun at 12 rev/s, and the axle is then placed in a horizontal position with one end resting on a pivot. (*a*) What is the angular momentum due to the spinning of the wheel? (Treat the wheel as a hoop.) (*b*) What is the angular velocity of precession? (*c*) How long does it take for the axle to swing through 360° around the pivot? (*d*) What is the angular momentum associated with the motion of the center of mass, that is, due to the precession? In what direction is this angular momentum?

Picture the Problem We can determine the angular momentum of the wheel and the angular velocity of its precession from their definitions. The period of the precessional motion can be found from its angular velocity and the angular momentum associated with the motion of the center of mass from its definition.

(*a*) Using the definition of angular momentum, express the angular momentum of the spinning wheel:

$$L = I\omega = MR^2\omega = \frac{w}{g}R^2\omega$$

Substitute numerical values and evaluate L:

$$L = \left(\frac{30\,\text{N}}{9.81\,\text{m/s}^2}\right)(0.28\,\text{m})^2 \times \left(12\frac{\text{rev}}{\text{s}} \times \frac{2\pi\,\text{rad}}{\text{rev}}\right) = 18.1\,\text{J}\cdot\text{s} = \boxed{18\,\text{J}\cdot\text{s}}$$

(*b*) Using its definition, express the angular velocity of precession:

$$\omega_\text{p} = \frac{d\phi}{dt} = \frac{MgD}{L}$$

Substitute numerical values and evaluate ω_p:

$$\boldsymbol{\omega}_p = \frac{(30\,\text{N})(0.25\,\text{m})}{18.1\,\text{J}\cdot\text{s}} = 0.414\,\text{rad/s} = \boxed{0.41\,\text{rad/s}}$$

(*c*) Express the period of the precessional motion as a function of the angular velocity of precession:

$$\boldsymbol{T} = \frac{2\pi}{\boldsymbol{\omega}_p} = \frac{2\pi}{0.414\,\text{rad/s}} = \boxed{15\,\text{s}}$$

(*d*) Express the angular momentum of the center of mass due to the precession:

$$L_p = I_{cm}\omega_p = MD^2\omega_p$$

Substitute numerical values and evaluate L_p:

$$L_p = \left(\frac{30\,\text{N}}{9.81\,\text{m/s}^2}\right)(0.25\,\text{m})^2(0.414\,\text{rad/s}) = \boxed{0.079\,\text{J}\cdot\text{s}}$$

The direction of L_p is either up or down, depending on the direction of L.

70 •• A uniform disk of mass 2.50 kg and radius 6.40 cm is mounted at the center of a 10.0-cm-long axle and spun at 700 rev/min. The axle is then placed in a horizontal position with one end resting on a pivot. The other end is given an initial horizontal speed such that the precession is smooth with no nutation. (*a*) What is the angular speed of precession? (*b*) What is the speed of the center of mass during the precession? (*c*) What is the acceleration (magnitude and direction) of the center of mass? (*d*) What are the vertical and horizontal components of the force exerted by the pivot on the axle?

Picture the Problem The angular speed of precession can be found from its definition. Both the speed and the magnitude of the acceleration of the center of mass during precession are related to the angular speed of precession. We can use Newton's 2nd law to find the vertical and horizontal components of the force exerted by the pivot on the axle.

(*a*) The angular speed of precession is given by:

$$\omega_p = \frac{d\phi}{dt} = \frac{MgD}{I_s\omega_s}$$

Substituting for I_s and simplifying yields:

$$\omega_p = \frac{MgD}{\frac{1}{2}MR^2\omega_s} = \frac{2gD}{R^2\omega_s}$$

Substitute numerical values and evaluate ω_p:

$$\omega_p = \frac{2\left(9.81\,\text{m/s}^2\right)\left(0.050\,\text{m}\right)}{\left(0.064\,\text{m}\right)^2\left(700\,\frac{\text{rev}}{\text{min}}\times\frac{2\pi\,\text{rad}}{\text{rev}}\times\frac{1\,\text{min}}{60\,\text{s}}\right)} = 3.27\,\text{rad/s} = \boxed{3.3\,\text{rad/s}}$$

(*b*) Express the speed of the center of mass in terms of its angular speed of precession:

$$\boldsymbol{v}_{\text{cm}} = \boldsymbol{D\omega}_{\text{p}} = \left(0.050\,\text{m}\right)\left(3.27\,\text{rad/s}\right) = \boxed{16\,\text{cm/s}}$$

(*c*) Relate the acceleration of the center of mass to its angular speed of precession:

$$\boldsymbol{a}_{\text{cm}} = \boldsymbol{D\omega}_{\text{p}}^2 = \left(0.050\,\text{m}\right)\left(3.27\,\text{rad/s}\right)^2 = 0.535\,\text{m/s}^2 = \boxed{54\,\text{cm/s}^2}$$

(*d*) Use Newton's 2nd law to relate the vertical component of the force exerted by the pivot to the weight of the disk:

$$\boldsymbol{F}_{\text{v}} = \boldsymbol{Mg} = \left(2.5\,\text{kg}\right)\left(9.81\,\text{m/s}^2\right) = \boxed{25\,\text{N}}$$

Relate the horizontal component of the force exerted by the pivot on the axle to the acceleration of the center of mass:

$$\boldsymbol{F}_{\text{H}} = \boldsymbol{Ma}_{\text{cm}} = \left(2.5\,\text{kg}\right)\left(0.535\,\text{m/s}^2\right) = \boxed{1.3\,\text{N}}$$

General Problems

71 • **[SSM]** A particle whose mass is 3.0 kg moves in the *xy* plane with velocity $\vec{v} = (3.0\text{ m/s})\hat{i}$ along the line $y = 5.3$ m. (*a*) Find the angular momentum $\vec{L}$ about the origin when the particle is at (12 m, 5.3 m). (*b*) A force $\vec{F} = (-3.9\text{ N})\hat{i}$ is applied to the particle. Find the torque about the origin due to this force as the particle passes through the point (12 m, 5.3 m).

Picture the Problem While the 3-kg particle is moving in a straight line, it has angular momentum given by $\vec{L} = \vec{r}\times\vec{p}$ where $\vec{r}$ is its position vector and $\vec{p}$ is its linear momentum. The torque due to the applied force is given by $\vec{\tau} = \vec{r}\times\vec{F}$.

(*a*) The angular momentum of the particle is given by:

$$\vec{L} = \vec{r}\times\vec{p}$$

Express the vectors $\vec{r}$ and $\vec{p}$:

$$\vec{r} = (12\,\text{m})\hat{i} + (5.3\,\text{m})\hat{j}$$

and

$$\vec{p} = mv\hat{i} = (3.0\,\text{kg})(3.0\,\text{m/s})\hat{i} = (9.0\,\text{kg}\cdot\text{m/s})\hat{i}$$

Substitute for $\vec{r}$ and $\vec{p}$:and simplify to find $\vec{L}$:

$$\vec{L} = \left[(12\,\text{m})\hat{i} + (5.3\,\text{m})\hat{j}\right]\times(9.0\,\text{kg}\cdot\text{m/s})\hat{i} = (47.7\,\text{kg}\cdot\text{m}^2/\text{s})(\hat{j}\times\hat{i}) = \boxed{-(48\,\text{kg}\cdot\text{m}^2/\text{s})\hat{k}}$$

(*b*) Using its definition, express the torque due to the force:

$$\vec{\tau} = \vec{r}\times\vec{F}$$

Substitute for $\vec{r}$ and $\vec{F}$ and simplify to find $\vec{\tau}$:

$$\vec{\tau} = \left[(12\,\text{m})\hat{i} + (5.3\,\text{m})\hat{j}\right]\times(-3.0\,\text{N})\hat{i} = -(15.9\,\text{N}\cdot\text{m})(\hat{j}\times\hat{i}) = \boxed{(16\,\text{N}\cdot\text{m})\hat{k}}$$

72 • The position vector of a particle whose mass is 3.0 kg is given by $\vec{r} = 4.0\hat{i} + 3.0t^2\hat{j}$, where $\vec{r}$ is in meters and t is in seconds. Determine the angular momentum and net torque, about the origin, acting on the particle.

Picture the Problem The angular momentum of the particle is given by $\vec{L} = \vec{r}\times\vec{p}$ where $\vec{r}$ is its position vector and $\vec{p}$ is its linear momentum. The torque acting on the particle is given by $\vec{\tau} = d\vec{L}/dt$.

The angular momentum of the particle is given by:

$$\vec{L} = \vec{r}\times\vec{p} = \vec{r}\times m\vec{v} = m\vec{r}\times\vec{v} = m\vec{r}\times\frac{d\vec{r}}{dt}$$

Evaluating $\frac{d\vec{r}}{dt}$ yields:

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}\left[4.0\hat{i} + 3.0t^2\hat{j}\right] = (6.0t)\hat{j}$$

Substitute for $m\vec{r}$ and $\frac{d\vec{r}}{dt}$ and simplify to find $\vec{L}$:

$$\vec{L} = \left[(3.0\,\text{kg})\left\{(4.0\,\text{m})\hat{i} + (3.0t^2\,\text{m/s}^2)\hat{j}\right\}\right]\times(6.0t\,\text{m/s})\hat{j} = \boxed{(72t\,\text{J}\cdot\text{s})\hat{k}}$$

Find the net torque due to the force:

$$\vec{\boldsymbol{\tau}}_{\text{net}} = \frac{d\vec{\boldsymbol{L}}}{dt} = \frac{d}{dt}\left[(72t\,\text{J}\cdot\text{s})\hat{\boldsymbol{k}}\right] = \boxed{(72\,\text{N}\cdot\text{m})\hat{\boldsymbol{k}}}$$

73 •• Two ice skaters, whose masses are 55 kg and 85 kg, hold hands and rotate about a vertical axes that passes between them, making one revolution in 2.5 s. Their centers of mass are separated by 1.7 m and their center of mass is stationary. Model each skater as a point particle and find (*a*) the angular momentum of the system about their center of mass and (*b*) the total kinetic energy of the system.

Picture the Problem The ice skaters rotate about their center of mass; a point we can locate using its definition. Knowing the location of the center of mass we can determine their moment of inertia with respect to an axis through this point. The angular momentum of the system is then given by $L = I_{\text{cm}}\omega$ and its kinetic energy can be found from $K = L^2/(2I_{\text{cm}})$.

(*a*) Express the angular momentum of the system about the center of mass of the skaters:

$$L = I_{\text{cm}}\omega$$

Using its definition, locate the center of mass, relative to the 85-kg skater, of the system:

$$x_{\text{cm}} = \frac{(55\,\text{kg})(1.7\,\text{m}) + (85\,\text{kg})(0)}{55\,\text{kg} + 85\,\text{kg}} = 0.668\,\text{m}$$

Calculate I_{cm}:

$$I_{\text{cm}} = (55\,\text{kg})(1.7\,\text{m} - 0.668\,\text{m})^2 + (85\,\text{kg})(0.668\,\text{m})^2 = 96.5\,\text{kg}\cdot\text{m}^2$$

Substitute to determine *L*:

$$L = (96.5\,\text{kg}\cdot\text{m}^2)\left(\frac{1\,\text{rev}}{2.5\,\text{s}} \times \frac{2\pi\,\text{rad}}{\text{rev}}\right) = 243\,\text{J}\cdot\text{s} = \boxed{0.24\,\text{kJ}\cdot\text{s}}$$

(*b*) Relate the total kinetic energy of the system to its angular momentum and evaluate *K*:

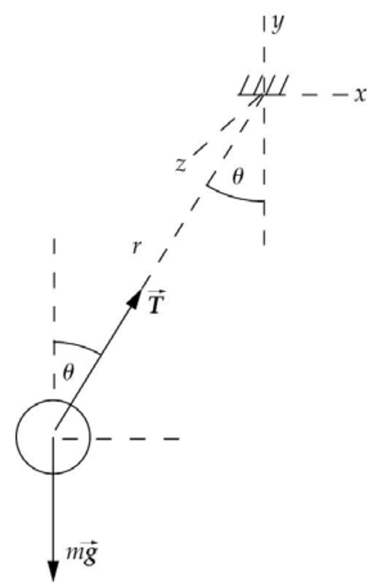
$$K = \frac{L^2}{2I_{\text{cm}}}$$

Substitute numerical values and evaluate *K*:

$$\boldsymbol{K} = \frac{(243\,\text{J}\cdot\text{s})^2}{2(96.5\,\text{kg}\cdot\text{m}^2)} = \boxed{0.31\,\text{kJ}}$$

74 •• A 2.0-kg ball attached to a string whose length is 1.5 m moves counterclockwise (as viewed from above) in a horizontal circle (Figure 10-56). The string makes an angle $\theta = 30°$ with the vertical. (*a*) Determine both the horizontal and vertical components of the angular momentum $\vec{L}$ of the ball about the point of support *P*. (*b*) Find the magnitude of $d\vec{L}/dt$ and verify that it equals the magnitude of the torque exerted by gravity about the point of support.

Picture the Problem Let the origin of the coordinate system be at the pivot. The diagram shows the forces acting on the ball. We'll apply Newton's 2nd law to the ball to determine its speed. We'll then use the derivative of its position vector to express its velocity and the definition of angular momentum to show that $\vec{L}$ has both horizontal and vertical components. We can use the derivative of $\vec{L}$ with respect to time to show that the rate at which the angular momentum of the ball changes is equal to the torque, relative to the pivot point, acting on it.



(*a*) Express the angular momentum of the ball about the point of support:

$$\vec{L} = \vec{r} \times \vec{p} = m\vec{r} \times \vec{v} \qquad (1)$$

Apply Newton's 2nd law to the ball:

$$\sum F_x = T\sin\theta = m\frac{v^2}{r\sin\theta}$$

and

$$\sum F_z = T\cos\theta - mg = 0$$

Eliminate *T* between these equations and solve for *v* to obtain:

$$v = \sqrt{rg\sin\theta\tan\theta}$$

Substitute numerical values and evaluate *v*:

$$v = \sqrt{(1.5\,\text{m})(9.81\,\text{m/s}^2)\sin 30°\tan 30°} = 2.06\,\text{m/s}$$

Express the position vector of the ball:

$$\vec{r} = (1.5\,\text{m})\sin 30°(\cos\omega t\,\hat{i} + \sin\omega t\hat{j}) - (1.5\,\text{m})\cos 30°\hat{k}$$

where $\omega = \omega\hat{k}$.

The velocity of the ball is:

$$\vec{v} = \frac{d\vec{r}}{dt} = (0.75\omega\,\text{m/s})\left(-\sin\omega t\,\hat{i} + \cos\omega t\,\hat{j}\right)$$

Evaluating ω yields:

$$\omega = \frac{2.06\,\text{m/s}}{(1.5\,\text{m})\sin 30°} = 2.75\,\text{rad/s}$$

Substitute for ω to obtain:

$$\vec{v} = (2.06\,\text{m/s})\left(-\sin\omega t\,\hat{i} + \cos\omega t\hat{j}\right)$$

Substitute in equation (1) and evaluate $\vec{L}$:

$$\begin{aligned}\vec{L} &= (2.0\,\text{kg})\left[(1.5\,\text{m})\sin 30°\left(\cos\boldsymbol{\omega t}\,\hat{i} + \sin\boldsymbol{\omega t}\,\hat{j}\right) - (1.5\,\text{m})\cos 30°\hat{k}\right] \\ &\quad\times\left[(2.06\,\text{m/s})\left(-\sin\boldsymbol{\omega t}\,\hat{i} + \cos\boldsymbol{\omega t}\,\hat{j}\right)\right] \\ &= (5.35\,\text{J}\cdot\text{s})\cos\boldsymbol{\omega t}\,\hat{i} + (5.35\,\text{J}\cdot\text{s})\sin\boldsymbol{\omega t}\,\hat{j} + (3.09\,\text{J}\cdot\text{s})\hat{k}\end{aligned}$$

The horizontal component of $\vec{L}$ is the component in the *xy* plane:

$$\vec{L}_{\text{hor}} = \boxed{(5.4\,\text{J}\cdot\text{s})\cos\boldsymbol{\omega t}\,\hat{i} + (5.4\,\text{J}\cdot\text{s})\sin\boldsymbol{\omega t}\,\hat{j}}$$

The vertical component of $\vec{L}$ is its z component:

$$\boldsymbol{L}_{\text{vertical}} = \boxed{(3.1\,\text{J}\cdot\text{s})\hat{k}}$$

(*b*) Evaluate $\frac{d\vec{L}}{dt}$:

$$\frac{d\vec{L}}{dt} = \left[5.36\omega\left(-\sin\omega t\,\hat{i} + \cos\omega t\hat{j}\right)\right]\text{J}$$

Evaluate the magnitude of $\frac{d\vec{L}}{dt}$:

$$\left|\frac{d\vec{L}}{dt}\right| = (5.36\,\text{N}\cdot\text{m}\cdot\text{s})(2.75\,\text{rad/s}) = \boxed{15\,\text{N}\cdot\text{m}}$$

Express the magnitude of the torque exerted by gravity about the point of support:

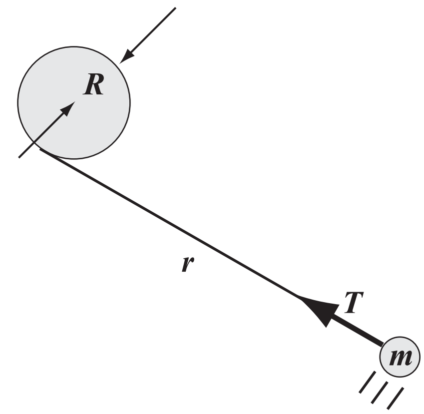
$$\tau = mgr\sin\theta$$

Substitute numerical values and evaluate τ:

$$\tau = (2.0\,\text{kg})(9.81\,\text{m/s}^2)(1.5\,\text{m})\sin 30° = \boxed{15\,\text{N}\cdot\text{m}}$$

75 •• A compact object whose mass is m resting on a horizontal, frictionless surface is attached to a string that wraps around a vertical cylindrical post attached to the surface so that when the object is set into motion, it follows a path that spirals inward. (*a*) Is the angular momentum of the object about the axis of the post conserved? Explain your answer. (*b*) Is the energy of the object conserved? Explain your answer. (*c*) If the speed of the object is v_0 when the unwrapped length of the string is r, what is its speed when the unwrapped length has shortened to $r/2$?

Picture the Problem The pictorial representation depicts the object rotating counterclockwise around the cylindrical post. Let the system be the object. In Part (*a*) we need to decide whether a net torque acts on the object and in Part (*b*) the issue is whether any external forces act on the object. In Part (*c*) we can apply the definition of kinetic energy to find the speed of the object when the unwrapped length has shortened to $r/2$.



(*a*) The net torque acting on the object is given by:

$$\tau_{\text{net}} = \frac{dL}{dt} = RT$$

Because $\tau_{\text{net}} \neq 0$, angular momentum is not conserved.

(*b*) Because, in this frictionless environment, the net external force acting on the object is the tension force and it acts at right angles to the object's velocity, the energy of the object is conserved.

(*c*) Apply conservation of mechanical energy to the object to obtain:

$$\Delta E = \Delta K + \Delta U = 0$$

or, because $\Delta U = 0$,

$$\Delta K_{\text{rot}} = 0$$

Substituting for the kinetic energies yields:

$$\tfrac{1}{2} I'\omega'^2 - \tfrac{1}{2} I\omega_0^2 = 0$$

or

$$I'\omega'^2 - I\omega_0^2 = 0$$

Substitute for I, I', ω', and ω_0 to obtain:

$$\tfrac{1}{2} m\left(\frac{r}{2}\right)^2\left(\frac{v'}{\frac{r}{2}}\right)^2 - \tfrac{1}{2} mr^2\left(\frac{v_0}{r}\right)^2 = 0$$

Solving for v' yields:

$$v' = \boxed{v_0}$$

76 •• Figure 10-57 shows a hollow cylindrical tube that has a mass M, a length L, and a moment of inertia $ML^2/10$. Inside the cylinder are two disks each of mass m and radius r, separated by a distance ℓ and tied to a central post by a thin string. The system can rotate about a vertical axis through the center of the cylinder. You are designing this cylinder-disk apparatus to shut down the rotations when the strings break by triggering an electronic ″shutoff″ signal (sent to the rotating motor) when the disks hit the ends of the cylinder. During development, you notice that with the system rotating at some critical angular speed ω, the string suddenly breaks. When the disks reach the ends of the cylinder, they stick. Obtain expressions for the final angular speed and the initial and final kinetic energies of the system. Assume that the inside walls of the cylinder are frictionless.

Picture the Problem Because the net torque acting on the system is zero; we can use conservation of angular momentum to relate the initial and final angular velocities of the system. See Table 9-1 for the moment of inertia of a disk.

Using conservation of angular momentum, relate the initial and final angular speeds to the initial and final moments of inertia:

$$\Delta \boldsymbol{L} = \boldsymbol{L}_\text{f} - \boldsymbol{L}_\text{i} = 0$$

or

$$\boldsymbol{I}_\text{f}\boldsymbol{\omega}_\text{f} - \boldsymbol{I}_\text{i}\boldsymbol{\omega}_\text{i} = 0$$

Solving for ω_f yields:

$$\omega_\text{f} = \frac{I_\text{i}}{I_\text{f}}\omega_\text{i} = \frac{I_\text{i}}{I_\text{f}}\omega \qquad (1)$$

Use the parallel-axis theorem to express the moment of inertia of each of the disks with respect to the axis of rotation:

$$\begin{aligned} I_\text{i, each disk} &= I_\text{cm} + m\left(\tfrac{1}{2}\ell\right)^2 \\ &= \tfrac{1}{4}mr^2 + \tfrac{1}{4}m\ell^2 \\ &= \tfrac{1}{4}m\left(r^2 + \ell^2\right) \end{aligned}$$

Express the initial moment of inertia I_i of the cylindrical tube plus disks system:

$$\begin{aligned} I_\text{i} &= I_{\substack{\text{cylindrical}\\\text{tube}}} + 2I_\text{i, each disk} \\ &= \tfrac{1}{10}ML^2 + 2\left[\tfrac{1}{4}m\left(r^2 + \ell^2\right)\right] \\ &= \tfrac{1}{10}ML^2 + \tfrac{1}{2}m\left(r^2 + \ell^2\right) \end{aligned}$$

When the disks have moved out to the end of the cylindrical tube:

$$I_\text{f} = \tfrac{1}{10}ML^2 + \tfrac{1}{2}m\left(r^2 + L^2\right)$$

Substitute for I_i and I_f in equation (1) and simplify to obtain:

$$\omega_f = \frac{\frac{1}{10}ML^2 + \frac{1}{2}m\left(r^2 + \ell^2\right)}{\frac{1}{10}ML^2 + \frac{1}{2}m\left(r^2 + L^2\right)}\omega$$
$$= \boxed{\frac{ML^2 + 5m\left(r^2 + \ell^2\right)}{ML^2 + 5m\left(r^2 + L^2\right)}\omega}$$

The initial kinetic energy of the system is:

$$K_i = \tfrac{1}{2} I_i \omega^2$$

Substituting for I_i and simplifying yields:

$$K_i = \tfrac{1}{2}\left[\tfrac{1}{10}ML^2 + \tfrac{1}{2}m\left(r^2 + \ell^2\right)\right]\omega^2$$
$$= \boxed{\left[\tfrac{1}{20}ML^2 + \tfrac{1}{4}m\left(r^2 + \ell^2\right)\right]\omega^2}$$

The final kinetic energy of the system is:

$$K_f = \tfrac{1}{2} I_f \omega_f^2$$

Substitute for I_f and ω_f and simplify to obtain:

$$K_f = \tfrac{1}{2}\left[\tfrac{1}{10}ML^2 + \tfrac{1}{2}m\left(r^2 + L^2\right)\right]\left(\frac{ML^2 + 5m\left(r^2 + \ell^2\right)}{ML^2 + 5m\left(r^2 + L^2\right)}\omega\right)^2$$
$$= \boxed{\frac{1}{20}\left[\frac{\left[ML^2 + 5m\left(r^2 + \ell^2\right)\right]^2}{ML^2 + 5m\left(r^2 + L^2\right)}\right]\omega^2}$$

77 •• **[SSM]** Repeat Problem 76, this time friction between the disks and the walls of the cylinder is not negligible. However, the coefficient of friction is not great enough to prevent the disks from reaching the ends of the cylinder. Can the final kinetic energy of the system be determined without knowing the coefficient of kinetic friction?

Determine the Concept Yes. The solution depends only upon conservation of angular momentum of the system, so it depends only upon the initial and final moments of inertia.

78 •• Suppose that in Figure 10-57 $\ell = 0.60$ m, $L = 2.0$ m, $M = 0.80$ kg, and $m = 0.40$ kg. The string breaks when the system's angular speed approaches the critical angular speed ω_i, at which time the tension in the string is 108 N. The masses then move radially outward until they undergo perfectly inelastic collisions with the ends of the cylinder. Determine the critical angular speed and the angular speed of the system after the inelastic collisions. Find the total kinetic energy of the system at the critical angular speed, and again after the inelastic collisions. Assume that the inside walls of the cylinder are frictionless.

Picture the Problem Because the net torque acting on the system is zero; we can use conservation of angular momentum to relate the initial and final angular speeds of the system.

Using conservation of angular momentum, relate the initial and final angular speeds to the initial and final moments of inertia:

$$\Delta L = L_f - L_i = 0$$
or
$$I_f\omega_f - I_i\omega_i = 0 \Rightarrow \omega_f = \frac{I_i}{I_f}\omega_i \quad (1)$$

Express the tension in the string as a function of the critical angular speed of the system:

$$T = mr\omega_i^2 = m\frac{\ell}{2}\omega_i^2 \Rightarrow \omega_i = \sqrt{\frac{2T}{m\ell}}$$

Substitute numerical values and evaluate ω_i :

$$\omega_i = \sqrt{\frac{2(108\,\text{N})}{(0.40\,\text{kg})(0.60\,\text{m})}} = 30.0\,\text{rad/s} = \boxed{30\,\text{rad/s}}$$

Express I_i:

$$I_i = \tfrac{1}{10}ML^2 + 2\left(\tfrac{1}{4}m\ell^2\right)$$

Substitute numerical values and evaluate I_i:

$$I_i = \tfrac{1}{10}(0.80\,\text{kg})(2.0\,\text{m})^2 + \tfrac{1}{2}(0.40\,\text{kg})(0.60\,\text{m})^2 = 0.392\,\text{kg}\cdot\text{m}^2$$

Express I_f:

$$I_f = \tfrac{1}{10}ML^2 + 2\left(\tfrac{1}{4}mL^2\right)$$

Substitute numerical values and evaluate I_f:

$$I_f = \tfrac{1}{10}(0.80\,\text{kg})(2.0\,\text{m})^2 + \tfrac{1}{2}(0.40\,\text{kg})(2.0\,\text{m})^2 = 1.12\,\text{kg}\cdot\text{m}^2$$

Substitute numerical values in equation (1) and evaluate ω_f:

$$\omega_f = \frac{0.392\,\text{kg}\cdot\text{m}^2}{1.12\,\text{kg}\cdot\text{m}^2}(30.0\,\text{rad/s}) = 10.5\,\text{rad/s} = \boxed{11\,\text{rad/s}}$$

The total kinetic energy of the system at the critical angular speed is:

$$K_i = \tfrac{1}{2}I_i\omega_i^2$$

Substitute numerical values and evaluate K_i:

$$K_i = \tfrac{1}{2}\left(0.392\,\text{kg}\cdot\text{m}^2\right)\left(30.0\,\text{rad/s}\right)^2 = 176\,\text{J} = \boxed{0.18\,\text{kJ}}$$

The total kinetic energy of the system after the inelastic collisions is:

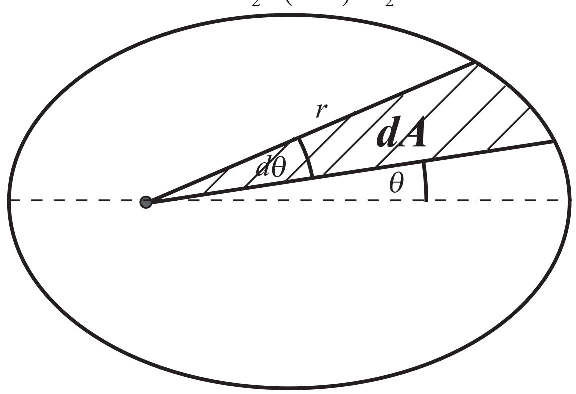
$$K_f = \tfrac{1}{2} I_f \omega_f^2$$

Substitute numerical values and evaluate K_f:

$$K_f = \tfrac{1}{2}\left(1.12\,\text{kg}\cdot\text{m}^2\right)\left(10.5\,\text{rad/s}\right)^2 = \boxed{62\,\text{J}}$$

79 •• **[SSM]** Kepler's second law states: *The line from the center of the Sun to the center of a planet sweeps out equal areas in equal times.* Show that this law follows directly from the law of conservation of angular momentum and the fact that the force of gravitational attraction between a planet and the Sun acts along the line joining the centers of the two celestial objects.

Picture the Problem The pictorial representation shows an elliptical orbit. The triangular element of the area is $dA = \tfrac{1}{2} r(r\,d\theta) = \tfrac{1}{2} r^2 d\theta$.



Differentiate dA with respect to t to obtain:

$$\frac{dA}{dt} = \tfrac{1}{2} r^2 \frac{d\theta}{dt} = \tfrac{1}{2} r^2 \omega \qquad (1)$$

Because the gravitational force acts along the line joining the two objects, $\tau = 0$. Hence:

$$L = mr^2\omega = \text{constant} \qquad (2)$$

Eliminate $r^2\omega$ between equations (1) and (2) to obtain:

$$\frac{dA}{dt} = \boxed{\frac{L}{2m} = \text{constant}}$$

80 •• Consider a cylindrical turntable whose mass is M and radius is R, turning with an initial angular speed ω_1. (*a*) A parakeet of mass m, hovering in flight above the outer edge of the turntable, gently lands on it and stays in one place on it as shown in Figure 10-58. What is the angular speed of the turntable after the parakeet lands? (*b*) Becoming dizzy, the parakeet jumps off (not flies off) with a velocity $\vec{v}$ relative to the turntable. The direction of $\vec{v}$ is tangent to the edge of the turntable, and in the direction of its rotation. What will be the angular speed of the turntable afterwards? Express your answer in terms of the two masses *m* and *M*, the radius *R*, the parakeet speed *v* and the initial angular speed ω_1.

Picture the Problem The angular momentum of the turntable-parakeet is conserved in both parts of this problem.

(*a*) Apply conservation of angular momentum to the turntable-parakeet system as the parakeet lands to obtain:

$$\Delta \boldsymbol{L} = \boldsymbol{L}_\text{f} - \boldsymbol{L}_\text{i} = 0 \qquad (1)$$

The final angular momentum of the system is given by:

$$\begin{aligned}\boldsymbol{L}_\text{f} &= \boldsymbol{L}_\text{turntable} + \boldsymbol{L}_\text{parakeet} \\ &= \boldsymbol{I}_\text{turntable}\boldsymbol{\omega}_\text{f} + \left|\vec{r} \times \vec{p}_\text{parakeet}\right|\end{aligned}$$

Because $I_\text{turntable} = \frac{1}{2}MR^2$ and $\left|\vec{r} \times \vec{p}_\text{parakeet}\right| = \boldsymbol{Rmv}_\text{parakeet}$:

$$\begin{aligned}L_\text{f} &= \tfrac{1}{2}MR^2\omega_\text{f} + Rmv_\text{parakeet} \\ &= \tfrac{1}{2}MR^2\omega_\text{f} + Rm(R\omega_\text{f}) \\ &= \tfrac{1}{2}MR^2\omega_\text{f} + mR^2\omega_\text{f}\end{aligned}$$

The initial angular momentum of the system is given by:

$$L_\text{i} = I_\text{turntable}\omega_\text{i} = \tfrac{1}{2}MR^2\omega_\text{i}$$

Substituting for L_f and L_i in equation (1) yields:

$$\tfrac{1}{2}MR^2\omega_\text{f} + mR^2\omega_\text{f} - \tfrac{1}{2}MR^2\omega_\text{i} = 0$$

Solve for ω_f to obtain:

$$\omega_\text{f} = \boxed{\frac{M}{M+2m}\omega_\text{i}}$$

(*b*) Apply conservation of angular momentum to the turntable-parakeet system as the parakeet jumps off to obtain:

$$\Delta L = L_\text{f} - L_\text{i} = 0 \qquad (2)$$

The final angular momentum of the system is given by:

$$\begin{aligned}\boldsymbol{L}_\text{f} &= \boldsymbol{L}_\text{turntable} + \boldsymbol{L}_\text{parakeet} \\ &= \boldsymbol{I}_\text{turntable}\boldsymbol{\omega}_\text{f} + \left|\vec{r} \times \vec{p}_\text{parakeet}\right|\end{aligned}$$

Because $\boldsymbol{I}_{\text{turntable}} = \frac{1}{2}\boldsymbol{MR}^2$ and $\left|\vec{r} \times \vec{p}_{\text{parakeet}}\right| = \boldsymbol{Rmv}_{\text{parakeet}}$:

$$L_{\text{f}} = \tfrac{1}{2}MR^2\omega_{\text{f}} + Rmv_{\text{parakeet}} \qquad (3)$$

Express the speed of the parakeet relative to the turntable:

$$\boldsymbol{v}_{\text{parakeet}} = \boldsymbol{v}_{\text{turntable}} + \boldsymbol{v} = \boldsymbol{R\omega}_{\text{f}} + \boldsymbol{v}$$

Using the expression derived in (*a*), substitute for ω_{f} to obtain:

$$v_{\text{parakeet}} = \frac{M}{M+2m}R\omega_{\text{i}} + v$$

Substituting for v_{parakeet} in equation (3) and simplifying yields:

$$L_{\text{f}} = \tfrac{1}{2}MR^2\omega_{\text{f}} + mR\left(\frac{M}{M+2m}R\omega_{\text{i}} + v\right)$$

The initial angular momentum of the system is the same as the final angular momentum in (*a*):

$$L_{\text{i}} = \tfrac{1}{2}MR^2\omega_{\text{i}}$$

Substituting for L_{f} and L_{i} in equation (2) yields:

$$\tfrac{1}{2}MR^2\omega_{\text{f}} + mR\left[\left(\frac{M}{M+2m}\right)R\omega_{\text{i}} + v\right] - \tfrac{1}{2}MR^2\omega_{\text{i}} = 0$$

Solving for ω_{f} yields:

$$\omega_{\text{f}} = \boxed{\frac{2m}{M}\left[\left(\frac{M^2}{2m(M+2m)}\right)\omega_{\text{i}} - \frac{v}{R}\right]}$$

81 •• You are given a heavy but thin metal disk (like a coin, but larger; Figure 10-59). (Objects like this are called *Euler disks.*) Placing the disk on a turntable, you spin the disk, on edge, about a vertical axis through a diameter of the disk and the center of the turntable. As you do this, you hold the turntable still with your other hand, letting it go immediately after you spin the disk. The turntable is a uniform solid cylinder with a radius equal to 0.250 m and a mass equal to 0.735 kg and rotates on a frictionless bearing. The disk has an initial angular speed of 30.0 rev/min. (*a*) The disk spins down and falls over, finally coming to rest on the turntable with its symmetry axis coinciding with the turntable's. What is the final angular speed of the turntable? (*b*) What will be the final angular speed if the disk's symmetry axis ends up 0.100 m from the axis of the turntable?

Picture the Problem Let the letters *d*, *m*, and *r* denote the disk and the letters *t*, *M*, and *R* the turntable. We can use conservation of angular momentum to relate the final angular speed of the turntable to the initial angular speed of the Euler disk and the moments of inertia of the turntable and the disk. In part (*b*) we'll need to use the parallel-axis theorem to express the moment of inertia of the disk with respect to the rotational axis of the turntable. You can find the moments of inertia of the disk in its two orientations and that of the turntable in Table 9-1.

(*a*) Use conservation of angular momentum to relate the initial and final angular momenta of the system:

$$I_{\rm df}\omega_{\rm df} + I_{\rm tf}\omega_{\rm tf} - I_{\rm di}\omega_{\rm di} = 0$$

Because $\omega_{\rm tf} = \omega_{\rm df}$:

$$I_{\rm df}\omega_{\rm tf} + I_{\rm tf}\omega_{\rm tf} - I_{\rm di}\omega_{\rm di} = 0$$

Solving for $\omega_{\rm tf}$ yields:

$$\omega_{\rm tf} = \frac{I_{\rm di}}{I_{\rm df} + I_{\rm tf}}\omega_{\rm di} \qquad (1)$$

Ignoring the negligible thickness of the disk, express its initial moment of inertia:

$$I_{\rm di} = \tfrac{1}{4}mr^2$$

Express the final moment of inertia of the disk:

$$I_{\rm df} = \tfrac{1}{2}mr^2$$

Express the final moment of inertia of the turntable:

$$I_{\rm tf} = \tfrac{1}{2}MR^2$$

Substitute in equation (1) and simplify to obtain:

$$\omega_{\rm tf} = \frac{\tfrac{1}{4}mr^2}{\tfrac{1}{2}mr^2 + \tfrac{1}{2}MR^2}\omega_{\rm di} = \frac{1}{2 + 2\dfrac{MR^2}{mr^2}}\omega_{\rm di} \qquad (2)$$

Express $\omega_{\rm di}$ in rad/s:

$$\boldsymbol{\omega}_{\rm di} = 30.0\frac{\rm rev}{\rm min}\times\frac{2\boldsymbol{\pi}\,{\rm rad}}{\rm rev}\times\frac{1\,{\rm min}}{60\,{\rm s}} = \boldsymbol{\pi}\,{\rm rad/s}$$

Substitute numerical values in equation (2) and evaluate $\omega_{\rm tf}$:

$$\boldsymbol{\omega}_{\rm tf} = \frac{\boldsymbol{\pi}\,{\rm rad/s}}{2 + 2\dfrac{(0.735\,{\rm kg})(0.250\,{\rm m})^2}{(0.500\,{\rm kg})(0.125\,{\rm m})^2}} = \boxed{0.228\,{\rm rad/s}}$$

(*b*) Use the parallel-axis theorem to express the final moment of inertia of the disk when it is a distance *L* from the center of the turntable:

$$I_{\text{df}} = \tfrac{1}{2}mr^2 + mL^2 = m\left(\tfrac{1}{2}r^2 + L^2\right)$$

Substitute in equation (1) to obtain:

$$\omega_{\text{tf}} = \frac{\tfrac{1}{4}mr^2}{m\left(\tfrac{1}{2}r^2 + L^2\right) + \tfrac{1}{2}MR^2}\,\omega_{\text{di}} = \frac{1}{2 + 4\dfrac{L^2}{r^2} + 2\dfrac{MR^2}{mr^2}}\,\omega_{\text{di}}$$

Substitute numerical values and evaluate ω_{tf}:

$$\omega_{\text{tf}} = \frac{\pi\ \text{rad/s}}{2 + 4\dfrac{(0.100\,\text{m})^2}{(0.125\,\text{m})^2} + 2\dfrac{(0.735\,\text{kg})(0.250\,\text{m})^2}{(0.500\,\text{kg})(0.125\,\text{m})^2}} = \boxed{0.192\,\text{rad/s}}$$

82 •• (*a*) Assuming Earth to be a homogeneous sphere that has a radius *r* and a mass *m*, show that the period *T* (time for one daily rotation) of Earth's rotation about its axis is related to its radius by $T = br^2$, where $b = (4/5)\pi m/L$. Here *L* is the magnitude of the spin angular momentum of Earth. (*b*) Suppose that the radius *r* changes by a very small amount Δr due to some internal cause such as thermal expansion. Show that the fractional change in the period ΔT is given approximately by $\Delta T/T = 2\Delta r/r$. (*c*) By how many kilometers would *r* need to increase for the period to change by 0.25 d/y (so that leap years would no longer be necessary)?

Picture the Problem We can express the period of the earth's rotation in terms of its angular velocity of rotation and relate its angular velocity to its angular momentum and moment of inertia with respect to an axis through its center. We can differentiate this expression with respect to *T* and then use differentials to approximate the changes in *r* and *T*.

(*a*) Express the period of the earth's rotation in terms of its angular velocity of rotation:

$$T = \frac{2\pi}{\omega}$$

Relate the earth's angular velocity of rotation to its angular momentum and moment of inertia:

$$\omega = \frac{L}{I} = \frac{L}{\tfrac{2}{5}mr^2}$$

Substitute for ω and simplify to obtain:

$$T = \frac{2\pi\left(\frac{2}{5}mr^2\right)}{L} = \boxed{\frac{4\pi m}{5L}r^2}$$

(*b*) Find dT/dr:

$$\frac{dT}{dr} = \frac{d}{dr}\left(\frac{4\pi m}{5L}r^2\right) = 2\left(\frac{4\pi m}{5L}\right)r$$

$$= 2\left(\frac{T}{r^2}\right)r = \frac{2T}{r}$$

Solving for dT/T yields:

$$\frac{dT}{T} = 2\frac{dr}{r} \Rightarrow \frac{\Delta T}{T} \approx \boxed{2\frac{\Delta r}{r}}$$

(*c*) Using the equation we just derived, substitute for the change in the period of the earth:

$$\frac{\Delta T}{T} = \frac{\frac{1}{4}\text{d}}{\text{y}} \times \frac{1\,\text{y}}{365.24\,\text{d}} = \frac{1}{1460} = 2\frac{\Delta r}{r}$$

Solving for Δr yields:

$$\Delta \boldsymbol{r} = \frac{\boldsymbol{r}}{2(1460)}$$

Substitute numerical values and evaluate Δr:

$$\Delta \boldsymbol{r} = \frac{6.37\times10^3\ \text{km}}{2(1460)} = \boxed{2.18\,\text{km}}$$

83 •• **[SSM]** The term precession of the equinoxes refers to the fact that the Earth's spin axis does not stay fixed but moves with a period of about 26,000 y. (This explains why our pole star, Polaris, will not remain the pole star forever.) The reason for this instability is that Earth is a giant gyroscope. The spin axis of Earth precesses because of the torques exerted on it by the gravitational forces of the Sun and Moon. The angle between the direction of Earth's spin axis and the normal to the ecliptic plane (the plane of Earth's orbit) is 22.5 degrees. Calculate an approximate value for this torque, given that the period of rotation of the earth is 1.00 d and its moment of inertia is 8.03×10^{37} kg·m^2.

Picture the Problem Let ω_P be the angular velocity of precession of the earth-as-gyroscope, ω_s its angular velocity about its spin axis, and I its moment of inertia with respect to an axis through its poles, and relate ω_P to ω_s and I using its definition.

Use its definition to express the precession rate of the earth as a giant gyroscope:

$$\omega_P = \frac{\tau}{L}$$

Substitute for I and solve for τ to obtain:

$$\tau = L\omega_P = I\omega\omega_P$$

The angular velocity ω_S of the earth about its spin axis is given by: $\omega = \frac{2\pi}{T}$ where T is the period of rotation of the earth.

Substitute for ω to obtain: $\tau = \frac{2\pi I\omega_P}{T}$

Substitute numerical values and evaluate τ:

$$\tau = \frac{2\pi\left(8.03\times10^{37}\ \text{kg}\cdot\text{m}^2\right)\left(7.66\times10^{-12}\ \text{s}^{-1}\right)}{1\,\text{d}\times\frac{24\,\text{h}}{\text{d}}\times\frac{3600\,\text{s}}{\text{h}}} = \boxed{4.47\times10^{22}\ \text{N}\cdot\text{m}}$$

84 •• As indicated in the text, according to the Standard Model of Particle Physics, electrons are point-like particles having no spatial extent. (This assumption has been confirmed experimentally, and the radius of the electron has been shown to be less than 10^{-18} m.) The intrinsic spin of an electron could *in principle* be due to its rotation. Let's check to see if this conclusion is feasible. (*a*) Assuming that the electron is a uniform sphere whose radius is 1.00×10^{-18} m, what angular speed would be necessary to produce the observed intrinsic angular momentum of $\hbar/2$? (*b*) Using this value of angular speed, show that the speed of a point on the "equator" of a "spinning" electron would be moving faster than the speed of light. What is your conclusion about the spin angular momentum being analogous to a spinning sphere with spatial extent?

Picture the Problem We can use the definition of the angular momentum of a spinning sphere, together with the expression for its moment of inertia, to find the angular speed of a point on the surface of a spinning electron. The speed of such a point is directly proportional to the angular speed of the sphere.

(*a*) Express the angular momentum of the spinning electron: $L = I\omega = \frac{1}{2}\hbar$

Assuming a spherical electron of radius *R,* its moment of inertia, relative to its spin axis, is: $I = \frac{2}{5}MR^2$

Substituting for I yields: $\frac{2}{5}MR^2\omega = \frac{1}{2}\hbar \Rightarrow \omega = \frac{5\hbar}{4MR^2}$

Substitute numerical values and evaluate ω:

$$\omega = \frac{5\left(1.05\times10^{-34}\ \text{J}\cdot\text{s}\right)}{4\left(9.11\times10^{-31}\ \text{kg}\right)\left(10^{-18}\ \text{m}\right)^2} = \boxed{1.44\times10^{32}\ \text{rad/s}}$$

(*b*) The speed of a point on the "equator" of a spinning electron of radius R is given by:

$$v = R\omega$$

Substitute numerical values and evaluate v:

$$v = \left(10^{-18}\ \text{m}\right)\left(1.44\times10^{32}\ \text{rad/s}\right) = \boxed{1.44\times10^{14}\ \text{m/s}} > c$$

Given that our model predicts a value for the speed of a point on the "equator" of a spinning electron that is greater than the speed of light, the idea that the spin angular momentum of an electron is analogous to that of a spinning sphere with spatial extent lacks credibility.

85 •• An interesting phenomenon occurring in certain *pulsars* (see Problem 26) is an event known as a "spin glitch," that is, a quick change in the spin rate of the pulsar due to a shift in mass location and a resulting rotational inertia change. Imagine a pulsar whose radius is 10.0 km and whose period of rotation is 25.032 ms. The rotation period is observed to suddenly decrease from 25.032 ms to 25.028 ms. If that decrease was related to a contraction of the star, by what amount would the pulsar radius have had to change?

Picture the Problem We can apply the conservation of angular momentum to the shrinking pulsar to relate its radii to the observed periods.

The change in the radius of the pulsar is:

$$\Delta R = R_f - R_i \qquad (1)$$

Apply conservation of angular momentum to the shrinking pulsar to obtain:

$$\Delta L = L_f - L_i = 0$$

or

$$I_f\omega_f - I_i\omega_i = 0$$

Substituting for I_f and I_i yields:

$$\tfrac{2}{5}MR_f^2\omega_f - \tfrac{2}{5}MR_i^2\omega_i = 0$$

Solve for ω_f to obtain:

$$\omega_f = \frac{R_i^2}{R_f^2}\omega_i$$

Because $\omega = 2\pi/T$, where T is the rotation period:

$$\frac{2\pi}{T_f} = \frac{R_i^2}{R_f^2}\frac{2\pi}{T_i} \Rightarrow R_f = \sqrt{\frac{T_f}{T_i}}R_i$$

Substitute for R_f in equation (1) and simplify to obtain:

$$\Delta R = \sqrt{\frac{T_f}{T_i}}R_i - R_i = \left(\sqrt{\frac{T_f}{T_i}} - 1\right)R_i$$

Substitute numerical values and evaluate ΔR:

$$\Delta R = \left(\sqrt{\frac{25.028\text{ ms}}{25.032\text{ ms}}} - 1\right)(10.0\text{ km})$$
$$= \boxed{-79.9\text{ cm}}$$

86 ••• Figure 10.60 shows a pulley in the form of a uniform disk with a rope hanging over it. The circumference of the pulley is 1.2 m and its mass is 2.2 kg. The rope is 8.0 m long and its mass is 4.8 kg. At the instant shown in the figure, the system is at rest and the difference in height of the two ends of the rope is 0.60 m. (*a*) What is the angular speed of the pulley when the difference in height between the two ends of the rope is 7.2 m? (*b*) Obtain an expression for the angular momentum of the system as a function of time while neither end of the rope is above the center of the pulley. There is no slippage between rope and pulley wheel.

Picture the Problem Let the origin of the coordinate system be at the center of the pulley with the upward direction positive. Let λ be the linear density (mass per unit length) of the rope and L_1 and L_2 the lengths of the hanging parts of the rope. We can use conservation of mechanical energy to find the angular velocity of the pulley when the difference in height between the two ends of the rope is 7.2 m.

(*a*) Apply conservation of energy to relate the final kinetic energy of the system to the change in potential energy:

$$\Delta K + \Delta U = 0$$

or, because $K_i = 0$,

$$K + \Delta U = 0 \qquad (1)$$

Express the change in potential energy of the system:

$$\Delta U = U_f - U_i = -\tfrac{1}{2}L_{1f}(L_{1f}\lambda)g - \tfrac{1}{2}L_{2f}(L_{2f}\lambda)g - \left[-\tfrac{1}{2}L_{1i}(L_{1i}\lambda)g - \tfrac{1}{2}L_{2i}(L_{2i}\lambda)g\right]$$
$$= -\tfrac{1}{2}\left(L_{1f}^2 + L_{2f}^2\right)\lambda g + \tfrac{1}{2}\left(L_{1i}^2 + L_{2i}^2\right)\lambda g$$
$$= -\tfrac{1}{2}\lambda g\left[\left(L_{1f}^2 + L_{2f}^2\right) - \left(L_{1i}^2 + L_{2i}^2\right)\right]$$

Because $L_1 + L_2 = 7.4$ m, $L_{2i} - L_{1i} = 0.6$ m, and $L_{2f} - L_{1f} = 7.2$ m, we obtain:

$L_{1i} = 3.4$ m, $L_{2i} = 4.0$ m, $L_{1f} = 0.1$ m, and $L_{2f} = 7.3$ m.

Substitute numerical values and evaluate ΔU:

$$\Delta U = -\tfrac{1}{2}(0.60\,\text{kg/m})(9.81\,\text{m/s}^2)\left[(0.10\,\text{m})^2 + (7.3\,\text{m})^2 - (3.4\,\text{m})^2 - (4.0\,\text{m})^2\right]$$
$$= -75.75\,\text{J}$$

Express the kinetic energy of the system when the difference in height between the two ends of the rope is 7.2 m:

$$K = \tfrac{1}{2}I_\text{p}\omega^2 + \tfrac{1}{2}Mv^2 = \tfrac{1}{2}\left(\tfrac{1}{2}M_\text{p}R^2\right)\omega^2 + \tfrac{1}{2}MR^2\omega^2 = \tfrac{1}{2}\left(\tfrac{1}{2}M_\text{p} + M\right)R^2\omega^2$$

Substitute numerical values and simplify:

$$K = \tfrac{1}{2}\left[\tfrac{1}{2}(2.2\,\text{kg}) + 4.8\,\text{kg}\right]\left(\frac{1.2\,\text{m}}{2\pi}\right)^2\omega^2 = \left(0.1076\,\text{kg}\cdot\text{m}^2\right)\omega^2$$

Substitute in equation (1) and solve for ω:

$$\left(0.1076\,\text{kg}\cdot\text{m}^2\right)\omega^2 - 75.75\,\text{J} = 0$$

and

$$\boldsymbol{\omega} = \sqrt{\frac{75.75\,\text{J}}{0.1076\,\text{kg}\cdot\text{m}^2}} = \boxed{27\,\text{rad/s}}$$

(*b*) Noting that the moment arm of each portion of the rope is the same, express the total angular momentum of the system:

$$L = L_\text{p} + L_\text{r} = I_\text{p}\omega + M_\text{r}R^2\omega = \left(\tfrac{1}{2}M_\text{p}R^2 + M_\text{r}R^2\right)\omega = \left(\tfrac{1}{2}M_\text{p} + M_\text{r}\right)R^2\omega \qquad (2)$$

Letting θ be the angle through which the pulley has turned, express $U(\theta)$:

$$U(\theta) = -\tfrac{1}{2}\left[(L_{1i} - R\theta)^2 + (L_{2i} + R\theta)^2\right]\lambda g$$

Express ΔU and simplify to obtain:

$$\Delta U = U_\text{f} - U_\text{i} = U(\theta) - U(0) = -\tfrac{1}{2}\left[(L_{1i} - R\theta)^2 + (L_{2i} + R\theta)^2\right]\lambda g + \tfrac{1}{2}\left(L_{1i}^2 + L_{2i}^2\right)\lambda g = -R^2\theta^2\lambda g + (L_{1i} - L_{2i})R\theta\lambda g$$

Assuming that, at $t = 0$, $L_{1i} \approx L_{2i}$:

$$\Delta U \approx -R^2\theta^2\lambda g$$

Substitute for K and ΔU in equation (1) to obtain:

$$\left(0.1076\,\text{kg}\cdot\text{m}^2\right)\omega^2 - R^2\theta^2\lambda g = 0$$

Solving for ω yields:

$$\omega = \sqrt{\frac{R^2\theta^2\lambda g}{0.1076\,\text{kg}\cdot\text{m}^2}}$$

Substitute numerical values to obtain:

$$\omega = \sqrt{\frac{\left(\frac{1.2\,\text{m}}{2\pi}\right)^2(0.6\,\text{kg/m})\left(9.81\,\text{m/s}^2\right)}{0.1076\,\text{kg}\cdot\text{m}^2}}\,\theta = \left(1.41\,\text{s}^{-1}\right)\theta$$

Express ω as the rate of change of θ:

$$\frac{d\theta}{dt} = \left(1.41\,\text{s}^{-1}\right)\theta \Rightarrow \frac{d\theta}{\theta} = \left(1.41\,\text{s}^{-1}\right)dt$$

Integrate θ from 0 to θ to obtain:

$$\ln\theta = \left(1.41\,\text{s}^{-1}\right)t$$

Transform from logarithmic to exponential form to obtain:

$$\theta(t) = e^{\left(1.41\,\text{s}^{-1}\right)t}$$

Differentiate to express ω as a function of time:

$$\omega(t) = \frac{d\theta}{dt} = \left(1.41\,\text{s}^{-1}\right)e^{\left(1.41\,\text{s}^{-1}\right)t}$$

Substitute for ω in equation (2) to obtain:

$$L = \left(\tfrac{1}{2}M_\text{p} + M_\text{r}\right)R^2\left(1.41\,\text{s}^{-1}\right)e^{\left(1.41\,\text{s}^{-1}\right)t}$$

Substitute numerical values and evaluate L:

$$\boldsymbol{L} = \left[\tfrac{1}{2}(2.2\,\text{kg}) + (4.8\,\text{kg})\right]\left(\frac{1.2\,\text{m}}{2\pi}\right)^2\left[\left(1.41\,\text{s}^{-1}\right)e^{\left(1.41\,\text{s}^{-1}\right)t}\right] = \boxed{\left(0.30\,\text{kg}\cdot\text{m}^2/\text{s}\right)e^{\left(1.41\,\text{s}^{-1}\right)t}}$$