5-5: Theorems About Roots of Polynomial Equations

Algebra 2

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Rational Root Theorem

Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial with integer coefficients. There are a limited number of possible roots of $P(x) = 0$.

- Integer roots must be factors of a_0.
- Rational roots must have reduced form $\frac{p}{q}$ where p is an integer factor of a_0 and q is an integer factor of a_n.

Factors of the leading coefficient: $\pm1, \pm3, \pm7, \pm21$

Factors of the constant: $\pm1, \pm2, \pm5, \pm10$

$$21x^2 + 29x + 10 = 0$$

$$\frac{\text{Factors of constant term } (10)}{\text{Factors of leading coefficient } (21)}$$

Use synthetic division and each combination of factors to find roots

Ex. 1: What are the rational roots of $x^3 - 2x^2 - 5x + 6 = 0$?

Factors of a_0 (6): $\pm 1, \pm 2, \pm 3, \pm 6$

Factors of a_n (1): ± 1

$\frac{\text{Factors of } a_0 \text{ (6)}}{\text{Factors of } a_n \text{ (1)}}$: $\pm\frac{1}{1}, \pm\frac{2}{1}, \pm\frac{3}{1}, \pm\frac{6}{1}$

Possible Roots: $\pm 1, \pm 2, \pm 3, \pm 6$

Test possible roots using synthetic division & $x^3 - 2x^2 - 5x + 6 = 0$?

-1	1	-2	-5	6
		-1	3	2
	1	-3	-2	8

Remainder ≠ 0; Not a root.

1	1	-2	-5	6
		1	-1	-6
	1	-1	-6	0

Remainder = 0; Is a root.

Rewrite $x^3 - 2x^2 - 5x + 6 = 0$ as $(x-1)(x^2 - x - 6) = 0$ and use appropriate method to solve the quadratic (factoring, Quadratic Formula, etc.).

Factor $x^2 - x - 6 \Rightarrow (x-3)(x+2)$

Linear Factors: $(x-1)(x-3)(x+2)$

Zeroes: x = -2, 1, 3

Complete Got It? #2 p.313

$-{}^{3}/_{2}, -1, 2$

Rational Root Theorem Summary

1. Find all factors of constant term and lead coefficient.

2. Put them all in the ratio $\frac{Constant\ Factors}{Lead\ Coeff.Factors}$

3. Use synthetic division to find the roots

   a) Repeat until you can solve the remaining polynomial or until you have a second-degree polynomial and can solve by factoring or the Quadratic Formula.

Conjugate Root Theorem

If $P(x)$ is a polynomial with *rational* coefficients, then irrational roots of $P(x) = 0$ that have the form $a + \sqrt{b}$ occur in conjugate pairs. That is, if $a + \sqrt{b}$ is an irrational root with a and b rational, then $a - \sqrt{b}$ is also a root.

If $P(x)$ is a polynomial with *real* coefficients, then the complex roots of $P(x) = 0$ occur in conjugate pairs. That is, if $a + bi$ is an irrational root with a and b real, then $a - bi$ is also a root.

Ex. 2: A quintic polynomial $P(x)$ has rational coefficients. If $1, \sqrt{3}$ and $2 - 3i$ are roots of $P(x) = 0$, what are the remaining roots?

$-\sqrt{3}$ and $2 + 31$. 1 is neither irrational or complex so doesn't have a conjugate.

Got It? #3 p. 314 $3 + 2i$

Ex. 3: What polynomial function $P(x)$ with rational coefficients so that $P(x) = 0$ has roots 4 and $3i$? Since $3i$ is a root, so is $-3i$.

Find the linear factors and write the polynomial in factored form.

$$(x-4)(x-3i)(x+3i)=0$$

Multiply the linear factors and write the polynomial in standard form.

$$(x-4)(x-3i)(x+3i)=0$$

$$(x-4)(x^2+9)=0$$

$$x^3-4x^2+9x-36=0$$

Got It? #4 p. 314 $x^4 - 6x^3 + 16x^2 - 26x + 15$

Descartes' Rule of Signs

Let $P(x)$ be a polynomial with <u>*real*</u> coefficients written in ***standard form***.

- The number of *positive* real roots of $P(x) = 0$ is either equal to the number of sign changes between consecutive coefficients of $P(x)$or is less than that by an even number.
- The number of *negative* real roots of $P(x) = 0$ is either equal to the number of sign changes between consecutive coefficients of $P(-x)$ or is less than that by an even number.

In both cases, count multiple roots according to their multiplicity.

Ex. 4: What does Descartes' Rule of Signs tell you about the real roots of $P(x) = -x^4 + x^3 - 2x^2 + x + 1 = 0$?

$P(x) = -x^4 + x^3 - 2x^2 + x + 1 = 0$

1 1 1 0

3 sign changes

3 or 1 *positive* real roots

$P(-x) = -(-x)^4 + (-x)^3 - 2(-x)^2 + (-)x + 1 = 0$

$P(x) = -x^4 - x^3 - 2x^2 - x + 1 = 0$

0 0 0 1

1 sign change

1 *negative* real root

Complete Got It? #5 p. 315

a. 3 or 1 positive real roots; 1 negative real root
b. Real roots can but complex roots have an imaginary component so can't.

Homework: p.316 #9-21 odd, 25, 26, 30, 32, 50-54

Ex. 5: Find the zeroes for $P(x) = 4x^4 + 4x^3 + 17x^2 + 16x + 4 = 0$

$P(x) = 4x^4 + 4x^3 + 17x^2 + 16x + 4 = 0$ 0 sign changes

0 0 0 0

0 *positive* real roots

$P(-x) = 4x^4 - 4x^3 + 17x^2 - 16x + 4 = 0$ 4 sign changes

1 1 1 1

4, 2 or 0 *positive* real roots

Factors of a_0 (4): $\pm 1, \pm 2, \pm 4$

Factors of a_n (4): $\pm 1, \pm 2, \pm 4$

Possible Roots: $\pm \frac{1}{4}, \pm \frac{1}{2}, \pm 1, \pm 2, \pm 4$

-1	4	4	17	16	4
		-4	0	-17	1
	4	0	17	-1	3

Remainder ≠ 0; Not a root.

$-\frac{1}{2}$	4	4	17	16	4
		-2	-1	-8	-4
	4	2	16	8	0

Remainder = 0; Is a root.

$$x = -\frac{1}{2}$$

$$2x + 1 = 0$$

$$\begin{array}{r|rrrrr} -\frac{1}{2} & 4 & 4 & 17 & 16 & 4 \\ & & -2 & -1 & -8 & -4 \\ \hline & 4 & 2 & 16 & 8 & 0 \end{array}$$

$$(2x+1)\left((4x^3 + 2x^2) + (16x + 8)\right) = 0$$

$$(2x+1)\left(x^2(2x+1) + 4(2x+1)\right) = 0$$

$$(2x+1)\left(x^2 + 4\right)(2x+1) = 0$$

$$2x + 1 = 0 \qquad x = -\frac{1}{2}$$

$$2x + 1 = 0 \qquad x = -\frac{1}{2}$$

$$x^2 + 4 = 0$$

$$x^2 = -4$$

$$x = \pm\sqrt{-4}$$

$$x = \pm 2i$$