## Chapter 28 Magnetic Induction

## Conceptual Problems

1 - [SSM] (a) The magnetic equator is a line on the surface of Earth on which Earth's magnetic field is horizontal. At the magnetic equator, how would you orient a flat sheet of paper so as to create the maximum magnitude of magnetic flux through it? (b) How about the minimum magnitude of magnetic flux?

## Determine the Concept

(a) Orient the sheet so the normal to the sheet is both horizontal and perpendicular to the local tangent to the magnetic equator.
(b) Orient the sheet of paper so the normal to the sheet is perpendicular to the direction of the normal described in the answer to Part (a).

2 - At one of Earth's magnetic poles, how would you orient a flat sheet of paper so as to create the maximum magnitude of magnetic flux through it?

## Determine the Concept

(a) Orient the sheet so the normal to the sheet is vertical.
(b) Any orientation as long as the paper's plane is perpendicular to Earth's surface at that location.

3 - [SSM] Show that the following combination of SI units is equivalent to the volt: $\mathrm{T} \cdot \mathrm{m}^{2} / \mathrm{s}$.

Determine the Concept Because a volt is a joule per coulomb, we can show that the SI units $\frac{\mathrm{T} \cdot \mathrm{m}^{2}}{\mathrm{~s}}$ are equivalent to a volt by making a series of substitutions and simplifications that reduces these units to a joule per coulomb.

The units of a tesla are $\frac{\mathrm{N}}{\mathrm{A} \cdot \mathrm{m}}$ :

$$
\frac{\mathrm{T} \cdot \mathrm{~m}^{2}}{\mathrm{~s}}=\frac{\frac{\mathrm{N}}{\mathrm{~A} \cdot \mathrm{~m}} \cdot \mathrm{~m}^{2}}{\mathrm{~s}}=\frac{\frac{\mathrm{N} \cdot \mathrm{~m}}{\mathrm{~A}}}{\mathrm{~s}}
$$

Substitute the units of an ampere (C/s), replace $N \cdot m$ with $J$, and simplify to obtain:

$$
\frac{\mathrm{T} \cdot \mathrm{~m}^{2}}{\mathrm{~s}}=\frac{\frac{\mathrm{J}}{\mathrm{C}}}{\mathrm{~s}}=\frac{\mathrm{J}}{\mathrm{C}}
$$

$\begin{aligned} & \text { Finally, because a joule per coulomb } \\ & \text { is a volt: }\end{aligned} \quad \frac{\mathrm{T} \cdot \mathrm{m}^{2}}{\mathrm{~s}}=\mathrm{V}$,
4 - Show that the following combination of SI units is equivalent to the ohm: $\frac{\mathrm{Wb}}{\mathrm{A} \cdot \mathrm{s}}$.

Determine the Concept Because a weber is a newton meter per ampere, we can show that the SI units $\frac{\mathrm{Wb}}{\mathrm{A} \cdot \mathrm{s}}$ are equivalent to an ohm by making a series of substitutions and simplifications that reduces these units to a volt per ampere.

Because a weber is a $\frac{N \cdot m}{A}: \quad \frac{W b}{A \cdot s}=\frac{\frac{N \cdot m}{A}}{A \cdot s}=\frac{J}{A^{2} \cdot s}$
$\begin{aligned} & \text { Substitute the units of an ampere and } \\ & \text { simplify to obtain: }\end{aligned} \quad \frac{\mathrm{Wb}}{\mathrm{A} \cdot \mathrm{s}}=\frac{\mathrm{J}}{\mathrm{A} \cdot \frac{\mathrm{C}}{\mathrm{S}} \cdot \mathrm{s}}=\frac{\mathrm{J}}{\mathrm{A} \cdot \mathrm{C}}$
$\begin{aligned} & \text { Finally, because a joule per coulomb } \\ & \text { is a volt: }\end{aligned} \quad \frac{\mathrm{Wb}}{\mathrm{A} \cdot \mathrm{s}}=\frac{\mathrm{V}}{\mathrm{A}}=\Omega$
5 - [SSM] A current is induced in a conducting loop that lies in a horizontal plane and the induced current is clockwise when viewed from above. Which of the following statements could be true? (a) A constant magnetic field is directed vertically downward. (b) A constant magnetic field is directed vertically upward. (c) A magnetic field whose magnitude is increasing is directed vertically downward. (d) A magnetic field whose magnitude is decreasing is directed vertically downward. (e) A magnetic field whose magnitude is decreasing is directed vertically upward.

Determine the Concept We know that the magnetic flux (in this case the magnetic field because the area of the conducting loop is constant and its orientation is fixed) must be changing so the only issues are whether the field is increasing or decreasing and in which direction. Because the direction of the magnetic field associated with the clockwise current is vertically downward, the changing field that is responsible for it must be either increasing vertically upward (not included in the list of possible answers) or a decreasing field directed into the page. $(d)$ is correct.

6 - Give the direction of the induced current in the circuit, shown on the right in Figure 28-37, when the resistance in the circuit on the left is suddenly (a) increased and (b) decreased. Explain your answer.

Determine the Concept The induced emf and induced current in the circuit on the right are in such a direction as to oppose the change that produces them (Lenz's Law). We can determine the direction of the induced current in the circuit. Note that when $R$ is constant, $\overrightarrow{\boldsymbol{B}}$ in the circuit to the right points out of the paper.
(a) If $R$ increases, $I$ decreases and $B$ in the circuit to the right decreases. Lenz's law tells us that the induced current is counterclockwise.
(b) If $R$ decreases, $I$ increases and $B$ in the circuit to the right increases. Lenz's law tells us that the induced current is clockwise.

7 - [SSM] The planes of the two circular loops in Figure 28-38, are parallel. As viewed from the left, a counterclockwise current exists in loop A. If the magnitude of the current in loop A is increasing, what is the direction of the current induced in loop B? Do the loops attract or repel each other? Explain your answer.

Determine the Concept Clockwise as viewed from the left. The loops repel each other.

8 - A bar magnet moves with constant velocity along the axis of a loop, as shown in Figure 28-39, (a) Make a graph of the magnetic flux through the loop as a function of time. Indicate on the graph when the magnet is halfway through the loop by designating this time $t_{1}$. Choose the direction of the normal to the flat surface bounded by the flat surface to be to the right. (b) Make a graph of the induced current in the loop as a function of time. Choose the positive direction for the current to be clockwise as viewed from the left.

Determine the Concept We know that, as the magnet moves to the right, the flux through the loop first increases until the magnet is half way through the loop and then decreases. Because the flux first increases and then decreases, the current will change directions, having its maximum values when the flux is changing most rapidly.
(a) and (b) The following graph shows the flux and the induced current as a function of time as the bar magnet passes through the coil. When the center of the magnet passes through the plane of the coil $d \phi_{\mathrm{m}} / d t=0$ and the current is zero.


9 - A bar magnet is mounted on the end of a coiled spring and is oscillating in simple harmonic motion along the axis of a loop, as shown in Figure 28-40. The magnet is in its equilibrium position when its midpoint is in the plane of the loop. (a) Make a graph of the magnetic flux through the loop as a function of time. Indicate when the magnet is halfway through the loop by designating these times $t_{1}$ and $t_{2}$. (b) Make a graph of the induced current in the loop as a function of time, choosing the current to be positive when it is clockwise as viewed from above.

Determine the Concept Because the magnet moves with simple harmonic motion, the flux and the induced current will vary sinusoidally. The current will be a maximum wherever the flux is changing most rapidly and will be zero wherever the flux is momentarily constant. (a), (b) The following graph shows the flux, $\phi_{\mathrm{m}}$, and the induced current (proportional to $-d \phi_{\mathrm{m}} / d t$ ) in the loop as a function of time.


10 - A pendulum is fabricated from a thin, flat piece of aluminum. At the bottom of its arc, it passes between the poles of a strong permanent magnet. In Figure 28-41a, the metal sheet is continuous, whereas in Figure 28-41b, there are slots in the sheet. When released from the same angle, the pendulum that has slots swings back and forth many times, but the pendulum that does not have slots stops swinging after no more than one complete oscillation. Explain why.

Determine the Concept In the configuration shown in $(a)$, energy is dissipated by eddy currents from the emf induced by the pendulum movement. In the configuration shown in (b), the slits inhibit the eddy currents and the braking effect is greatly reduced.

11 - A bar magnet is dropped inside a long vertical tube. If the tube is made of metal, the magnet quickly approaches a terminal speed, but if the tube is made of cardboard, the magnet falls with constant acceleration. Explain why the magnet falls differently in the metal tube than it does in the cardboard tube.

Determine the Concept The magnetic field of the falling magnet sets up eddy currents in the metal tube. The eddy currents establish a magnetic field that exerts a force on the magnet opposing its motion; thus the magnet is slowed down. If the tube is made of a nonconducting material, there are no eddy currents.

12 - A small square wire loop lies in the plane of this page, and a constant magnetic field is directed into the page. The loop is moving to the right, which is the $+x$ direction. Find the direction of the induced current, if any, in the loop if (a) the magnetic field is uniform, $(b)$ the magnetic field strength increases as $x$ increases, and (c) the magnetic field strength decreases as $x$ increases.

Determine the Concept The direction of the induced current is in such a direction as to oppose, or tend to oppose, the change that produces it (Lenz's Law).
(a) Because the applied field is constant and uniform, there is no change in flux through the loop and, in accord with Faraday's law, no induced current in the loop.
(b) Let the positive normal direction on the flat surface bounded by the loop be into the page. Because the strength of the applied field increases to the right, the flux through the loop increases as it moves to the right. In accord with Lenz's law, the direction of the induced current will be such that the flux through the loop due to its magnetic field will be opposite in sign the change in flux of the applied field. Thus, on the flat surface bounded by the loop the magnetic field due to the induced current is out of the page. Using the right-hand rule, the induced current must be counterclockwise.
(c) Let the positive normal direction on the flat surface bounded by the loop be into the page. Because the strength of the applied field increases to the right, the flux through the loop decreases as it moves to the right. In accord with Lenz's law, the direction of the induced current will be such that the flux through the loop due to its magnetic field will be opposite in sign the change in flux of the applied field. Thus, on the flat surface bounded by the loop the magnetic field due to the induced current is into the page. Using the right-hand rule, the induced current must be clockwise.

13 - If the current in an inductor doubles, the energy stored in the inductor will (a) remain the same, (b) double, (c) quadruple, (d) halve.

Determine the Concept The magnetic energy stored in an inductor is given by $U_{\mathrm{m}}=\frac{1}{2} L I^{2}$. Doubling $I$ quadruples $U_{\mathrm{m}} .(c)$ is correct.

14 - Two solenoids are equal in length and radius, and the cores of both are identical cylinders of iron. However, solenoid A has three times the number of turns per unit length as solenoid B. (a) Which solenoid has the larger selfinductance? (b) What is the ratio of the self-inductance of solenoid A to the selfinductance of solenoid B?

Determine the Concept The self-inductance of a coil is given by $L=\mu_{0} \boldsymbol{n}^{2} A \ell$, where $n$ is the number of turns per unit length and $\ell$ is the length of the coil.
(a) Because the two solenoids are equal in length and radius and have identical cores, their self-inductances are proportional to the square of their number of turns per unit length. Hence A has the larger self-inductance.
(b) The self-inductances of the two coils are given by:

Divide the self-inductance of coil A by the self-inductance of coil B and simplify to obtain:

$$
\begin{aligned}
& L_{\mathrm{A}}=\mu_{0} n_{\mathrm{A}}^{2} A_{\mathrm{A}} \ell_{\mathrm{A}} \\
& \text { and } \\
& L_{\mathrm{B}}=\mu_{0} n_{\mathrm{B}}^{2} A_{\mathrm{B}} \ell_{\mathrm{B}}
\end{aligned}
$$

$$
\frac{L_{\mathrm{A}}}{L_{\mathrm{B}}}=\frac{\mu_{0} n_{\mathrm{A}}^{2} A_{\mathrm{A}} \ell_{\mathrm{A}}}{\mu_{0} n_{\mathrm{B}}^{2} A_{\mathrm{B}} \ell_{\mathrm{B}}}=\frac{n_{\mathrm{A}}^{2} A_{\mathrm{A}} \ell_{\mathrm{A}}}{n_{\mathrm{B}}^{2} A_{\mathrm{B}} \ell_{\mathrm{B}}}
$$

or, because the coils have the same lengths and radii (hence, the same cross-sectional areas),

$$
\frac{L_{\mathrm{A}}}{L_{\mathrm{B}}}=\frac{n_{\mathrm{A}}^{2}}{n_{\mathrm{B}}^{2}}=\left(\frac{n_{\mathrm{A}}}{n_{\mathrm{B}}}\right)^{2}
$$

If $n$ increases by a factor of $3, \ell$ will
decrease by the same factor, because

$$
\frac{L_{\mathrm{A}}}{L_{\mathrm{B}}}=\left(\frac{3 n_{\mathrm{B}}}{n_{\mathrm{B}}}\right)^{2}=9
$$

the inductors are made from the same length of wire. Hence:

15 - [SSM] True or false:
(a) The induced emf in a circuit is equal to the negative of the magnetic flux through the circuit.
(b) There can be a non-zero induced emf at an instant when the flux through the circuit is equal to zero.
(c) The self inductance of a solenoid is proportional to the rate of change of the current in the solenoid.
(d) The magnetic energy density at some point in space is proportional to the square of the magnitude of the magnetic field at that point.
(e) The inductance of a solenoid is proportional to the current in it.
(a) False. The induced emf in a circuit is equal to the rate of change of the magnetic flux through the circuit.
(b) True. The rate of change of the magnetic flux can be non-zero when the flux through the circuit is momentarily zero
(c) False. The self inductance of a solenoid is determined by its length, crosssectional area, number of turns per unit length, and the permeability of the matter in its core.
(d) True. The magnetic energy density at some point in space is given by Equation 28-20: $u_{\mathrm{m}}=\frac{B^{2}}{2 \mu_{0}}$.
(e) False. The inductance of a solenoid is determined by its length, cross-sectional area, number of turns per unit length, and the permeability of the matter in its core.

## Estimation and Approximation

16 - Your baseball teammates, having just studied this chapter, are concerned about generating enough voltage to shock them while swinging aluminum bats at fast balls. Estimate the maximum possible motional emf measured between the ends of an aluminum baseball bat during a swing. Do you think your team should switch to wooden bats to avoid electrocution?

Picture the Problem The bat is swung in Earth's magnetic field. We'll assume that the batter swings such that the maximum linear velocity of the bat occurs at
an angle such that it is moving perpendicular to Earth's field (i.e. when the bat is aligned north-south and moving east-west). The induced emf in the bat is given by $\mathcal{E}=v B \ell$. A bat is roughly 1 m long, and at most its center is probably moving at 75 mph , or about $33 \mathrm{~m} / \mathrm{s}$. Earth's magnetic field is about 0.3 G .

The emf induced in the bat is given $\quad \mathcal{E}=v B \ell$ by:

Under the conditions resulting in a maximum induced emf outlined above:

$$
\begin{aligned}
\mathcal{E} & =(33 \mathrm{~m} / \mathrm{s})\left(0.3 \mathrm{G} \times \frac{1 \mathrm{~T}}{10^{4} \mathrm{G}}\right)(1 \mathrm{~m}) \\
& \approx 1 \mathrm{mV}
\end{aligned}
$$

Because 1 mV is so low, there is no danger of being shocked and no reason to switch to wooden bats.

17 - Compare the energy density stored in Earth's electric field near its surface to that stored in Earth's magnetic field near its surface.

Picture the Problem We can compare the energy density stored in Earth's electric field to that of Earth's magnetic field by finding their ratio. We'll take Earth's magnetic field to be 0.3 G and its electric field to be $100 \mathrm{~V} / \mathrm{m}$.

The energy density in an electric

$$
u_{\mathrm{e}}=\frac{1}{2} \epsilon_{0} E^{2}
$$

field $E$ is given by:

The energy density in a magnetic field $B$ is given by:

$$
u_{\mathrm{m}}=\frac{B^{2}}{2 \mu_{0}}
$$

Express the ratio of $u_{\mathrm{m}}$ to $u_{\mathrm{e}}$ to obtain:

$$
\frac{u_{\mathrm{m}}}{u_{\mathrm{e}}}=\frac{\frac{B^{2}}{2 \mu_{0}}}{\frac{1}{2} \epsilon_{0} E^{2}}=\frac{B^{2}}{\mu_{0} \epsilon_{0} E^{2}}
$$

Substitute numerical values and evaluate $u_{\mathrm{m}} / u_{\mathrm{e}}$ :

$$
\frac{u_{\mathrm{m}}}{u_{\mathrm{e}}}=\frac{\left(0.3 \mathrm{G} \times \frac{1 \mathrm{~T}}{10^{4} \mathrm{G}}\right)^{2}}{\left(4 \pi \times 10^{-7} \mathrm{~N} / \mathrm{A}^{2}\right)\left(8.854 \times 10^{-12} \mathrm{C}^{2} / \mathrm{N} \cdot \mathrm{~m}^{2}\right)(100 \mathrm{~V} / \mathrm{m})^{2}}=8.09 \times 10^{3}
$$

or

$$
u_{\mathrm{m}} \approx\left(8 \times 10^{3}\right) u_{\mathrm{e}}
$$

18 •• A physics teacher does the following emf demonstration. She has two students hold a long wire connected to a voltmeter. The wire is held slack, so that it sags with a large arc in it. When she says "start, "the students begin rotating the wire as if they were playing jump rope. The students stand 3.0 m apart, and the sag in the wire is about 1.5 m . The motional emf from the "jump rope" is then measured on the voltmeter. (a) Estimate a reasonable value for the maximum angular speed that the students can rotate the wire. (b) From this, estimate the maximum motional emf in the wire. HINT: What field is involved in creating the induced emf?

Picture the Problem We can use Faraday's law to relate the motional emf in the wire to the angular speed with which the students turn the jump rope. Assume that Earth's magnetic field is 0.3 G .
(a) It seems unlikely that the students could turn the "jump rope" wire faster than $5.0 \mathrm{rev} / \mathrm{s}$.
(b) The magnetic flux $\phi_{\mathrm{m}}$ through the rotating circular loop of wire varies

$$
\phi_{\mathrm{m}}=B A \sin \omega t \Rightarrow \frac{d \phi_{\mathrm{m}}}{d t}=B A \omega \cos \omega t
$$ sinusoidally with time according to:

Because the average value of the cosine function, over one revolution, is $1 / 2$, the average rate at which the

$$
\left.\frac{d \phi_{\mathrm{m}}}{d t}\right|_{\mathrm{av}}=\frac{1}{2} B A \omega=\frac{1}{2} \pi r^{2} B \omega
$$ flux changes through the circular loop is:

From Faraday's law, the magnitude of the average motional emf in the

$$
\mathcal{E}=\frac{d \phi_{\mathrm{m}}}{d t}=\frac{1}{2} \pi r^{2} B \omega
$$ loop is:

Substitute numerical values and evaluate $\varepsilon$ :

$$
\mathcal{E}=\frac{1}{2} \pi\left(\frac{1.5 \mathrm{~m}}{2}\right)^{2}\left(0.3 \mathrm{G} \times \frac{1 \mathrm{~T}}{10^{4} \mathrm{G}}\right)(31.4 \mathrm{rad} / \mathrm{s}) \approx 0.8 \mathrm{mV}
$$

19 ••(a) Estimate the maximum possible motional emf between the wingtips of a typical commercial airliner in flight. (b) Estimate the magnitude of the electric field between the wingtips.

Picture the Problem The motional emf between the wingtips of an airliner is given by $\mathcal{E}=v B \ell$. Assume a speed, relative to Earth's magnetic field, of $500 \mathrm{mi} / \mathrm{h}$ or about $220 \mathrm{~m} / \mathrm{s}$ and a wingspan of 70 m . Assume that Earth's magnetic field is 0.3 G.
(a) The motional emf between the $\quad \mathcal{E}=v B \ell$ wingtips is given by:

Substitute numerical values and evaluate $\varepsilon$ :

$$
\begin{aligned}
\boldsymbol{\varepsilon} & =(220 \mathrm{~m} / \mathrm{s})\left(0.3 \mathrm{G} \times \frac{1 \mathrm{~T}}{10^{4} \mathrm{G}}\right)(70 \mathrm{~m}) \\
& \approx 0.5 \mathrm{~V}
\end{aligned}
$$

(b) The magnitude of the electric field between the wingtips is the ratio of the potential difference between them and their separation:

## Magnetic Flux

20 - A uniform magnetic field of magnitude 0.200 T is in the $+x$ direction. A square coil that has $5.00-\mathrm{cm}$ long sides has a single turn and makes an angle $\theta$ with the z axis, as shown in Figure 28-42. Find the magnetic flux through the coil when $\theta$ is $(a) 0^{\circ}$, (b) $30^{\circ}$, (c) $60^{\circ}$, and (d) $90^{\circ}$.

Picture the Problem Because the surface is a plane with area $A$ and $\overrightarrow{\boldsymbol{B}}$ is constant in magnitude and direction over the surface and makes an angle $\theta$ with the unit normal vector, we can use $\phi_{\mathrm{m}}=B A \cos \theta$ to find the magnetic flux through the coil.

The magnetic flux through the coil is

$$
\phi_{\mathrm{m}}=B A \cos \theta
$$

given by:

Substitute for $B$ and $A$ to obtain:

$$
\phi_{\mathrm{m}}=\left(2000 \mathrm{G} \cdot \frac{1 \mathrm{~T}}{10^{4} \mathrm{G}}\right)\left(5.00 \times 10^{-2} \mathrm{~m}\right)^{2} \cos \theta=\left(5.00 \times 10^{-4} \mathrm{~Wb}\right) \cos \theta
$$

(a) For $\theta=0^{\circ}$ :

$$
\begin{aligned}
\phi_{\mathrm{m}} & =\left(5.00 \times 10^{-4} \mathrm{~Wb}\right) \cos 0^{\circ} \\
& =5.00 \times 10^{-4} \mathrm{~Wb}=0.50 \mathrm{mWb}
\end{aligned}
$$

(b) For $\theta=30^{\circ}$ :

$$
\begin{aligned}
\phi_{\mathrm{m}} & =\left(5.00 \times 10^{-4} \mathrm{~Wb}\right) \cos 30^{\circ} \\
& =4.33 \times 10^{-4} \mathrm{~Wb}=0.43 \mathrm{mWb}
\end{aligned}
$$

(c) For $\theta=60^{\circ}$ :

$$
\begin{aligned}
\phi_{\mathrm{m}} & =\left(5.00 \times 10^{-4} \mathrm{~Wb}\right) \cos 60^{\circ} \\
& =2.50 \times 10^{-4} \mathrm{~Wb}=0.25 \mathrm{mWb}
\end{aligned}
$$

(d) For $\theta=90^{\circ}$ :

$$
\phi_{\mathrm{m}}=\left(5.00 \times 10^{-4} \mathrm{~Wb}\right) \cos 90^{\circ}=0
$$

21 - [SSM] A circular coil has 25 turns and a radius of 5.0 cm . It is at the equator, where Earth's magnetic field is 0.70 G , north. The axis of the coil is the line that passes through the center of the coil and is perpendicular to the plane of the coil. Find the magnetic flux through the coil when the axis of the coil is (a) vertical, (b) horizontal with the axis pointing north, (c) horizontal with the axis pointing east, and $(d)$ horizontal with the axis making an angle of $30^{\circ}$ with north.

Picture the Problem Because the coil defines a plane with area $A$ and $\overrightarrow{\boldsymbol{B}}$ is constant in magnitude and direction over the surface and makes an angle $\theta$ with the unit normal vector, we can use $\phi_{\mathrm{m}}=N B A \cos \theta$ to find the magnetic flux through the coil.

The magnetic flux through the

$$
\phi_{\mathrm{m}}=N B A \cos \theta=N B \pi r^{2} \cos \theta
$$ coil is given by:

Substitute for numerical values to obtain:

$$
\phi_{\mathrm{m}}=25\left(0.70 \mathrm{G} \cdot \frac{1 \mathrm{~T}}{10^{4} \mathrm{G}}\right) \pi\left(5.0 \times 10^{-2} \mathrm{~m}\right)^{2} \cos \theta=(13.7 \mu \mathrm{~Wb}) \cos \theta
$$

(a) When the plane of the coil is horizontal, $\theta=90^{\circ}$ :

$$
\phi_{\mathrm{m}}=(13.7 \mu \mathrm{~Wb}) \cos 90^{\circ}=0
$$

(b) When the plane of the coil is vertical with its axis pointing north, $\theta=0^{\circ}$ :
(c) When the plane of the coil is vertical with its axis pointing east, $\theta=90^{\circ}$ :
(d) When the plane of the coil is vertical with its axis making an angle

$$
\phi_{\mathrm{m}}=(13.7 \mu \mathrm{~Wb}) \cos 30^{\circ}=12 \mu \mathrm{~Wb}
$$ of $30^{\circ}$ with north, $\theta=30^{\circ}$ :

22 - A magnetic field of 1.2 T is perpendicular to the plane of a14 turn square coil with sides $5.0-\mathrm{cm}$ long. (a) Find the magnetic flux through the coil. (b) Find the magnetic flux through the coil if the magnetic field makes an angle of $60^{\circ}$ with the normal to the plane of the coil.

$$
\phi_{\mathrm{m}}=(13.7 \mu \mathrm{~Wb}) \cos 90^{\circ}=0
$$

$$
\phi_{\mathrm{m}}=(13.7 \mu \mathrm{~Wb}) \cos 0^{\circ}=14 \mu \mathrm{~Wb}
$$

Picture the Problem Because the square coil defines a plane with area $A$ and $\overrightarrow{\boldsymbol{B}}$ is constant in magnitude and direction over the surface and makes an angle $\theta$ with the unit normal vector, we can use $\phi_{\mathrm{m}}=N B A \cos \theta$ to find the magnetic flux through the coil.

The magnetic flux through the coil

$$
\phi_{\mathrm{m}}=N B A \cos \theta
$$

is given by:

Substitute numerical values for $N, B$, and $A$ to obtain:

$$
\begin{aligned}
\phi_{\mathrm{m}} & =14(1.2 \mathrm{~T})\left(5.0 \times 10^{-2} \mathrm{~m}\right)^{2} \cos \theta \\
& =(42.0 \mathrm{mWb}) \cos \theta
\end{aligned}
$$

(a) For $\theta=0^{\circ}$ :

$$
\phi_{\mathrm{m}}=(42.0 \mathrm{mWb}) \cos 0^{\circ}=42 \mathrm{mWb}
$$

(b) For $\theta=60^{\circ}$ :

$$
\phi_{\mathrm{m}}=(42.0 \mathrm{mWb}) \cos 60^{\circ}=21 \mathrm{mWb}
$$

23 - A uniform magnetic field $\overrightarrow{\boldsymbol{B}}$ is perpendicular to the base of a hemisphere of radius $R$. Calculate the magnetic flux (in terms of $B$ and $R$ ) through the spherical surface of the hemisphere.

Picture the Problem Noting that the flux through the base must also penetrate the spherical surface (the $\pm$ in the answer below), we can apply its definition to express $\phi_{\mathrm{m}}$.

Apply the definition of magnetic

$$
\phi_{\mathrm{m}}= \pm A B= \pm \pi R^{2} B
$$

flux to obtain:

24 - Find the magnetic flux through a 400-turn solenoid that has a length equal to 25.0 cm , has a radius equal to 1.00 cm , and carries a current of 3.00 A .

Picture the Problem We can use $\phi_{\mathrm{m}}=N B A \cos \theta$ to express the magnetic flux through the solenoid and $B=\mu_{0} n I$ to relate the magnetic field in the solenoid to the current in its coils. Assume that the magnetic field in the solenoid is constant.

Express the magnetic flux through a $\quad \phi_{\mathrm{m}}=N B A \cos \theta$ coil with $N$ turns:

Express the magnetic field inside a long solenoid:

$$
\phi_{\mathrm{m}}=N B A \cos \theta
$$

$$
B=\mu_{0} n I
$$

where $n$ is the number of turns per unit length.

Substitute to obtain:

$$
\begin{aligned}
& \phi_{\mathrm{m}}=N \mu_{0} n I A \cos \theta \\
& \text { or, because } n=N / L \text { and } \theta=0^{\circ}, \\
& \phi_{\mathrm{m}}=\frac{N^{2} \mu_{0} I A}{L}=\frac{N^{2} \mu_{0} I \pi r^{2}}{L}
\end{aligned}
$$

Substitute numerical values and evaluate $\phi_{\mathrm{m}}$ :

$$
\phi_{\mathrm{m}}=\frac{(400)^{2}\left(4 \pi \times 10^{-7} \mathrm{~N} / \mathrm{A}^{2}\right)(3.00 \mathrm{~A}) \pi(0.0100 \mathrm{~m})^{2}}{0.250 \mathrm{~m}}=758 \mu \mathrm{~Wb}
$$

25 - Find the magnetic flux through a 800-turn solenoid that has a length equal to 30.0 cm , has a radius equal to 1.00 cm , and carries a current of 2.00 A .

Picture the Problem We can use $\phi_{\mathrm{m}}=N B A \cos \theta$ to express the magnetic flux through the solenoid and $B=\mu_{0} n I$ to relate the magnetic field in the solenoid to the current in its coils. Assume that the magnetic field in the solenoid is constant.

Express the magnetic flux through a

$$
\phi_{\mathrm{m}}=N B A \cos \theta
$$

coil with $N$ turns:

Express the magnetic field inside a long solenoid:

$$
B=\mu_{0} n I
$$

where $n$ is the number of turns per unit length.

Substitute for $B$ to obtain:

$$
\begin{aligned}
& \phi_{\mathrm{m}}=N \mu_{0} n I A \cos \theta \\
& \text { or, because } n=N / L \text { and } \theta=0^{\circ}, \\
& \phi_{\mathrm{m}}=\frac{N^{2} \mu_{0} I A}{L}=\frac{N^{2} \mu_{0} I \pi r^{2}}{L}
\end{aligned}
$$

Substitute numerical values and evaluate $\phi_{\mathrm{m}}$ :

$$
\phi_{\mathrm{m}}=\frac{(800)^{2}\left(4 \pi \times 10^{-7} \mathrm{~N} / \mathrm{A}^{2}\right)(2.00 \mathrm{~A}) \pi(0.0200 \mathrm{~m})^{2}}{0.300 \mathrm{~m}}=6.74 \mathrm{mWb}
$$

26 •• A circular coil has 15.0 turns, has a radius 4.00 cm , and is in a uniform magnetic field of 4.00 kG in the $+x$ direction. Find the flux through the coil when the unit normal to the plane of the coil is $(a) \hat{\boldsymbol{i}},(b) \hat{\boldsymbol{j}},(c)(\hat{\boldsymbol{i}}+\hat{\boldsymbol{j}}) / \sqrt{2},(d) \hat{\boldsymbol{k}}$, and (e) $0.60 \hat{\mathbf{i}}+0.80 \hat{\boldsymbol{j}}$.

Picture the Problem We can apply the definitions of magnet flux and of the dot product to find the flux for the given unit vectors.

Apply the definition of magnetic flux to the coil to obtain:

$$
\phi_{\mathrm{m}}=N \int_{S} \overrightarrow{\boldsymbol{B}} \cdot \hat{\mathbf{n}} d A
$$

Because $\overrightarrow{\boldsymbol{B}}$ is constant:

$$
\begin{aligned}
\phi_{\mathrm{m}} & =N \overrightarrow{\boldsymbol{B}} \cdot \hat{\boldsymbol{n}} \int_{S} d A=N(\overrightarrow{\boldsymbol{B}} \cdot \hat{\boldsymbol{n}}) A \\
& =N(\overrightarrow{\boldsymbol{B}} \cdot \hat{\boldsymbol{n}}) \tau r^{2}
\end{aligned}
$$

Evaluate $\overrightarrow{\boldsymbol{B}}$ :

$$
\overrightarrow{\boldsymbol{B}}=(0.400 \boldsymbol{T}) \hat{\boldsymbol{i}}
$$

Substitute numerical values and simplify to obtain:

$$
\begin{aligned}
\boldsymbol{\phi}_{\mathrm{m}} & =(15.0)[(0.400 \mathrm{~T})] \pi(0.0400 \mathrm{~m})^{2} \\
& =\left(0.03016 \mathrm{~T} \cdot \mathrm{~m}^{2}\right) \hat{\boldsymbol{i}} \cdot \hat{\boldsymbol{n}}
\end{aligned}
$$

(a) Evaluate $\phi_{\mathrm{m}}$ for $\hat{\boldsymbol{n}}=\hat{\boldsymbol{i}}$ :

$$
\boldsymbol{\phi}_{\mathrm{m}}=\left(0.03016 \mathrm{~T} \cdot \mathrm{~m}^{2}\right) \hat{\boldsymbol{i}} \cdot \hat{\boldsymbol{i}}=30.2 \mathrm{mWb}
$$

(b) Evaluate $\phi_{\mathrm{m}}$ for $\hat{\boldsymbol{n}}=\hat{\boldsymbol{j}}$ :

$$
\boldsymbol{\phi}_{\mathrm{m}}=\left(0.03016 \mathrm{~T} \cdot \mathrm{~m}^{2}\right) \hat{\boldsymbol{i}} \cdot \hat{\boldsymbol{j}}=0
$$

(c) Evaluate $\phi_{\mathrm{m}}$ for $\hat{\boldsymbol{n}}=(\hat{\boldsymbol{i}}+\hat{\boldsymbol{j}}) / \sqrt{2}$ :

$$
\begin{aligned}
\boldsymbol{\phi}_{\mathrm{m}} & =\left(0.03016 \mathrm{~T} \cdot \mathrm{~m}^{2}\right) \hat{\boldsymbol{i}} \cdot \frac{(\hat{\boldsymbol{i}}+\hat{\boldsymbol{j}})}{\sqrt{2}} \\
& =\frac{0.03016 \mathrm{~T} \cdot \mathrm{~m}^{2}}{\sqrt{2}}=21.3 \mathrm{mWb}
\end{aligned}
$$

(d) Evaluate $\phi_{\mathrm{m}}$ for $\hat{\boldsymbol{n}}=\hat{\boldsymbol{k}}$ :

$$
\boldsymbol{\phi}_{\mathrm{m}}=\left(0.03016 \mathrm{~T} \cdot \mathrm{~m}^{2}\right) \hat{\boldsymbol{i}} \cdot \hat{\boldsymbol{k}}=0
$$

(e) Evaluate $\phi_{\mathrm{m}}$ for $\hat{\boldsymbol{n}}=0.60 \hat{\boldsymbol{i}}+0.80 \hat{\boldsymbol{j}}$ :

$$
\phi_{\mathrm{m}}=\left(0.03016 \mathrm{~T} \cdot \mathrm{~m}^{2}\right) \hat{\mathbf{i}} \cdot(0.60 \hat{\mathbf{i}}+0.80 \hat{\boldsymbol{j}})=18 \mathrm{mWb}
$$

27 •• [SSM] A long solenoid has $n$ turns per unit length, has a radius $R_{1}$, and carries a current $I$. A circular coil with radius $R_{2}$ and with $N$ total turns is coaxial with the solenoid and equidistant from its ends. (a) Find the magnetic flux through the coil if $R_{2}>R_{1}$. (b) Find the magnetic flux through the coil if $R_{2}<R_{1}$.

Picture the Problem The magnetic field outside the solenoid is, to a good approximation, zero. Hence, the flux through the coil is the flux in the core of the solenoid. The magnetic field inside the solenoid is uniform. Hence, the flux through the circular coil is given by the same expression with $R_{2}$ replacing $R_{1}$ :
(a) The flux through the large

$$
\phi_{\mathrm{m}}=N B A
$$

circular loop outside the solenoid is given by:

Substituting for $B$ and $A$ and

$$
\phi_{\mathrm{m}}=N\left(\mu_{0} n I\right)\left(\pi R_{1}^{2}\right)=\mu_{0} n I N \pi R_{1}^{2}
$$ simplifying yields:

(b) The flux through the coil when

$$
\phi_{\mathrm{m}}=N\left(\mu_{0} n I\right)\left(\pi R_{2}^{2}\right)=\mu_{0} n I N \pi R_{2}^{2}
$$

$R_{2}<R_{1}$ is given by:
28 ••• (a) Compute the magnetic flux through the rectangular loop shown in Figure 28-45. (b) Evaluate your answer for $a=5.0 \mathrm{~cm}, b=10 \mathrm{~cm}, d=2.0 \mathrm{~cm}$, and $I=20 \mathrm{~A}$.

Picture the Problem We can use the hint to set up the element of area $d A$ and express the flux $d \phi_{\mathrm{m}}$ through it and then carry out the details of the integration to express $\phi_{\mathrm{m}}$.
(a) The flux through the strip of area $d A$ is given by:

Express $B$ at a distance $x$ from a long, straight wire:

Substitute to obtain:

$$
d \phi_{\mathrm{m}}=B d A
$$

$$
\text { where } d A=b d x
$$

$$
B=\frac{\mu_{0}}{4 \pi} \frac{2 I}{x}=\frac{\mu_{0}}{2 \pi} \frac{I}{x}
$$

$$
d \phi_{\mathrm{m}}=\frac{\mu_{0}}{2 \pi} \frac{I}{x} b d x=\frac{\mu_{0} I b}{2 \pi} \frac{d x}{x}
$$

Integrate from $x=d$ to $x=d+a$ :

$$
\phi_{\mathrm{m}}=\frac{\mu_{0} I b}{2 \pi} \int_{d}^{d+a} \frac{d x}{x}=\frac{\mu_{0} I b}{2 \pi} \ln \left(\frac{d+a}{d}\right)
$$

(b) Substitute numerical values and evaluate $\phi_{\mathrm{m}}$ :

$$
\phi_{\mathrm{m}}=\frac{\left(4 \pi \times 10^{-7} \mathrm{~N} / \mathrm{A}^{2}\right)(20 \mathrm{~A})(0.10 \mathrm{~m})}{2 \pi} \ln \left(\frac{7.0 \mathrm{~cm}}{2.0 \mathrm{~cm}}\right)=0.50 \mu \mathrm{~Wb}
$$

29 ••• A long cylindrical conductor with a radius $R$ and a length $L$ carries a current $I$. Find the magnetic flux per unit length through the area indicated in Figure 28-44.

Picture the Problem Consider an element of area $d A=L d r$ where $r \leq R$. We can use its definition to express $d \phi_{\mathrm{m}}$ through this area in terms of $B$ and Ampere's law to express $B$ as a function of $I$. The fact that the current is uniformly distributed
over the cross-sectional area of the conductor allows us to set up a proportion from which we can obtain $I$ as a function of $r$. With these substitutions in place we can integrate $d \phi_{\mathrm{m}}$ to obtain $\phi_{\mathrm{m}} / L$.

Noting that $\vec{B}$ is parallel to $\hat{n} \quad d \phi_{\mathrm{m}}=B d A=B L d r$
over the entire area, express the
flux $d \phi_{\mathrm{m}}$ through an area $L d r$ :
Apply Ampere's law to the current

$$
\oint_{C} \overrightarrow{\boldsymbol{B}} \cdot \overrightarrow{d \ell}=2 \pi r B=\mu_{0} I_{C}
$$

contained inside a cylindrical region of radius $r<R$ :

Solving for $B$ yields:

$$
\begin{equation*}
B=\frac{\mu_{0} I_{C}}{2 \pi r} \tag{2}
\end{equation*}
$$

Using the fact that the current $I$ is uniformly distributed over the cross-

$$
\frac{I(r)}{I}=\frac{\pi r^{2}}{\pi R^{2}} \Rightarrow I(r)=I_{C}=I \frac{r^{2}}{R^{2}}
$$ sectional area of the conductor, express its variation with distance $r$ from the center of the conductor:

Substitute for $I_{C}$ in equation (2) and simplify to obtain:

$$
B=\frac{\mu_{0} I}{2 \pi r} \frac{r^{2}}{R^{2}}=\frac{\mu_{0} I}{2 \pi R^{2}} r
$$

Substituting for $B$ in equation (1) yields:

$$
d \phi_{\mathrm{m}}=\frac{\mu_{0} L I}{2 \pi R^{2}} r d r
$$

Integrate $d \phi_{\mathrm{m}}$ from $r=0$ to $r=R$ to obtain:

$$
\phi_{\mathrm{m}}=\frac{\mu_{0} L I}{2 \pi R^{2}} \int_{0}^{R} r d r=\frac{\mu_{0} L I}{4 \pi}
$$

Divide both sides of this equation by $L$ to express the magnetic flux per

$$
\frac{\phi_{\mathrm{m}}}{L}=\frac{\mu_{0} I}{4 \pi}
$$ unit length:

## Induced EMF and Faraday's Law

30 - The flux through a loop is given by $\phi_{\mathrm{m}}=0.10 t^{2}-0.40 t$, where $\phi_{\mathrm{m}}$ is in webers and $t$ is in seconds. (a) Find the induced emf as a function of time.
(b) Find both $\phi_{\mathrm{m}}$ and $\varepsilon$ at $t=0, t=2.0 \mathrm{~s}, t=4.0 \mathrm{~s}$, and $t=6.0 \mathrm{~s}$.

Picture the Problem Given $\phi_{\mathrm{m}}$ as a function of time, we can use Faraday's law to express $\mathcal{E}$ as a function of time.
(a) Apply Faraday's law to express the induced emf in the loop in terms of the rate of change of the magnetic flux:

$$
\begin{aligned}
\mathcal{E}(t) & =-\frac{d \phi_{\mathrm{m}}}{d t}=-\frac{d}{d t}\left[0.10\left(t^{2}-0.40 t\right)\right] \\
& =-(0.20 t-0.40) \frac{\mathrm{Wb}}{\mathrm{~s}} \\
& =-(0.20 t-0.40) \mathrm{V}
\end{aligned}
$$

where $\mathcal{E}$ is in volts and $t$ is in seconds.
(b) Evaluate $\phi_{\mathrm{m}}$ at $t=0$ :
$\phi_{\mathrm{m}}(0 \mathrm{~s})=0.10(0)^{2}-(0.40)(0)=0$

Evaluate $\mathcal{E}$ at $t=0$ :
$\varepsilon(0 \mathrm{~s})=-[0.20(0)-0.40] \mathrm{V}=0.40 \mathrm{~V}$

Proceed as above to complete the table to the right:

| $t$ | $\phi_{\mathrm{m}}$ | $\varepsilon$ |
| :---: | :---: | :---: |
| $(\mathrm{s})$ | $(\mathrm{Wb})$ | $(\mathrm{V})$ |
| 0 | 0 | 0.40 |
| 2.0 | -0.40 | 0.00 |
| 4.0 | 0.0 | -0.40 |
| 6.0 | 1.2 | -0.80 |

31 - The flux through a loop is given by $\phi_{\mathrm{m}}=0.10 t^{2}-0.40 t$, where $\phi_{\mathrm{m}}$ is in webers and $t$ is in seconds. (a) Sketch graphs of magnetic flux and induced emf as a function of time. (b) At what time(s) is the flux minimum? What is the induced emf at that (those) time(s)? (c) At what time(s) is the flux zero? What is (are) the induced emf(s) at those time(s)?

Picture the Problem We can find the time at which the flux is a minimum by looking for the lowest point on the graph of $\mathcal{E}$ versus $t$ and the emf at this time by determining the value of $V$ at this time from the graph. We can interpret the graphs to find the times at which the flux is zero and the corresponding values of the emf.
(a) The flux, $\phi_{\mathrm{m}}$, and the induced emf, $\mathcal{E}$, are shown as functions of $t$ in the following graph. The solid curve represents $\phi_{\mathrm{m}}$, the dashed curve represents $\boldsymbol{\varepsilon}$.

(b) Referring to the graph, we see that the flux is a minimum when $t=2.0 \mathrm{~s}$ and that $V(2.0 \mathrm{~s})=0$.
(c) The flux is zero when $t=0$ and $t=4.0 \mathrm{~s} . \boldsymbol{E}(0)=0.40 \mathrm{~V}$ and $\mathcal{E}(4.0 \mathrm{~s})=-0.40 \mathrm{~V}$.

32 - A solenoid that has a length equal to 25.0 cm , a radius equal to 0.800 cm , and 400 turns is in a region where a magnetic field of 600 G exists and makes an angle of $50^{\circ}$ with the axis of the solenoid. (a) Find the magnetic flux through the solenoid. (b) Find the magnitude of the average emf induced in the solenoid if the magnetic field is reduced to zero in 1.40 s .

Picture the Problem We can use its definition to find the magnetic flux through the solenoid and Faraday's law to find the emf induced in the solenoid when the external field is reduced to zero in 1.4 s .
(a) Express the magnetic flux through the solenoid in terms of $N, B, A$, and $\theta$ :

Substitute numerical values and evaluate $\phi_{\mathrm{m}}$ :

$$
\begin{aligned}
\phi_{\mathrm{m}} & =N B A \cos \theta \\
& =N B \pi R^{2} \cos \theta
\end{aligned}
$$

$$
\begin{aligned}
\phi_{\mathrm{m}} & =(400)(60.0 \mathrm{mT}) \pi(0.00800 \mathrm{~m})^{2} \cos 50^{\circ} \\
& =3.10 \mathrm{mWb}=3.1 \mathrm{mWb}
\end{aligned}
$$

(b) Apply Faraday’s law to obtain:

$$
\begin{aligned}
\varepsilon & =-\frac{\Delta \phi_{\mathrm{m}}}{\Delta t}=-\frac{0-3.10 \mathrm{mWb}}{1.40 \mathrm{~s}} \\
& =2.2 \mathrm{mV}
\end{aligned}
$$

33 •• [SSM] A 100-turn circular coil has a diameter of 2.00 cm , a resistance of $50.0 \Omega$, and the two ends of the coil are connected together. The plane of the coil is perpendicular to a uniform magnetic field of magnitude 1.00 T . The direction of the field is reversed. (a) Find the total charge that passes through a cross section of the wire. If the reversal takes 0.100 s , find $(b)$ the average current and $(c)$ the average emf during the reversal.

Picture the Problem We can use the definition of average current to express the total charge passing through the coil as a function of $I_{\mathrm{av}}$. Because the induced current is proportional to the induced emf and the induced emf, in turn, is given by Faraday's law, we can express $\Delta Q$ as a function of the number of turns of the coil, the magnetic field, the resistance of the coil, and the area of the coil. Knowing the reversal time, we can find the average current from its definition and the average emf from Ohm's law.
(a) Express the total charge that

$$
\begin{equation*}
\Delta Q=I_{\mathrm{av}} \Delta t \tag{1}
\end{equation*}
$$ passes through the coil in terms of the induced current:

Relate the induced current to the induced emf:

$$
I=I_{\mathrm{av}}=\frac{\varepsilon}{R}
$$

Using Faraday's law, express the induced emf in terms of $\phi_{\mathrm{m}}$ :

$$
\varepsilon=-\frac{\Delta \phi_{\mathrm{m}}}{\Delta t}
$$

Substitute in equation (1) and simplify to obtain:

$$
\begin{aligned}
\Delta Q & =\frac{|\varepsilon|}{R} \Delta t=\frac{\frac{\Delta \phi_{\mathrm{m}}}{\Delta t}}{R} \Delta t=\frac{2 \phi_{\mathrm{m}}}{R} \\
& =\frac{2 N B A}{R}=\frac{2 N B\left(\frac{\pi}{4} d^{2}\right)}{R} \\
& =\frac{N B \pi d^{2}}{2 R}
\end{aligned}
$$

where $d$ is the diameter of the coil.

Substitute numerical values and evaluate $\Delta Q$ :

$$
\begin{aligned}
\Delta Q & =\frac{(100)(1.00 \mathrm{~T}) \pi(0.0200 \mathrm{~m})^{2}}{2(50.0 \Omega)} \\
& =1.257 \mathrm{mC}=1.26 \mathrm{mC}
\end{aligned}
$$

(b) Apply the definition of average current to obtain:

$$
\begin{aligned}
I_{\mathrm{av}} & =\frac{\Delta Q}{\Delta t}=\frac{1.257 \mathrm{mC}}{0.100 \mathrm{~s}}=12.57 \mathrm{~mA} \\
& =12.6 \mathrm{~mA} \\
\mathcal{E}_{\mathrm{av}} & =I_{\mathrm{av}} R=(12.57 \mathrm{~mA})(50.0 \Omega) \\
& =628 \mathrm{mV}
\end{aligned}
$$

(c) Using Ohm's law, relate the average emf in the coil to the average current:

34 •• At the equator, a 1000-turn coil that has a cross-sectional area of $300 \mathrm{~cm}^{2}$ and a resistance of $15.0 \Omega$ is aligned so that its plane is perpendicular to Earth's magnetic field of 0.700 G . (a) If the coil is flipped over in 0.350 s , what is the average induced current in it during the 0.350 s ? (b) How much charge flows through a cross section of the coil wire during the 0.350 s ?

Picture the Problem (a) Because the average induced current is proportional to the induced emf and the induced emf, in turn, is given by Faraday's law, we can find $I_{\mathrm{av}}$ from the change in the magnetic flux through the coil, the resistance of the coil, and the time required for the flipping of the coil. (b) Knowing the average current, we can use its definition to find the charge flowing in the coil.
(a) The average induced current is given by:

$$
I_{\mathrm{av}}=\frac{\varepsilon}{R}
$$

The induced emf in the coil is the rate at which the magnetic flux is

$$
\varepsilon=\frac{\Delta \phi_{\mathrm{m}}}{\Delta t}=\frac{2 \phi_{\mathrm{m}}}{\Delta t}=\frac{2 N B A}{\Delta t}
$$ changing:

Substituting for $\mathcal{E}$ yields: $\quad I_{\mathrm{av}}=\frac{2 N B A}{R \Delta t}$

Substitute numerical values and evaluate $I_{\mathrm{av}}$ :

$$
I_{\mathrm{av}}=\frac{2(1000)\left(0.700 \mathrm{G} \times \frac{1 \mathrm{~T}}{10^{4} \mathrm{G}}\right)\left(300 \mathrm{~cm}^{2} \times\left(\frac{1 \mathrm{~m}}{10^{2} \mathrm{~cm}}\right)^{2}\right)}{(15.0 \Omega)(0.350 \mathrm{~s})}=800 \mu \mathrm{~A}
$$

(b) The average current is also given by:

Substitute numerical values and evaluate $\Delta Q$ :

$$
\begin{aligned}
I_{\mathrm{av}} & =\frac{\Delta Q}{\Delta t} \Rightarrow \Delta Q=I_{\mathrm{av}} \Delta t \\
\Delta Q & =(0.800 \mathrm{~mA})(0.350 \mathrm{~s}) \\
& =280 \mu \mathrm{C}
\end{aligned}
$$

35 •• A current integrator measures the current as a function of time and integrates (adds) the current to find the total charge passing through it. (Because $I=d q / d t$, the integrator calculates the integral of the current or $Q=\int I d t$. ) A circular coil that has 300 turns and a radius equal to 5.00 cm is connected to such an instrument. The total resistance of the circuit is $20.0 \Omega$. The plane of the coil is originally aligned perpendicular to Earth's magnetic field at some point. When the coil is rotated through $90^{\circ}$ about an axis that is in the plane of the coil, a charge of $9.40 \mu \mathrm{C}$ passes through the current integrator is measured to be $9.40 \mu \mathrm{C}$.
Calculate the magnitude of Earth's magnetic field at that point.
Picture the Problem We can use Faraday's law to express Earth's magnetic field at this location in terms of the induced emf and Ohm's law to relate the induced emf to the charge that passes through the current integrator.

Using Faraday's law, express the induced emf in terms of the change in the magnetic flux as the coil is rotated through $90^{\circ}$ :

Solving for $B$ yields:

$$
\varepsilon=\left|-\frac{\Delta \phi_{\mathrm{m}}}{\Delta t}\right|=\frac{N B A}{\Delta t}=\frac{N B \pi r^{2}}{\Delta t}
$$

$$
B=\frac{\varepsilon \Delta t}{N \pi r^{2}}
$$

Using Ohm's law, relate the induced emf to the induced current:

Substitute for $\mathcal{E}$ and simplify to obtain:

Substitute numerical values and evaluate $B$ :

$$
B=\frac{\frac{\Delta Q}{\Delta t} R \Delta t}{N \pi r^{2}}=\frac{\Delta Q R}{N \pi r^{2}}
$$

$$
\varepsilon=I R=\frac{\Delta Q}{\Delta t} R
$$

where $\Delta Q$ is the charge that passes through the current integrator.

$$
B=\frac{(9.40 \mu \mathrm{C})(20.0 \Omega)}{(300) \pi(0.0500 \mathrm{~m})^{2}}=79.8 \mu \mathrm{~T}
$$

## Motional EMF

36 •• A $30.0-\mathrm{cm}$ long rod moves steadily at $8.00 \mathrm{~m} / \mathrm{s}$ in a plane that is perpendicular to a magnetic field of 500 G . The velocity of the rod is perpendicular to its length. Find $(a)$ the magnetic force on an electron in the rod, (b) the electrostatic field in the rod, and (c) the potential difference between the ends of the rod.

Picture the Problem We can apply the equation for the force on a charged particle moving in a magnetic field to find the magnetic force acting on an electron in the rod. We can use $\overrightarrow{\boldsymbol{E}}=\overrightarrow{\boldsymbol{v}} \times \overrightarrow{\boldsymbol{B}}$ to find $E$ and $V=E \ell$, where $\ell$ is the length of the rod, to find the potential difference between its ends.
(a) Relate the magnetic force on an electron in the rod to the speed of the rod, the electronic charge, and the magnetic field in which the rod is moving:

Substitute numerical values and evaluate $F$ :

$$
\begin{aligned}
& \overrightarrow{\boldsymbol{F}}=q \overrightarrow{\boldsymbol{v}} \times \overrightarrow{\boldsymbol{B}} \\
& \text { and } \\
& F=q v B \sin \theta
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{F} & =\left(1.602 \times 10^{-19} \mathrm{C}\right)(8.00 \mathrm{~m} / \mathrm{s}) \\
& \times(0.0500 \mathrm{~T}) \sin 90^{\circ} \\
& =6.4 \times 10^{-20} \mathrm{~N}
\end{aligned}
$$

(b) Express the electrostatic field
$\overrightarrow{\boldsymbol{E}}$ in the rod in terms of the magnetic field $\overrightarrow{\boldsymbol{B}}$ :

Substitute numerical values and evaluate $B$ :
(c) Relate the potential difference between the ends of the rod to its length $\ell$ and the electric field $E$ :

Substitute numerical values and evaluate $V$ :
$\overrightarrow{\boldsymbol{E}}=\overrightarrow{\boldsymbol{v}} \times \overrightarrow{\boldsymbol{B}}$ and $E=v B \sin \theta$ where $\theta$ is the angle between $\overrightarrow{\boldsymbol{v}}$ and $\overrightarrow{\boldsymbol{B}}$.

$$
\begin{aligned}
E & =(8.00 \mathrm{~m} / \mathrm{s})(0.0500 \mathrm{~T}) \sin 90^{\circ} \\
& =0.400 \mathrm{~V} / \mathrm{m}=0.40 \mathrm{~V} / \mathrm{m}
\end{aligned}
$$

$$
V=E \ell
$$

$$
V=(0.400 \mathrm{~V} / \mathrm{m})(0.300 \mathrm{~m})=0.12 \mathrm{~V}
$$

37 •• A $30.0-\mathrm{cm}$ long rod moves in a plane that is perpendicular to a magnetic field of 500 G . The velocity of the rod is perpendicular to its length. Find the speed of the rod if the potential difference between the ends is 6.00 V .

Picture the Problem We can use $\overrightarrow{\boldsymbol{E}}=\overrightarrow{\boldsymbol{v}} \times \overrightarrow{\boldsymbol{B}}$ to relate the speed of the rod to the electric field in the rod and magnetic field in which it is moving and $V=E \ell$ to relate the electric field in the rod to the potential difference between its ends.

Express the electrostatic field $\overrightarrow{\boldsymbol{E}}$ in the rod in terms of the magnetic field $\overrightarrow{\boldsymbol{B}}$ and solve for $v$ :

Relate the potential difference between the ends of the rod to its length $\ell$ and the electric field $E$ and solve for $E$ :

Substitute for $E$ to obtain:

$$
v=\frac{V}{B \ell \sin \theta}
$$

$$
v=\frac{6.00 \mathrm{~V}}{(0.0500 \mathrm{~T})(0.300 \mathrm{~m})}=400 \mathrm{~m} / \mathrm{s}
$$

38 •• In Figure 28-45, let the magnetic field strength be 0.80 T , the rod speed be $10 \mathrm{~m} / \mathrm{s}$, the rod length be 20 cm , and the resistance of the resistor be $2.0 \Omega$. (The resistance of the rod and rails are negligible.) Find (a) the induced emf in the circuit, (b) the induced current in the circuit (including direction), and (c) the force needed to move the rod with constant speed (assuming negligible friction). Find (d) the power delivered by the force found in Part (c) and (e) the rate of Joule heating in the resistor.

Picture the Problem Because the speed of the rod is constant, an external force must act on the rod to counter the magnetic force acting on the induced current. We can use the motional-emf equation $\mathcal{E}=v B \ell$ to evaluate the induced emf, Ohm's law to find the current in the circuit, Newton's $2^{\text {nd }}$ law to find the force needed to move the rod with constant speed, and $P=F v$ to find the power input by the force.
(a) Relate the induced emf in the circuit to the speed of the rod, the magnetic field, and the length of the rod:

$$
\begin{aligned}
\mathcal{E} & =v B \ell=(10 \mathrm{~m} / \mathrm{s})(0.80 \mathrm{~T})(0.20 \mathrm{~m}) \\
& =1.60 \mathrm{~V}=1.6 \mathrm{~V}
\end{aligned}
$$

(b) Using Ohm's law, relate the current in the circuit to the induced emf and the resistance of the circuit:
(c) Because the rod is moving with
constant speed in a straight line, the
net force acting on it must be zero.
Apply Newton's $2^{\text {nd }}$ law to relate $F$
to the magnetic force $F_{\mathrm{m}}$ :

Solving for $F$ and substituting for $F_{\mathrm{m}} \quad F=F_{\mathrm{m}}=B I \ell$ yields:

Substitute numerical values and evaluate $F$ :

$$
\begin{aligned}
F & =(0.80 \mathrm{~T})(0.80 \mathrm{~A})(0.20 \mathrm{~m})=0.128 \mathrm{~N} \\
& =0.13 \mathrm{~N}
\end{aligned}
$$

(d) Express the power input by the force in terms of the force and the velocity of the rod:
(e) The rate of Joule heat production

$$
I=\frac{\mathcal{E}}{R}=\frac{1.60 \mathrm{~V}}{2.0 \Omega}=0.80 \mathrm{~A}
$$

Note that, because the rod is moving to the right, the flux in the region defined by the rod, the rails, and the resistor is increasing. Hence, in accord with Lenz's law, the current must be counterclockwise if its magnetic field is to counter this increase in flux.

$$
\sum F_{x}=F-F_{\mathrm{m}}=0
$$

$$
F=F_{\mathrm{m}}=B I \ell
$$ is given by:

$$
P=F v=(0.128 \mathrm{~N})(10 \mathrm{~m} / \mathrm{s})=1.3 \mathrm{~W}
$$

$$
P=I^{2} R=(0.80 \mathrm{~A})^{2}(2.0 \Omega)=1.3 \mathrm{~W}
$$

39 •• A $10-\mathrm{cm}$ by $5.0-\mathrm{cm}$ rectangular loop (Figure 28-46) that has a resistance equal to $2.5 \Omega$ moves at a constant speed of $2.4 \mathrm{~cm} / \mathrm{s}$ through a region that has a uniform 1.7-T magnetic field directed out of the page as shown. The front of the loop enters the field region at time $t=0$. (a) Graph the flux through the loop as a function of time. (b) Graph the induced emf and the current in the loop as functions of time. Neglect any self-inductance of the loop and construct your graphs to include the interval $0 \leq t \leq 16 \mathrm{~s}$.

Picture the Problem We'll need to determine how long it takes for the loop to completely enter the region in which there is a magnetic field, how long it is in the region, and how long it takes to leave the region. Once we know these times, we can use its definition to express the magnetic flux as a function of time. We can use Faraday's law to find the induced emf as a function of time.
(a) Find the time required for the loop to enter the region where there is a uniform magnetic field:

Letting $w$ represent the width of the loop, express and evaluate $\phi_{\mathrm{m}}$ for $0<t<4.17 \mathrm{~s}$ :

Find the time during which the loop is fully in the region where there is a uniform magnetic field:

Express $\phi_{\mathrm{m}}$ for $4.17 \mathrm{~s}<t<8.33 \mathrm{~s}$ :

The left-end of the loop will exit the field when $t=12.5 \mathrm{~s}$. Express $\phi_{\mathrm{m}}$ for $8.33 \mathrm{~s}<t<12.5 \mathrm{~s}$ :

For $t=8.33 \mathrm{~s}$ and $\phi_{\mathrm{m}}=8.50 \mathrm{mWb}$ :

For $t=12.5 \mathrm{~s}$ and $\phi_{\mathrm{m}}=0$ :

Solve equations (1) and (2) simultaneously to obtain:

The loop will be completely out of the magnetic field when $t>12.5 \mathrm{~s}$ and:

$$
t=\frac{\ell_{\text {side of loop }}}{v}=\frac{10 \mathrm{~cm}}{2.4 \mathrm{~cm} / \mathrm{s}}=4.17 \mathrm{~s}
$$

$$
\begin{aligned}
\phi_{\mathrm{m}} & =N B A=N B w v t \\
& =(1.7 \mathrm{~T})(0.050 \mathrm{~m})(0.024 \mathrm{~m} / \mathrm{s}) t \\
& =(2.04 \mathrm{mWb} / \mathrm{s}) t
\end{aligned}
$$

$$
t=\frac{\ell_{\text {side of loop }}}{v}=\frac{10 \mathrm{~cm}}{2.4 \mathrm{~cm} / \mathrm{s}}=4.17 \mathrm{~s}
$$

i.e., the loop will begin to exit the region when $t=8.33 \mathrm{~s}$.

$$
\begin{aligned}
\phi_{\mathrm{m}} & =N B A=N B \ell w \\
& =(1.7 \mathrm{~T})(0.10 \mathrm{~m})(0.050 \mathrm{~m}) \\
& =8.50 \mathrm{mWb}
\end{aligned}
$$

$\phi_{\mathrm{m}}=m t+b$
where $m$ is the slope of the line and $b$ is the $\phi_{\mathrm{m}}$-intercept.
$8.50 \mathrm{mWb}=m(8.33 \mathrm{~s})+b$
$0=m(12.5 \mathrm{~s})+b$
$\phi_{\mathrm{m}}=-(2.04 \mathrm{mWb} / \mathrm{s}) t+25.5 \mathrm{mWb}$
$\phi_{\mathrm{m}}=0$

The following graph of $\phi_{\mathrm{m}}(t)$ was plotted using a spreadsheet program.

(b) Using Faraday's law, relate the induced emf to the magnetic flux:

$$
\varepsilon=-\frac{d \phi_{\mathrm{m}}}{d t}
$$

During the interval $0<t<4.17 \mathrm{~s}$ :

$$
\mathcal{E}=-\frac{d}{d t}[(2.04 \mathrm{mWb} / \mathrm{s}) t]=-2.04 \mathrm{mV}
$$

During the interval $4.17 \mathrm{~s}<t<8.33 \mathrm{~s}$ :

$$
\boldsymbol{\varepsilon}=-\frac{d}{d t}[8.50 \mathrm{mWb}]=0
$$

During the interval $8.33 \mathrm{~s}<t<12.5 \mathrm{~s}$ :

$$
\begin{aligned}
\mathcal{E} & =-\frac{d}{d t}[(-2.04 \mathrm{mWb} / \mathrm{s}) t+25.5 \mathrm{mWb}] \\
& =2.04 \mathrm{mV}
\end{aligned}
$$

For $t>12.5 \mathrm{~s}$ :

$$
\mathcal{E}=0
$$

The current in each of these intervals is given by Ohm's law:

$$
I=\frac{\mathcal{E}}{R}
$$

The following graph of $\boldsymbol{\varepsilon}(t)$ was plotted using a spreadsheet program.


The following graph of $I(t)$ was plotted using a spreadsheet program.

$40 \quad \bullet$ A uniform 1.2-T magnetic field is in the $+z$ direction. A conducting rod of length 15 cm lies parallel to the $y$ axis and oscillates in the $x$ direction with displacement given by $x=(2.0 \mathrm{~cm}) \cos (120 \pi t)$, where $120 \pi$ has units of $\mathrm{rad} / \mathrm{s}$. (a) Find an expression for the potential difference between the ends the rod as a function of time? (b) What is the maximum potential difference between the ends the rod?

Picture the Problem The rod is executing simple harmonic motion in the $x y$ plane, i.e., in a plane perpendicular to the magnetic field. The emf induced in the
rod is a consequence of its motion in this magnetic field and is given by $|\mathcal{E}|=v B \ell$. Because we're given the position of the oscillator as a function of time, we can differentiate this expression to obtain $v$.
(a) The potential difference between the ends of the rod is given by:

$$
\varepsilon=v B \ell=B \ell \frac{d x}{d t}
$$

Evaluate $d x / d t$ :

$$
\begin{aligned}
\frac{d x}{d t} & =\frac{d}{d t}[(2.0 \mathrm{~cm}) \cos 120 \pi t] \\
& =-(2.0 \mathrm{~cm})\left(120 \mathrm{~s}^{-1}\right) \pi \sin 120 \pi t \\
& =-(7.54 \mathrm{~m} / \mathrm{s}) \sin 120 \pi t
\end{aligned}
$$

Substitute numerical values and evaluate $\mathcal{E}$ :

$$
\boldsymbol{\varepsilon}=-(1.2 \mathrm{~T})(0.15 \mathrm{~m})(7.54 \mathrm{~m} / \mathrm{s}) \sin 120 \pi t=-(1.4 \mathrm{~V}) \sin 120 \pi t
$$

(b) The maximum potential difference between the ends the rod

$$
\mathcal{E}_{\max }=1.4 \mathrm{~V}
$$

is the amplitude of the expression for
$\varepsilon$ derived in Part (a):
41 •• [SSM] In Figure 28-47, the rod has a mass $m$ and a resistance $R$. The rails are horizontal, frictionless and have negligible resistances. The distance
between the rails is $\ell$. An ideal battery that has an emf $\boldsymbol{\varepsilon}$ is connected between points $a$ and $b$ so that the current in the rod is downward. The rod released from rest at $t=0$. (a) Derive an expression for the force on the rod as a function of the speed. (b) Show that the speed of the rod approaches a terminal speed and find an expression for the terminal speed. (c) What is the current when the rod is moving at its terminal speed?

Picture the Problem (a) The net force acting on the rod is the magnetic force it experiences as a consequence of carrying a current and being in a magnetic field. The net emf that drives $I$ in this circuit is the difference between the emf of the battery and the emf induced in the rod as a result of its motion. Applying a righthand rule to the rod reveals that the direction of this magnetic force is to the right. Hence the rod will accelerate to the right when it is released. (b) We can obtain the equation of motion of the rod by applying Newton's $2^{\text {nd }}$ law to relate its acceleration to $\mathcal{E}, B, I, R$ and $\ell$. (c) Letting $v=v_{\mathrm{t}}$ in the equation for the current in the circuit will yield current when the rod is at its terminal speed.
(a) Express the magnetic force on the $\quad F_{\mathrm{m}}=I \ell B$ current-carrying rod:

The current in the rod is given by: $\quad I=\frac{\mathcal{E}-B \ell v}{R}$

Substituting for $I$ yields:

$$
F_{\mathrm{m}}=\left(\frac{\varepsilon-B \ell v}{R}\right) \ell B=\frac{B \ell}{R}(\varepsilon-B \ell v)
$$

(b) Letting the direction of motion of the rod be the positive $x$ direction,

$$
\frac{B \ell}{R}(\varepsilon-B \ell v)=m \frac{d v}{d t}
$$ apply $\sum F_{x}=m a_{x}$ to the rod:

Solving for $d v / d t$ yields:

$$
\frac{d v}{d t}=\frac{B \ell}{m R}(\varepsilon-B \ell v)
$$

Note that as $v$ increases, $\mathcal{E}-B \ell v \rightarrow 0, d v / d t \rightarrow 0$ and the

$$
\frac{B \ell}{m R}\left(\varepsilon-B \ell v_{\mathrm{t}}\right)=0 \Rightarrow v_{\mathrm{t}}=\frac{\varepsilon}{B \ell}
$$

rod approaches its terminal speed $v_{\mathrm{t}}$.
Set $d v / d t=0$ to obtain:
(c) Substitute $v_{\mathrm{t}}$ for $v$ in equation
(1) to obtain:

$$
I=\frac{\varepsilon-B \ell \frac{\varepsilon}{B \ell}}{R}=0
$$

42 - A uniform magnetic field is established perpendicular to the plane of a loop that has a radius equal to 5.00 cm and a resistance equal to $0.400 \Omega$. The magnitude of the field is increasing at a rate of $40.0 \mathrm{mT} / \mathrm{s}$. Find (a) the magnitude of the induced emf in the loop, (b) the induced current in the loop, and (c) the rate of Joule heating in the loop.

Picture the Problem (a) We can find the magnitude of the induced emf by applying Faraday's law to the loop. (b) and (c) The application of Ohm's law will yield the induced current in the loop and we can find the rate of Joule heating using $P=I^{2} R$.
(a) Apply Faraday's law to express the induced emf in the loop in terms

$$
|\mathcal{E}|=\frac{d \phi_{\mathrm{m}}}{d t}=\frac{d}{d t}(A B)=A \frac{d B}{d t}=\pi R^{2} \frac{d B}{d t}
$$ of the rate of change of the magnetic field:

Substitute numerical values and evaluate $|\boldsymbol{E}|$ :

$$
\begin{aligned}
|\mathcal{E}| & =\pi(0.0500 \mathrm{~m})^{2}(40.0 \mathrm{mT} / \mathrm{s}) \\
& =0.3142 \mathrm{mV}=0.314 \mathrm{mV}
\end{aligned}
$$

(b) Using Ohm's law, relate the induced current to the induced voltage and the resistance of the loop and evaluate $I$ :

$$
\begin{aligned}
I & =\frac{\varepsilon}{R}=\frac{0.3142 \mathrm{mV}}{0.400 \Omega}=0.7854 \mathrm{~mA} \\
& =0.785 \mathrm{~mA}
\end{aligned}
$$

(c) Express the rate at which power is dissipated in a conductor in terms of the induced current and the

$$
\begin{aligned}
P & =I^{2} R=(0.7854 \mathrm{~mA})^{2}(0.400 \Omega) \\
& =0.247 \mu \mathrm{~W}
\end{aligned}
$$

resistance of the loop and evaluate
$P$ :

43 •• In Figure 28-48, a conducting rod that has a mass $m$ and a negligible resistance is free to slide without friction along two parallel frictionless rails that have negligible resistances separated by a distance $\ell$ and connected by a resistance $R$. The rails are attached to a long inclined plane that makes an angle $\theta$ with the horizontal. There is a magnetic field directed upward as shown. (a) Show that there is a retarding force directed up the incline given by
$F=\left(B^{2} \ell^{2} v \cos ^{2} \theta\right) / R$. (b) Show that the terminal speed of the rod is
$v_{\mathrm{t}}=(m g R \sin \theta) /\left(B^{2} \ell^{2} \cos ^{2} \theta\right)$.
Picture the Problem The free-body diagram shows the forces acting on the rod as it slides down the inclined plane. The retarding force is the component of $F_{\mathrm{m}}$ acting up the incline, i.e., in the $-x$ direction. We can express $F_{\mathrm{m}}$ using the expression for the force acting on a conductor moving in a magnetic field. Recognizing that only the horizontal component of the rod's velocity $\vec{v}$ produces an induced emf, we can apply the expression for a motional emf in conjunction with Ohm's law to find the induced current in the rod. In Part (b) we can apply Newton's $2^{\text {nd }}$ law to obtain an expression for $d v / d t$ and set this expression equal to zero to obtain $v_{\mathrm{t}}$.

(a) Express the retarding force acting on the rod:
$F=F_{\mathrm{m}} \cos \theta$
where

$$
F_{\mathrm{m}}=I \ell B
$$

and $I$ is the current induced in the rod as a consequence of its motion in the magnetic field.

Express the induced emf due to the motion of the rod in the magnetic field:

Using Ohm's law, relate the current $I$ in the circuit to the induced emf:

Substitute in equation (1) to obtain:
(b) Apply $\sum F_{x}=m a_{x}$ to the rod:

When the rod reaches its terminal speed $v_{\mathrm{t}}, d v / d t=0$ :

Solve for $v_{t}$ to obtain:

$$
0=g \sin \theta-\frac{B^{2} \ell^{2} v_{\mathrm{t}}}{m R} \cos ^{2} \theta
$$

$$
v_{\mathrm{t}}=\frac{m g R \sin \theta}{B^{2} \ell^{2} \cos ^{2} \theta}
$$

$44 \quad$ ••• A conducting rod of length $\ell$ rotates at constant angular speed $\omega$ about one end, in a plane perpendicular to a uniform magnetic field $B$ (Figure 28-49).
(a) Show that the potential difference between the ends of the rod is $\frac{1}{2} \ell^{2} \theta$. (b) Let the angle $\theta$ between the rotating rod and the dashed line be given by $\theta=\omega t$. Show that the area of the pie-shaped region swept out by the rod during time $t$ is $\frac{1}{2} \ell^{2} \theta$.
(c) Compute the flux $\phi_{\mathrm{m}}$ through this area, and apply $\mathcal{E}=-d \phi_{\mathrm{m}} / d t$ (Faraday's law) to show that the motional emf is given by $\frac{1}{2} B \omega \ell^{2}$.

Picture the Problem We can use $\overrightarrow{\boldsymbol{F}}=q \overrightarrow{\boldsymbol{v}} \times \overrightarrow{\boldsymbol{B}}$ to express the magnetic force acting on the moving charged body. Expressing the emf induced in a segment of the rod of length $d r$ and integrating this expression over the length of the rod will lead us to an expression for the induced emf.
(a) Use the motional emf equation to express the emf induced in a segment of the rod of length $d r$ and at a distance $r$ from the pivot:

Integrate this expression from $r=0$ to $r=\ell$ to obtain:

$$
\begin{aligned}
d \varepsilon & =B r d v \\
& =B r \omega d r
\end{aligned}
$$

$$
\int_{0}^{\varepsilon} d \boldsymbol{\varepsilon}=B \omega \int_{0}^{\ell} r d r \Rightarrow \varepsilon=\frac{1}{2} B \omega \ell^{2}
$$

(b) Using Faraday's law, relate the induced emf to the rate at which the

$$
|\boldsymbol{\varepsilon}|=\frac{d \phi_{\mathrm{m}}}{d t}
$$

flux changes:

Express the area $d A$, for any value of

$$
d A=r \theta d r
$$

$\theta$, between $r$ and $r+d r$ :

Integrate from $r=0$ to $r=\ell$ to obtain:

$$
A=\theta \int_{0}^{\ell} r d r=\frac{1}{2} \theta \ell^{2}
$$

(c) Using its definition, express the magnetic flux through this area:

Differentiate $\phi_{\mathrm{m}}$ with respect to time to obtain:

$$
\phi_{\mathrm{m}}=B A=\frac{1}{2} B \ell^{2} \theta
$$

45 - [SSM] A $2.00-\mathrm{cm}$ by $1.50-\mathrm{cm}$ rectangular coil has 300 turns and rotates in a region that has a magnetic field of 0.400 T . (a) What is the maximum emf generated when the coil rotates at $60 \mathrm{rev} / \mathrm{s}$ ? (b) What must its angular speed be to generate a maximum emf of 110 V ?

Picture the Problem We can use the relationship $\boldsymbol{\varepsilon}_{\text {max }}=2 \pi N B A f$ to relate the maximum emf generated to the area of the coil, the number of turns of the coil, the magnetic field in which the coil is rotating, and the angular speed at which it rotates.
(a) Relate the induced emf to the $\quad \boldsymbol{E}_{\max }=N B A \omega=2 \pi N B A f$ magnetic field in which the coil is rotating:

Substitute numerical values and evaluate $\mathcal{E}_{\text {max }}$ :

$$
\varepsilon_{\max }=2 \pi(300)(0.400 \mathrm{~T})\left(2.00 \times 10^{-2} \mathrm{~m}\right)\left(1.50 \times 10^{-2} \mathrm{~m}\right)\left(60 \mathrm{~s}^{-1}\right)=14 \mathrm{~V}
$$

(b) Solve equation (1) for $f$ : $\quad f=\frac{\mathcal{E}_{\text {max }}}{2 \pi N B A}$

Substitute numerical values and evaluate $f$ :

$$
f=\frac{110 \mathrm{~V}}{2 \pi(300)(0.400 \mathrm{~T})\left(2.00 \times 10^{-2} \mathrm{~m}\right)\left(1.50 \times 10^{-2} \mathrm{~m}\right)}=486 \mathrm{rev} / \mathrm{s}
$$

46 - The coil of Problem 45 rotates at $60 \mathrm{rev} / \mathrm{s}$ in a magnetic field. If the maximum emf generated by the coil is 24 V , what is the magnitude of the magnetic field?

Picture the Problem We can use the relationship $\boldsymbol{\varepsilon}_{\max }=N B A \omega$ to relate the maximum emf generated to the area of the coil, the number of turns of the coil, the magnitude of the magnetic field in which the coil is rotating, and the angular speed at which it rotates.

Relate the induced emf to the magnetic field in which the coil

$$
\varepsilon_{\max }=N B A \omega \Rightarrow B=\frac{\varepsilon_{\max }}{N A \omega}
$$ is rotating:

Substitute numerical values and evaluate $B$ :

$$
B=\frac{24 \mathrm{~V}}{2 \pi(300)\left(2.00 \times 10^{-2} \mathrm{~m}\right)\left(1.50 \times 10^{-2} \mathrm{~m}\right)(60 \mathrm{rev} / \mathrm{s})}=0.71 \mathrm{~T}
$$

## Inductance

47 - When the current in an $8.00-\mathrm{H}$ coil is equal to 3.00 A and is increasing at $200 \mathrm{~A} / \mathrm{s}$, find (a) the magnetic flux through the coil and $(b)$ the induced emf in the coil.

Picture the Problem We can use $\phi_{\mathrm{m}}=L I$ to find the magnetic flux through the
coil. We can apply Faraday's law to find the induced emf in the coil.
(a) The magnetic flux through

$$
\phi_{\mathrm{m}}=L I
$$

the coil is the product of the selfinductance of the coil and the current it is carrying:

When the current is 3.00 A :

$$
\phi_{\mathrm{m}}=(8.00 \mathrm{H})(3.00 \mathrm{~A})=24.0 \mathrm{~Wb}
$$

(b) Use Faraday's law to relate $\mathcal{E}, \quad \quad \varepsilon=-L \frac{d I}{d t}$
$L$, and $d I / d t$ :

Substitute numerical values and

$$
\varepsilon=-(8.00 \mathrm{H})(200 \mathrm{~A} / \mathrm{s})=-1.60 \mathrm{kV}
$$ evaluate $\mathcal{E}$ :

48 •• A 300-turn solenoid has a radius equal to 2.00 cm ; a length equal to 25.0 cm , and a 1000-turn solenoid has a radius equal to 5.00 cm and is also $25.0-\mathrm{cm}$ long. The two solenoids are coaxial, with one nested completely inside the other. What is their mutual inductance?

Picture the Problem We can find the mutual inductance of the two coaxial solenoids using $M_{2,1}=\frac{\phi_{\mathrm{m} 2}}{I_{1}}=\mu_{0} n_{2} n_{1} \ell \pi r_{1}^{2}$.

Substitute numerical values and evaluate $M_{2,1}$ :

$$
M_{2,1}=\left(4 \pi \times 10^{-7} \mathrm{~N} / \mathrm{A}^{2}\right)\left(\frac{300}{0.250 \mathrm{~m}}\right)\left(\frac{1000}{0.250 \mathrm{~m}}\right)(0.250 \mathrm{~m}) \pi(0.0200 \mathrm{~m})^{2}=1.89 \mathrm{mH}
$$

49 •• [SSM] An insulated wire that has a resistance of $18.0 \Omega / \mathrm{m}$ and a length of 9.00 m will be used to construct a resistor. First, the wire is bent in half and then the doubled wire is wound on a cylindrical form (Figure 28-50) to create a $25.0-\mathrm{cm}$-long helix that has a diameter equal to 2.00 cm . Find both the resistance and the inductance of this wire-wound resistor.

Picture the Problem Note that the current in the two parts of the wire is in opposite directions. Consequently, the total flux in the coil is zero. We can find the resistance of the wire-wound resistor from the length of wire used and the resistance per unit length.

Because the total flux in the coil is

$$
L=0
$$

Express the total resistance of the wire:

$$
R=\left(18.0 \frac{\Omega}{\mathrm{~m}}\right) L
$$

Substitute numerical values and evaluate $R$ :

$$
R=\left(18.0 \frac{\Omega}{\mathrm{~m}}\right)(9.00 \mathrm{~m})=162 \Omega
$$

$50 \quad$ •• You are given a length $\ell$ of wire that has radius $a$ and are told to wind it into an inductor in the shape of a helix that has a circular cross section of radius $r$. The windings are to be as close together as possible without overlapping. Show that the self-inductance of this inductor is $L=\frac{1}{4} \mu_{0} r \ell / a$.

Picture the Problem The wire of length $\ell$ and radius $a$ is shown in the diagram, as is the inductor constructed with this wire and whose inductance $L$ is to be found. We can use the equation for the self-inductance of a cylindrical inductor to derive an expression for $L$.


The self-inductance of an inductor with length $d$, cross-sectional area $A$, and number of turns per unit length $n$ is:

The number of turns $N$ is given by:

$$
N=\frac{d}{2 a} \Rightarrow n=\frac{N}{d}=\frac{1}{2 a}
$$

Assuming that $a \ll r$, the length of the wire $\ell$ is related to $N$ and $r$ :

Solving for $d$ yields:

$$
\ell=N(2 \pi r)=\left(\frac{d}{2 a}\right) 2 \pi r=\frac{\pi r}{a} d
$$

$$
d=\frac{a \ell}{\pi r}
$$

Substitute for $d, A$, and $n$ in equation
(1) to obtain:

$$
L=\mu_{0}\left(\frac{1}{2 a}\right)^{2}\left(\pi r^{2}\right)\left(\frac{a \ell}{\pi r}\right)=\frac{1}{4} \mu_{0} r \ell / a
$$

51 - Using the result of Problem 50, calculate the self-inductance of an inductor wound from 10 cm of wire that has a diameter of 1.0 mm into a coil that has a radius of 0.25 cm .

Picture the Problem We can substitute numerical values in the expression derived in Problem 50to find the self-inductance of the inductor.

From Problem 50 we have:

$$
L=\frac{\mu_{0} r d}{4 a}
$$

Substitute numerical values and evaluate $L$ :

$$
L=\frac{\left(4 \pi \times 10^{-7} \mathrm{~N} / \mathrm{A}^{2}\right)(0.25 \mathrm{~cm})(10 \mathrm{~cm})}{4(0.50 \mathrm{~mm})}=0.16 \mu \mathrm{H}
$$

52 ••• In Figure 28-51, circuit 2 has a total resistance of $300 \Omega$. After switch S is closed, the current in circuit 1 increases-reaching a value of 5.00 A after a long time. A charge of $200 \mu \mathrm{C}$ passes through the galvanometer in circuit 2 during the time that the current in circuit 1 is increasing. What is the mutual inductance between the two coils?

Picture the Problem We can apply Kirchhoff's loop rule to the galvanometer circuit to relate the potential difference across $L_{2}$ to the potential difference across $R_{2}$. Integration of this equation over time will yield an equation that relates the mutual inductance between the two coils to the steady-state current in circuit 1 and the charge that flows through the galvanometer.

Apply Kirchhoff's loop rule to the galvanometer circuit:

$$
M \frac{d I_{1}}{d t}+L_{2} \frac{d I_{2}}{d t}-R_{2} I_{2}=0
$$

or

$$
M d I_{1}+L_{2} d I_{2}-R_{2} I_{2} d t=0
$$

Integrate each term from $t=0$ to $t=\infty$ :

$$
\begin{aligned}
& M \int_{0}^{\infty} d I_{1}+L_{2} \int_{0}^{\infty} d I_{2}-R_{2} \int_{0}^{\infty} I_{2} d t=0 \\
& \text { and } \\
& M I_{1 \infty}+L_{2} I_{2 \infty}-R_{2} Q=0
\end{aligned}
$$

Because $I_{2 \infty}=0$ :

$$
M I_{1 \infty}-R_{2} Q=0 \Rightarrow M=\frac{R_{2} Q}{I_{1 \infty}}
$$

Substitute numerical values and evaluate $M$ :

$$
\begin{aligned}
M & =\frac{(300 \Omega)\left(2.00 \times 10^{-4} \mathrm{C}\right)}{5.00 \mathrm{~A}} \\
& =12.0 \mathrm{mH}
\end{aligned}
$$

53 ••• [SSM] Show that the inductance of a toroid of rectangular cross section, as shown in Figure 28-52 is given by $L=\frac{\mu_{0} N^{2} H \ln (b / a)}{2 \pi}$ where $N$ is the total number of turns, $a$ is the inside radius, $b$ is the outside radius, and $H$ is the height of the toroid.

Picture the Problem We can use Ampere's law to express the magnetic field inside the rectangular toroid and the definition of magnetic flux to express $\phi_{\mathrm{m}}$ through the toroid. We can then use the definition of self-inductance of a solenoid to express $L$.

Using the definition of the self-
inductance of a solenoid, express

$$
\begin{equation*}
L=\frac{N \phi_{\mathrm{m}}}{I} \tag{1}
\end{equation*}
$$

$L$ in terms of $\phi_{\mathrm{m}}, N$, and $I$ :
Apply Ampere's law to a closed path of radius $a<r<b$ :

$$
\oint_{\mathrm{C}} \overrightarrow{\boldsymbol{B}} \cdot \overrightarrow{d \ell}=B 2 \pi r=\mu_{0} I_{\mathrm{C}}
$$

or, because $I_{\mathrm{C}}=N I$,

$$
B 2 \pi r=\mu_{0} N I \Rightarrow B=\frac{\mu_{0} N I}{2 \pi r}
$$

Express the flux in a strip of height

$$
d \phi_{\mathrm{m}}=B H d r
$$

$H$ and width $d r$ :

Substituting for $B$ yields:

$$
d \phi_{\mathrm{m}}=\frac{\mu_{0} N I H}{2 \pi r} d r
$$

Integrate $d \phi_{\mathrm{m}}$ from $r=a$ to $r=b$ to obtain:

$$
\phi_{\mathrm{m}}=\frac{\mu_{0} N I H}{2 \pi} \int_{a}^{b} \frac{d r}{r}=\frac{\mu_{0} N I H}{2 \pi} \ln \left(\frac{b}{a}\right)
$$

Substitute for $\phi_{\mathrm{m}}$ in equation (1) and simplify to obtain:

$$
L=\frac{\mu_{0} N^{2} H}{2 \pi} \ln \left(\frac{b}{a}\right)
$$

## Magnetic Energy

54 - A coil that has a self-inductance of 2.00 H and a resistance of $12.0 \Omega$ is connected to an ideal $24.0-\mathrm{V}$ battery. (a) What is the steady-state current?
(b) How much energy is stored in the inductor when the steady-state current is established?

Picture the Problem The current in an $L R$ circuit, as a function of time, is given by $I=I_{\mathrm{f}}\left(1-e^{-t / \tau}\right)$, where $I_{\mathrm{f}}=\varepsilon_{0} / R$ and $\tau=L / R$. The energy stored in the inductor under steady-state conditions is stored in its magnetic field and is given by $U_{\mathrm{m}}=\frac{1}{2} L I_{\mathrm{f}}^{2}$.
(a) The final current is the quotient of the emf of the battery and the resistance

$$
I_{\mathrm{f}}=\frac{\varepsilon_{0}}{R}=\frac{24.0 \mathrm{~V}}{12.0 \Omega}=2.00 \mathrm{~A}
$$ of the coil:

(b) The energy stored in the inductor

$$
\begin{aligned}
U_{\mathrm{m}} & =\frac{1}{2} L I_{\mathrm{f}}^{2}=\frac{1}{2}(2.00 \mathrm{H})(2.00 \mathrm{~A})^{2} \\
& =4.00 \mathrm{~J}
\end{aligned}
$$

55 - [SSM] In a plane electromagnetic wave, the magnitudes of the electric fields and magnetic fields are related by $E=c B$, where $c=1 / \sqrt{\epsilon_{0} \mu_{0}}$ is the speed of light. Show that when $E=c B$ the electric and the magnetic energy densities are equal.

Picture the Problem We can examine the ratio of $u_{\mathrm{m}}$ to $u_{\mathrm{E}}$ with $E=c B$ and $c=1 / \sqrt{\epsilon_{0} \mu_{0}}$ to show that the electric and magnetic energy densities are equal.

Express the ratio of the energy density in the magnetic field to the energy density in the electric field:

Because $E=c B$ :

$$
\frac{u_{\mathrm{m}}}{u_{\mathrm{E}}}=\frac{\frac{B^{2}}{2 \mu_{0}}}{\frac{1}{2} \epsilon_{0} E^{2}}=\frac{B^{2}}{\mu_{0} \epsilon_{0} E^{2}}
$$

$$
\frac{u_{\mathrm{m}}}{u_{\mathrm{E}}}=\frac{B^{2}}{\mu_{0} \in_{0} c^{2} B^{2}}=\frac{1}{\mu_{0} \in_{0} c^{2}}
$$

Substituting for $c^{2}$ and simplifying yields:

$$
\frac{u_{\mathrm{m}}}{u_{\mathrm{E}}}=\frac{\mu_{0} \in_{0}}{\mu_{0} \epsilon_{0}}=1 \Rightarrow u_{\mathrm{m}}=u_{\mathrm{E}}
$$

56 -. A 2000-turn solenoid has a cross-sectional area equal to $4.0 \mathrm{~cm}^{2}$ and length equal to 30 cm . The solenoid carries a current of 4.0 A . (a) Calculate the magnetic energy stored in the solenoid using $U=\frac{1}{2} L I^{2}$, where $L=\mu_{0} n^{2} A \ell$.
(b) Divide your answer in Part (a) by the volume of the region inside the solenoid to find the magnetic energy per unit volume in the solenoid. (c) Check your Part (b) result by computing the magnetic energy density from $\mu_{\mathrm{m}}=B^{2} / 2 \mu_{0}$ where $B=\mu_{0} n I$.

Picture the Problem We can use $L=\mu_{0} n^{2} A \ell$ to find the inductance of the solenoid and $B=\mu_{0} n I$ to find the magnetic field inside it.
(a) Express the magnetic energy $\quad U_{\mathrm{m}}=\frac{1}{2} L I^{2}$
stored in the solenoid:

Relate the inductance of the solenoid

$$
L=\mu_{0} n^{2} A \ell
$$ to its dimensions and properties:

Substitute for $L$ to obtain: $\quad U_{\mathrm{m}}=\frac{1}{2} \mu_{0} n^{2} A \ell I^{2}$

Substitute numerical values and evaluate $U_{\mathrm{m}}$ :

$$
\begin{aligned}
U_{\mathrm{m}}= & \frac{1}{2}\left(4 \pi \times 10^{-7} \mathrm{~N} / \mathrm{A}^{2}\right)\left(\frac{2000}{0.30 \mathrm{~m}}\right)^{2} \\
& \times\left(4.0 \times 10^{-4} \mathrm{~m}^{2}\right)(0.30 \mathrm{~m})(4.0 \mathrm{~A})^{2} \\
= & 53.6 \mathrm{~mJ}=54 \mathrm{~mJ}
\end{aligned}
$$

(b) The magnetic energy per unit volume in the solenoid is:

$$
\begin{aligned}
\frac{U_{\mathrm{m}}}{V} & =\frac{U_{\mathrm{m}}}{A \ell}=\frac{53.6 \mathrm{~mJ}}{\left(4.0 \times 10^{-4} \mathrm{~m}^{2}\right)(0.30 \mathrm{~m})} \\
& =0.45 \mathrm{~kJ} / \mathrm{m}^{3}
\end{aligned}
$$

(c) The magnetic energy density in the solenoid is given by:

$$
u_{\mathrm{m}}=\frac{B^{2}}{2 \mu_{0}}
$$

Substituting for $B$ and simplifying yields:

$$
\begin{aligned}
u_{\mathrm{m}} & =\frac{\left(\mu_{0} n I\right)^{2}}{2 \mu_{0}}=\frac{\left(\mu_{0} \frac{N}{\ell} I\right)^{2}}{2 \mu_{0}} \\
& =\frac{\mu_{0} N^{2} I^{2}}{2 \ell^{2}}
\end{aligned}
$$

Substitute numerical values and evaluate $u_{\mathrm{m}}$ :

$$
\begin{aligned}
u_{\mathrm{m}} & =\frac{\left(4 \pi \times 10^{-7} \mathrm{~N} / \mathrm{A}^{2}\right)(2000)^{2}(4.0 \mathrm{~A})^{2}}{2(0.30 \mathrm{~m})^{2}} \\
& =0.45 \mathrm{~kJ} / \mathrm{m}^{3}
\end{aligned}
$$

57 •• A long cylindrical wire has a radius equal to 2.0 cm and carries a current of 80 A uniformly distributed over its cross-sectional area. Find the magnetic energy per unit length within the wire.

Picture the Problem Consider a cylindrical annulus of thickness $d r$ at a radius $r<a$. We can use its definition to express the total magnetic energy $d U_{\mathrm{m}}$ inside the cylindrical annulus and divide both sides of this expression by the length of the wire to express the magnetic energy per unit length $d U_{m}^{\prime}$. Integration of this expression will give us the magnetic energy per unit length within the wire.


Express the magnetic energy within the cylindrical annulus:

$$
\begin{aligned}
d U_{\mathrm{m}} & =\frac{B^{2}}{2 \mu_{0}} V_{\text {annulus }}=\frac{B^{2}}{2 \mu_{0}} 2 \pi r \ell d r \\
& =\frac{B^{2}}{\mu_{0}} \pi r \ell d r
\end{aligned}
$$

Divide both sides of the equation by $\ell$ to express the magnetic

$$
\begin{equation*}
d U_{m}^{\prime}=\frac{B^{2}}{\mu_{0}} \pi r d r \tag{1}
\end{equation*}
$$ energy per unit length $d U_{m}^{\prime}$ :

Use Ampere's law to express the magnetic field inside the wire at a distance $r<a$ from its center:

Because the current is uniformly distributed over the cross-

$$
2 \pi r B=\mu_{0} I_{\mathrm{C}} \Rightarrow B=\frac{\mu_{0} I_{\mathrm{C}}}{2 \pi r}
$$

where $I_{\mathrm{C}}$ is the current inside the cylinder of radius $r$.
sectional area of the wire:
Substitute for $I_{\mathrm{C}}$ to obtain:

$$
\frac{I_{\mathrm{C}}}{I}=\frac{\pi r^{2}}{\pi a^{2}} \Rightarrow I_{\mathrm{C}}=\frac{r^{2}}{a^{2}} I
$$

$$
B=\frac{\mu_{0} r I}{2 \pi a^{2}}
$$

Substituting for $B$ in equation (1) and simplifying yields:

$$
d U_{m}^{\prime}=\frac{\left(\frac{\mu_{0} r I}{2 \pi a^{2}}\right)^{2}}{\mu_{0}} \pi r d r=\frac{\mu_{0} I^{2}}{4 \pi a^{4}} r^{3} d r
$$

Integrate $d U^{\prime}$ from $r=0$ to $r=a$ :

$$
U_{m}^{\prime}=\frac{\mu_{0} I^{2}}{4 \pi a^{4}} \int_{0}^{a} r^{3} d r=\frac{\mu_{0} I^{2}}{4 \pi a^{4}} \cdot \frac{a^{4}}{4}=\frac{\mu_{0} I^{2}}{16 \pi}
$$

## Remarks: Note that the magnetic energy per unit length is independent of the radius of the cylinder and depends only on the total current.

58 •• A toroid that has a mean radius equal to 25.0 cm and a circular loops with radii equal to 2.00 cm is wound with a superconducting wire. The wire has a length equal to 1000 m and carries a current of 400 A . (a) What is the number of turns of the wire? (b) What is the magnetic field strength and magnetic energy density at the mean radius? (c) Estimate the total energy stored in this toroid by assuming that the energy density is uniformly distributed in the region inside the toroid.

Picture the Problem We can find the number of turns on the coil from the length of the superconducting wire and the cross-sectional radius of the coil. We can use $B=\left(\mu_{0} N I\right) /\left(2 \pi r_{\text {mean }}\right)$ to find the magnetic field at the mean radius. We can find the energy density in the magnetic field from $u_{\mathrm{m}}=B^{2} /\left(2 \mu_{0}\right)$ and the total energy stored in the toroid by multiplying $u_{\mathrm{m}}$ by the volume of the toroid.
(a) Express the number of turns in terms of the length of the wire $L$

$$
N=\frac{L}{2 \pi r}
$$

and length required per turn $2 \pi r$ :

Substitute numerical values and evaluate $N$ :

$$
\begin{aligned}
N & ==\frac{1000 \mathrm{~m}}{2 \pi(0.0200 \mathrm{~m})}=7958 \\
& =7.96 \times 10^{3}
\end{aligned}
$$

(b) $B$ inside a tightly wound toroid or radius $r$ is given by:

$$
B=\frac{\mu_{0} N I}{2 \pi r}
$$

Substitute numerical values and evaluate the magnetic field at the mean radius:

$$
\begin{aligned}
B & =\frac{\left(4 \pi \times 10^{-7} \mathrm{~N} / \mathrm{A}^{2}\right)(7958)(400 \mathrm{~A})}{2 \pi(0.250 \mathrm{~m})} \\
& =2.547 \mathrm{~T}=2.55 \mathrm{~T}
\end{aligned}
$$

The energy density in the magnetic field is given by:

$$
u_{\mathrm{m}}=\frac{B^{2}}{2 \mu_{0}}
$$

Substitute numerical values and evaluate $u_{\mathrm{m}}$ :

$$
\begin{aligned}
\boldsymbol{u}_{\mathrm{m}} & =\frac{(2.547 \mathrm{~T})^{2}}{2\left(4 \pi \times 10^{-7} \mathrm{~N} / \mathrm{A}^{2}\right)}=2.580 \mathrm{MJ} / \mathrm{m}^{3} \\
& =2.58 \mathrm{MJ} / \mathrm{m}^{3}
\end{aligned}
$$

(c) Relate the total energy stored

$$
U_{\mathrm{m}}=u_{\mathrm{m}} V_{\text {toroid }}
$$

in the toroid to the energy density in its magnetic field and the volume of the toroid:

Think of the toroid as a cylinder of

$$
V_{\text {toroid }}=\pi r^{2}\left(2 \pi r_{\text {mean }}\right)=2 \pi^{2} r^{2} r_{\text {mean }}
$$ radius $r$ and height $2 \pi r_{\text {mean }}$ to obtain:

Substitute for $V_{\text {toroid }}$ to obtain: $\quad U_{\mathrm{m}}=2 \pi^{2} r^{2} r_{\text {mean }} u_{\mathrm{m}}$

Substitute numerical values and evaluate $U_{\mathrm{m}}$ :

$$
U_{\mathrm{m}}=2 \pi^{2}(0.0200 \mathrm{~m})^{2}(0.250 \mathrm{~m})\left(2.580 \mathrm{MJ} / \mathrm{m}^{3}\right)=5.09 \mathrm{~kJ}
$$

## RL Circuits

59 - [SSM] A circuit consists of a coil that has a resistance equal to $8.00 \Omega$ and a self-inductance equal to 4.00 mH , an open switch and an ideal $100-\mathrm{V}$ battery-all connected in series. At $t=0$ the switch is closed. Find the current and its rate of change at times (a) $t=0$, (b) $t=0.100 \mathrm{~ms},(c) t=0.500 \mathrm{~ms}$, and (d) $t=1.00 \mathrm{~ms}$.

Picture the Problem We can find the current using $I=I_{\mathrm{f}}\left(1-e^{-t / \tau}\right)$ where $I_{\mathrm{f}}=\varepsilon_{0} / R$ and $\tau=L / R$ and its rate of change by differentiating this expression with respect to time.

Express the dependence of the

$$
I=I_{\mathrm{f}}\left(1-e^{-t / \tau}\right)
$$ current on $I_{\mathrm{f}}$ and $\tau$.

Evaluating $I_{\mathrm{f}}$ and $\tau$ yields:

$$
I_{\mathrm{f}}=\frac{\varepsilon_{0}}{R}=\frac{100 \mathrm{~V}}{8.00 \Omega}=12.5 \mathrm{~A}
$$

and

$$
\tau=\frac{L}{R}=\frac{4.00 \mathrm{mH}}{8.00 \Omega}=0.500 \mathrm{~ms}
$$

Substitute for $I_{\mathrm{f}}$ and $\tau$ to obtain:

Express $d I / d t$ :

$$
I=(12.5 \mathrm{~A})\left(1-e^{-t / 0.500 \mathrm{~ms}}\right)
$$

$$
\begin{aligned}
\frac{d I}{d t} & =(12.5 \mathrm{~A})\left(-e^{-t / 0.500 \mathrm{~ms}}\right)\left(-2000 \mathrm{~s}^{-1}\right) \\
& =(25.0 \mathrm{kA} / \mathrm{s}) e^{-t / 0.500 \mathrm{~ms}}
\end{aligned}
$$

(a) Evaluate $I$ and $d I / d t$ at $t=0$ :
(b) Evaluating $I$ and $d I / d t$ at $t=0.100 \mathrm{~ms}$ yields:
(c) Evaluate $I$ and $d I / d t$ at $t=0.500 \mathrm{~ms}$ to obtain:
(d) Evaluating $I$ and $d I / d t$ at $t=1.00 \mathrm{~ms}$ yields:
$I(0)=(12.5 \mathrm{~A})\left(1-e^{0}\right)=0$
and
$\left.\frac{d I}{d t}\right|_{t=0}=(25.0 \mathrm{kA} / \mathrm{s}) e^{0}=25.0 \mathrm{kA} / \mathrm{s}$

$$
\begin{aligned}
I(0.100 \mathrm{~ms}) & =(12.5 \mathrm{~A})\left(1-e^{-0.100 \mathrm{~ms} / 0.500 \mathrm{~ms}}\right) \\
& =2.27 \mathrm{~A}
\end{aligned}
$$

and

$$
\begin{aligned}
\left.\frac{d I}{d t}\right|_{t=0.500 \mathrm{~ms}} & =(25.0 \mathrm{kA} / \mathrm{s}) e^{-0.100 \mathrm{~ms} / 0.500 \mathrm{~ms}} \\
& =20.5 \mathrm{kA} / \mathrm{s}
\end{aligned}
$$

$$
\begin{aligned}
I(0.500 \mathrm{~ms}) & =(12.5 \mathrm{~A})\left(1-e^{-0.500 \mathrm{~ms} / 0.500 \mathrm{~ms}}\right) \\
& =7.90 \mathrm{~A}
\end{aligned}
$$

and

$$
\begin{aligned}
\left.\frac{d I}{d t}\right|_{t=0.500 \mathrm{~ms}} & =(25.0 \mathrm{kA} / \mathrm{s}) e^{-0.500 \mathrm{~ms} / 0.500 \mathrm{~ms}} \\
& =9.20 \mathrm{kA} / \mathrm{s}
\end{aligned}
$$

$$
\begin{aligned}
I(1.00 \mathrm{~ms}) & =(12.5 \mathrm{~A})\left(1-e^{-1.00 \mathrm{~ms} / 0.500 \mathrm{~ms}}\right) \\
& =10.8 \mathrm{~A}
\end{aligned}
$$

and

$$
\begin{aligned}
\left.\frac{d I}{d t}\right|_{t=1.00 \mathrm{~ms}} & =(25.0 \mathrm{kA} / \mathrm{s}) e^{-1.00 \mathrm{~ms} / 0.500 \mathrm{~ms}} \\
& =3.38 \mathrm{kA} / \mathrm{s}
\end{aligned}
$$

60 - In the circuit shown in Figure 28-53, the throw of the make-beforebreak switch has been at contact $a$ for a long time and the current in the 1.00 mH coil is equal to 2.00 A . At $t=0$ the throw is quickly moved to contact $b$. The total resistance $R+r$ of the coil and the resistor is $10.0 \Omega$. Find the current when (a) $t=0.500 \mathrm{~ms}$, and (b) $t=100 \mathrm{~ms}$.

Picture the Problem We can find the current using $I(t)=I_{0} e^{-t / \tau}$, where $I_{0}$ is the current at time $t=0$ and $\tau=L / R$.

Express the current as a function of $\quad I(t)=I_{0} e^{-t / \tau}=(2.00 \mathrm{~A}) e^{-t / \tau}$ time:
Evaluating $\tau$ yields:

$$
\tau=\frac{L}{R}=\frac{1.00 \mathrm{mH}}{10.0 \Omega}=0.100 \mathrm{~ms}
$$

Substitute for $\tau$ to obtain:

$$
I(t)=(2.00 \mathrm{~A}) e^{-0.100 \mathrm{~ms} t}
$$

(a) When $t=0.500 \mathrm{~ms}$ :

$$
\begin{aligned}
I(0.500 \mathrm{~ms}) & =(2.00 \mathrm{~A}) e^{-0.500 \mathrm{~ms} / 0.100 \mathrm{~ms}} \\
& =13.5 \mathrm{~mA}
\end{aligned}
$$

(b) When $t=10.0 \mathrm{~ms}$ :

$$
\begin{aligned}
I(10.0 \mathrm{~ms}) & =(2 \mathrm{~A}) e^{-10.0 \mathrm{~ms} / 0.100 \mathrm{~ms}} \\
& =(2.00 \mathrm{~A}) e^{-100}=7.44 \times 10^{-44} \mathrm{~A} \\
& \approx 0
\end{aligned}
$$

61 •• [SSM] In the circuit shown in Figure 28-54, let $\boldsymbol{E}_{0}=12.0 \mathrm{~V}$, $R=3.00 \Omega$, and $L=0.600 \mathrm{H}$. The switch, which was initially open, is closed at time $t=0$. At time $t=0.500 \mathrm{~s}$, find (a) the rate at which the battery supplies energy, $(b)$ the rate of Joule heating in the resistor, and $(c)$ the rate at which energy is being stored in the inductor.

Picture the Problem We can find the current using $I=I_{\mathrm{f}}\left(1-e^{-t / \tau}\right)$, where $I_{\mathrm{f}}=\varepsilon_{0} / R$, and $\tau=L / R$, and its rate of change by differentiating this expression with respect to time.

Express the dependence of the

$$
I(t)=I_{\mathrm{f}}\left(1-e^{-t / \tau}\right)
$$ current on $I_{\mathrm{f}}$ and $\tau$ :

Evaluating $I_{\mathrm{f}}$ and $\tau$ yields:

Substitute for $I_{\mathrm{f}}$ and $\tau$ to obtain:

Express $d I / d t$ :
(a) The rate at which the battery supplies energy is given by:

Substituting for $I$ and $\mathcal{E}_{0}$ yields:

The rate at which the battery supplies energy at $t=0.500 \mathrm{~s}$ is:
(b) The rate of Joule heating is:

Substitute for $I$ and $R$ and simplify to obtain:

The rate of Joule heating at $t=0.500 \mathrm{~s}$ is:
(c) Use the expression for the magnetic energy stored in an inductor to express the rate at which energy is being stored:

$$
I_{\mathrm{f}}=\frac{\varepsilon_{0}}{R}=\frac{12.0 \mathrm{~V}}{3.00 \Omega}=4.00 \mathrm{~A}
$$

and

$$
\tau=\frac{L}{R}=\frac{0.600 \mathrm{H}}{3.00 \Omega}=0.200 \mathrm{~s}
$$

$$
I(t)=(4.00 \mathrm{~A})\left(1-e^{-t / 0.200 \mathrm{~s}}\right)
$$

$$
\begin{aligned}
\frac{d I}{d t} & =(4.00 \mathrm{~A})\left(-e^{-t / 0.200 \mathrm{~s}}\right)\left(-5.00 \mathrm{~s}^{-1}\right) \\
& =(20.0 \mathrm{~A} / \mathrm{s}) e^{-t / 0.200 \mathrm{~s}}
\end{aligned}
$$

$$
P=I \varepsilon_{0}
$$

$$
\begin{aligned}
P(t) & =(4.00 \mathrm{~A})\left(1-e^{-t / 0.200 \mathrm{~s}}\right)(12.0 \mathrm{~V}) \\
& =(48.0 \mathrm{~W})\left(1-e^{-t / 0.200 \mathrm{~s}}\right)
\end{aligned}
$$

$$
\begin{aligned}
P(0.500 \mathrm{~s}) & =(48.0 \mathrm{~W})\left(1-e^{-0.500 / 0.200 \mathrm{~s}}\right) \\
& =44.1 \mathrm{~W}
\end{aligned}
$$

$$
P_{\mathrm{J}}=I^{2} R
$$

$$
\begin{aligned}
& P_{\mathrm{J}}=\left[(4.00 \mathrm{~A})\left(1-e^{-t / 0.200 \mathrm{~s}}\right)\right]^{2}(3.00 \Omega) \\
& =(48.0 \mathrm{~W})\left(1-e^{-t / 0.200 \mathrm{~s}}\right)^{2} \\
& \begin{aligned}
P_{\mathrm{J}}(0.500 \mathrm{~s}) & =(48.0 \mathrm{~W})\left(1-e^{-0.500 \mathrm{~s} / 0.200 \mathrm{~s}}\right)^{2} \\
& =40.4 \mathrm{~W}
\end{aligned}
\end{aligned}
$$

$$
\frac{d U_{\mathrm{L}}}{d t}=\frac{d}{d t}\left[\frac{1}{2} L I^{2}\right]=L I \frac{d I}{d t}
$$

Substitute for $L, I$, and $d I / d t$ to obtain:

$$
\begin{aligned}
\frac{d U_{\mathrm{L}}}{d t} & =(0.600 \mathrm{H})(4.00 \mathrm{~A})\left(1-e^{-t / 0.200 \mathrm{~s}}\right)(20.0 \mathrm{~A} / \mathrm{s}) e^{-t / 0.200 \mathrm{~s}} \\
& =(48.0 \mathrm{~W})\left(1-e^{-t / 0.200 \mathrm{~s}}\right) e^{-t / 0.200 \mathrm{~s}}
\end{aligned}
$$

Evaluate this expression for $t=0.500 \mathrm{~s}$ :

$$
\left.\frac{d U_{\mathrm{L}}}{d t}\right]_{t=0.500 \mathrm{~s}}=(48.0 \mathrm{~W})\left(1-e^{-0.500 \mathrm{~s} / 0.200 \mathrm{~s}}\right) e^{-0.500 \mathrm{~s} / 0.200 \mathrm{~s}}=3.62 \mathrm{~W}
$$

## Remarks: Note that, to a good approximation, $d U_{\mathrm{L}} / d t=P-P_{\mathrm{J}}$.

62 •• How many time constants must elapse before the current in an $R L$ circuit (Figure 28-54) that is initially zero reaches (a) 90 percent, (b) 99 percent, and (c) 99.9 percent of its steady-state value?

Picture the Problem If the current is initially zero in an $L R$ circuit, its value at some later time $t$ is given by $I=I_{\mathrm{f}}\left(1-e^{-t / \tau}\right)$, where $I_{\mathrm{f}}=\mathcal{E}_{0} / R$ and $\tau=L / R$ is the time constant for the circuit. We can find the number of time constants that must elapse before the current reaches any given fraction of its final value by solving this equation for $t / \tau$.

Express the fraction of its final value to which the current has risen as a

$$
\frac{I}{I_{\mathrm{f}}}=1-e^{-t / \tau} \Rightarrow \frac{t}{\tau}=-\ln \left(1-\frac{I}{I_{\mathrm{f}}}\right)
$$ function of time:

(a) Evaluate $t / \tau$ for $I / I_{\mathrm{f}}=0.90$ :

$$
\left.\frac{t}{\tau}\right|_{90 \%}=-\ln (1-0.90)=2.3
$$

(b) Evaluate $t / \tau$ for $I / I_{\mathrm{f}}=0.99$ :
(c) Evaluate $t / \tau$ for $I / I_{\mathrm{f}}=0.999$ :

$$
\left.\frac{t}{\tau}\right|_{99 \%}=-\ln (1-0.99)=4.6
$$

$$
\left.\frac{t}{\tau}\right|_{99.9 \%}=-\ln (1-0.999)=6.91
$$

63 -• [SSM] A circuit consists of a $4.00-\mathrm{mH}$ coil, a $150-\Omega$ resistor, a $12.0-\mathrm{V}$ ideal battery and an open switch-all connected in series. After the switch is closed: (a) What is the initial rate of increase of the current? (b) What is the rate
of increase of the current when the current is equal to half its steady-state value? (c) What is the steady-state value of the current? (d) How long does it take for the current to reach 99 percent of its steady state value?

Picture the Problem If the current is initially zero in an $L R$ circuit, its value at some later time $t$ is given by $I=I_{\mathrm{f}}\left(1-e^{-t / \tau}\right)$, where $I_{\mathrm{f}}=\mathcal{E}_{0} / R$ and $\tau=L / R$ is the time constant for the circuit. We can find the rate of increase of the current by differentiating $I$ with respect to time and the time for the current to reach any given fraction of its initial value by solving for $t$.
(a) Express the current in the circuit as a function of time:

$$
I=\frac{\varepsilon_{0}}{R}\left(1-e^{-t / \tau}\right)
$$

Express the initial rate of increase of the current by differentiating this

$$
\begin{aligned}
\frac{d I}{d t} & =\frac{\varepsilon_{0}}{R} \frac{d}{d t}\left(1-e^{-t / \tau}\right) \\
& =\frac{\varepsilon_{0}}{R}\left(-e^{-t / \tau}\right)\left(-\frac{1}{\tau}\right)=\frac{\varepsilon_{0}}{\tau R} e^{-\frac{R}{L} t} \\
& =\frac{\varepsilon_{0}}{L} e^{-\frac{R}{L} t}
\end{aligned}
$$

Evaluate $d I / d t$ at $t=0$ to obtain:
(b) When $I=0.5 I_{\mathrm{f}}$ :

Evaluate $d I / d t$ with $e^{-t / \tau}=0.5$ to obtain:

$$
\left.\frac{d I}{d t}\right|_{t=0}=\frac{\varepsilon_{0}}{L} e^{0}=\frac{12.0 \mathrm{~V}}{4.00 \mathrm{mH}}=3.00 \mathrm{kA} / \mathrm{s}
$$

$$
0.5=1-e^{-t / \tau} \Rightarrow e^{-t / \tau}=0.5
$$

$$
\begin{aligned}
\left.\frac{d I}{d t}\right|_{e^{-l /}=0.5} & =0.5 \frac{\varepsilon_{0}}{L}=0.5\left(\frac{12.0 \mathrm{~V}}{4.00 \mathrm{mH}}\right) \\
& =1.50 \mathrm{kA} / \mathrm{s}
\end{aligned}
$$

(c) Calculate $I_{\mathrm{f}}$ from $\mathcal{E}_{0}$ and $R$ :

$$
I_{\mathrm{f}}=\frac{\varepsilon_{0}}{R}=\frac{12.0 \mathrm{~V}}{150 \Omega}=80.0 \mathrm{~mA}
$$

(d) When $I=0.99 I_{\mathrm{f}}$ :
$0.99=1-e^{-t / \tau} \Rightarrow e^{-t / \tau}=0.01$

Solving for $t$ and substituting for $\tau$ yields:

$$
t=-\tau \ln (0.01)=-\frac{L}{R} \ln (0.01)
$$

Substitute numerical values and evaluate $t$ :

$$
t=-\frac{4.00 \mathrm{mH}}{150 \Omega} \ln (0.01)=0.123 \mathrm{~ms}
$$

64 -. A circuit consists of a large electromagnet that has an inductance of 50.0 H and a resistance of $8.00 \Omega$, a dc $250-\mathrm{V}$ power source and an open switchall connected in series. How long after the switch is closed is the current equal to (a) 10 A , and (b) 30 A .

Picture the Problem If the current is initially zero in an $L R$ circuit, its value at some later time $t$ is given by $I=I_{\mathrm{f}}\left(1-e^{-t / \tau}\right)$, where $I_{\mathrm{f}}=\mathcal{E}_{0} / R$ and $\tau=L / R$ is the time constant for the circuit. We can find the time for the current to reach any given value by solving this equation for $t$.

Evaluate $I_{\mathrm{f}}$ and $\tau$ :

$$
I_{\mathrm{f}}=\frac{\varepsilon_{0}}{R}=\frac{250 \mathrm{~V}}{8.00 \Omega}=31.25 \mathrm{~A}
$$

and

$$
\tau=\frac{L}{R}=\frac{50.0 \mathrm{H}}{8.00 \Omega}=6.25 \mathrm{~s}
$$

Solve $I=I_{\mathrm{f}}\left(1-e^{-t / \tau}\right)$ for $t$ :

$$
t=-\tau \ln \left(1-\frac{I}{I_{\mathrm{f}}}\right)
$$

Substituting for $\tau$ and $I_{\mathrm{f}}$ yields:

$$
t=-(6.25 \mathrm{~s}) \ln \left(1-\frac{I}{31.25 \mathrm{~A}}\right)
$$

(a) Evaluate $t$ for $I=10 \mathrm{~A}$ :

$$
\begin{aligned}
\left.\boldsymbol{t}\right|_{10 \mathrm{~A}} & =-(6.25 \mathrm{~s}) \ln \left(1-\frac{10 \mathrm{~A}}{31.25 \mathrm{~A}}\right) \\
& =2.4 \mathrm{~s}
\end{aligned}
$$

(b) Evaluate $t$ for $I=30 \mathrm{~A}$ :

$$
\begin{aligned}
\left.\boldsymbol{t}\right|_{30 \mathrm{~A}} & =-(6.25 \mathrm{~s}) \ln \left(1-\frac{30 \mathrm{~A}}{31.25 \mathrm{~A}}\right) \\
& =20 \mathrm{~s}
\end{aligned}
$$

65 •• [SSM] Given the circuit shown in Figure 28-55, assume that the inductor has negligible internal resistance and that the switch S has been closed for a long time so that a steady current exists in the inductor. (a) Find the battery current, the current in the $100 \Omega$ resistor, and the current in the inductor.
(b) Find the potential drop across the inductor immediately after the switch S is opened. (c) Using a spreadsheet program, make graphs of the current in the inductor and the potential drop across the inductor as functions of time for the period during which the switch is open.

Picture the Problem The self-induced emf in the inductor is proportional to the rate at which the current through it is changing. Under steady-state conditions, $d I / d t=0$ and so the self-induced emf in the inductor is zero. We can use Kirchhoff's loop rule to obtain the current through and the voltage across the inductor as a function of time.
(a) Because, under steady-state

$$
10 \mathrm{~V}-(10 \Omega) I_{10-\Omega}=0
$$

conditions, the self-induced emf in the inductor is zero and because the inductor has negligible resistance, we can apply Kirchhoff's loop rule to the loop that includes the source, the $10-\Omega$ resistor, and the $2-\mathrm{H}$ inductor to find the current drawn from the battery and flowing through the inductor and the $10-\Omega$ resistor:

Solving for $I_{10-\Omega}$ yields:

$$
\begin{aligned}
& I_{10-\Omega}=I_{2-\mathrm{H}}=1.0 \mathrm{~A} \\
& I_{100-\Omega}=I_{\text {battery }}-I_{2-\mathrm{H}}=0
\end{aligned}
$$

By applying Kirchhoff's junction rule at the junction between the resistors, we can conclude that:
(b) When the switch is opened, the current cannot immediately go to zero in the circuit because of the inductor. For a time, a current will circulate in the circuit loop between the inductor and the $100-\Omega$ resistor. Because the current flowing through this circuit is initially 1 A , the voltage drop across the $100-\Omega$ resistor is initially 100 V . Conservation of energy (Kirchhoff's loop rule) requires that the voltage drop across the 2-H inductor is $V_{2-\mathrm{H}}=100 \mathrm{~V}$.
(c) Apply Kirchhoff's loop rule to the $R L$ circuit to obtain:

$$
L \frac{d I}{d t}+I R=0
$$

The solution to this differential equation is:

$$
I(t)=I_{0} e^{-\frac{R}{L} t}=I_{0} e^{-\frac{t}{\tau}}
$$

where $\tau=\frac{L}{R}=\frac{2.0 \mathrm{H}}{100 \Omega}=0.020 \mathrm{~s}$

A spreadsheet program to generate the data for graphs of the current and the voltage across the inductor as functions of time is shown below. The formulas used to calculate the quantities in the columns are as follows:

| Cell | Formula/Content | Algebraic Form |
| :---: | :---: | :---: |
| B1 | 2.0 | $L$ |
| B2 | 100 | $R$ |
| B3 | 1 | $I_{0}$ |
| A6 | 0 | $t_{0}$ |
| B6 | $\$ \mathrm{~B} \$ 3 * \operatorname{EXP}((-\$ \mathrm{~B} \$ 2 / \$ \mathrm{~B} \$ 1) * \mathrm{~A} 6)$ | $I_{0} e^{-\frac{R}{L} t}$ |


|  | A | B | C |
| :---: | :---: | :--- | :--- |
| 1 | $L=$ | 2 | H |
| 2 | $R=$ | 100 | ohms |
| 3 | $I_{0}=$ | 1 | A |
| 4 |  |  |  |
| 5 | $t$ | $I(t)$ | $V(t)$ |
| 6 | 0.000 | $1.00 \mathrm{E}+00$ | 100.00 |
| 7 | 0.005 | $7.79 \mathrm{E}-01$ | 77.88 |
| 8 | 0.010 | $6.07 \mathrm{E}-01$ | 60.65 |
| 9 | 0.015 | $4.72 \mathrm{E}-01$ | 47.24 |
| 10 | 0.020 | $3.68 \mathrm{E}-01$ | 36.79 |
| 11 | 0.025 | $2.87 \mathrm{E}-01$ | 28.65 |
| 12 | 0.030 | $2.23 \mathrm{E}-01$ | 22.31 |
|  |  |  |  |
| 32 | 0.130 | $1.50 \mathrm{E}-03$ | 0.15 |
| 33 | 0.135 | $1.17 \mathrm{E}-03$ | 0.12 |
| 34 | 0.140 | $9.12 \mathrm{E}-04$ | 0.09 |
| 35 | 0.145 | $7.10 \mathrm{E}-04$ | 0.07 |
| 36 | 0.150 | $5.53 \mathrm{E}-04$ | 0.06 |

The following graph of the current in the inductor as a function of time was plotted using the data in columns A and B of the spreadsheet program.


The following graph of the voltage across the inductor as a function of time was plotted using the data in columns A and C of the spreadsheet program.


66 •• Given the circuit shown in Figure 28-56, the inductor has negligible internal resistance and the switch S has been open for a long time. The switch is then closed. (a) Find the current in the battery, the current in the $100-\Omega$ resistor, and the current in the inductor immediately after the switch is closed. (b) Find the current in the battery, the current in the $100-\Omega$ resistor, and the current in the inductor a long time after the switch is closed. After being closed for a long time the switch is now opened. (c) Find the current in the battery, the current in the $100-\Omega$ resistor, and the current in the inductor immediately after the switch is opened. (d) Find the current in the battery, the current in the $100-\Omega$ resistor, and the current in the inductor after the switch is opened for a long time.

Picture the Problem Let the current supplied by the battery be $I_{\text {battery }}$, the current through the inductor be $I_{L}$, and the current in the $100-\Omega$ resistor be $I_{100-\Omega}$. (a) We simplify our calculations by using the fact that the current in an inductor cannot change abruptly. Thus, the current in the inductor must be zero just after the
switch is closed, because the current is zero before the switch is closed. (b) When the current reaches its final value, $d I / d t$ equals zero, and there is no potential drop across the inductor. The inductor thus acts like a short circuit; that is, the inductor acts like a wire with zero resistance. (c) Immediately after the switch is opened, the current in the inductor is the same as it was before. (d) A long time after the switch is opened, all the currents must be zero.
(a) The switch is just opened. The current through the inductor is zero, just as it was before the switch was closed. Apply the junction rule to relate $I_{\text {battery }}$ and $I_{100-\Omega}$ :

Apply Kirchhoff’s loop rule to the loop on the right to obtain:

Solving for $I_{\text {battery }}$ yields:
(b) After a long time, the currents are steady and the inductor acts like a short circuit, so the potential drop across the $100-\Omega$ resistor is zero. Apply the loop rule to the loop to the right to obtain:

Because $\frac{d I_{\mathrm{L}}}{d t}=0$ :

Apply the loop rule to the loop to the right to obtain:

$$
\text { or, because } I_{\text {battery }}=I_{100-\Omega} \text {, }
$$

$$
10.0 \mathrm{~V}-I_{\text {battery }}(10.0 \Omega)
$$

$$
-I_{\text {battery }}(100 \Omega)=0
$$

$$
-(2.00 \mathrm{H}) \frac{d I_{\mathrm{L}}}{d t}+I_{100-\Omega}(100 \Omega)=0
$$

$$
I_{100-\Omega}(100 \Omega)=0
$$

and

$$
I_{100-\Omega}=0
$$

$$
\begin{aligned}
& I_{L}=0 \\
& I_{\text {battery }}=I_{100-\Omega}+I_{L} \Rightarrow I_{\text {battery }}=I_{100-\Omega}
\end{aligned}
$$

$$
\begin{aligned}
10.0 \mathrm{~V}-I_{\text {battery }} & (10.0 \Omega) \\
& -I_{100-\Omega}(100 \Omega)=0
\end{aligned}
$$

$$
\begin{aligned}
& I_{\text {battery }}=I_{100-\Omega}=\frac{10.0 \mathrm{~V}}{10.0 \Omega+100 \Omega} \\
& =90.9 \mathrm{~mA}
\end{aligned}
$$

$$
\begin{aligned}
10.0 \mathrm{~V} & -I_{\text {battery }}(10.0 \mathrm{~V}) \\
& -I_{100-\Omega}(100 \Omega)=0
\end{aligned}
$$

Because $I_{100-\Omega}=0$ :

$$
10.0 \mathrm{~V}-I_{\text {battery }}(10.0 \mathrm{~V})=0
$$

and

$$
I_{\text {battery }}=1.00 \mathrm{~A}
$$

Apply the junction rule to the three currents to obtain:

$$
\begin{aligned}
& I_{\text {battery }}=I_{L}+I_{100-\Omega} \Rightarrow 1.00 \mathrm{~A}=I_{L}+0 \\
& \text { and so } I_{L}=1.00 \mathrm{~A}
\end{aligned}
$$

(c) When the switch is reopened, $I_{1}$ instantly becomes zero. The current $I_{L}$ in the inductor changes continuously, so at that instant
$I_{L}=1.00 \mathrm{~A}$. Apply the junction rule to obtain:
$I_{L}=1.00 \mathrm{~A}$,
$I_{\text {battery }}=I_{100-\Omega}+I_{L}$,
and
$I_{100-\Omega}=I_{\text {battery }}-I_{L}$

Substitute numerical values for $I_{\text {battery }}$ and $I_{\mathrm{L}}$ and evaluate $I_{100-\Omega}$ :

$$
I_{100-\Omega}=0-1.00 \mathrm{~A}=-1.00 \mathrm{~A}
$$

(d) A long time after the switch is opened, all the currents must be zero:

$$
I_{\text {battery }}=I_{L}=I_{100-\Omega}=0
$$

67 •• An inductor, two resistors, a make-before-break switch an a battery are connected as shown in Figure 28-57. The switch throw has been at contact $e$ for a long time and the current in the inductor is 2.5 A . Then, at $t=0$, the throw is quickly moved to contact $f$. During the next 45 ms the current in the inductor drops to 1.5 A . (a) What is the time constant for this circuit? (b) If the resistance $R$ is equal to $0.40 \Omega$, what is the value of the inductance $L$ ?

Picture the Problem The current in an initially energized but source-free $R L$ circuit is given by $I=I_{0} e^{-t / \tau}$. We can find $\tau$ from this equation and then use its definition to evaluate $L$.
(a) Express the current in the $R L$ circuit as a function of time:

$$
I=I_{0} e^{-t / \tau} \Rightarrow \tau=-\frac{t}{\ln \left(\frac{I}{I_{0}}\right)}
$$

Substitute numerical values and evaluate $\tau$ :

$$
\tau=-\frac{45 \mathrm{~ms}}{\ln \left(\frac{1.5 \mathrm{~A}}{2.5 \mathrm{~A}}\right)}=88.1 \mathrm{~ms}=88 \mathrm{~ms}
$$

(b) Using the definition of the

$$
L=\tau R
$$

inductive time constant, relate $L$ to
$R$ :

Substitute numerical values and evaluate $L$ :

68 •• A circuit consists of a coil that has a self-inductance equal to 5.00 mH and an internal resistance equal to $15.0 \Omega$, an ideal $12.0-\mathrm{V}$ battery and an open switch-all connected in series (Figure 28-58), At $t=0$ the switch is closed. Find the time at which the rate at which energy is dissipated in the coil equals the rate at which magnetic energy is stored in the coil.

Picture the Problem If the current is initially zero in an $L R$ circuit, its value at some later time $t$ is given by $I=I_{\mathrm{f}}\left(1-e^{-t / \tau}\right)$, where $I_{\mathrm{f}}=\mathcal{E}_{0} / R$ and $\tau=L / R$ is the time constant for the circuit. We can find the time at which the power dissipation in the resistor equals the rate at which magnetic energy is stored in the inductor by equating expressions for these rates and using the expression for $I$ and its rate of change.

Express the rate at which magnetic energy is stored in the inductor:

$$
\frac{d U_{L}}{d t}=\frac{d}{d t}\left[\frac{1}{2} L I^{2}\right]=L I \frac{d I}{d t}
$$

Express the rate at which power is dissipated in the resistor:

Equate these expressions to obtain:

Express the current and its rate of change:

$$
\begin{equation*}
I^{2} R=L I \frac{d I}{d t} \Rightarrow I=\tau \frac{d I}{d t} \tag{1}
\end{equation*}
$$

$$
I=I_{\mathrm{f}}\left(1-e^{-t / \tau}\right)
$$

and

$$
\begin{aligned}
\frac{d I}{d t} & =I_{\mathrm{f}} \frac{d}{d t}\left(1-e^{-t / \tau}\right)=-I_{\mathrm{f}} e^{-t / \tau}\left(-\frac{1}{\tau}\right) \\
& =\frac{I_{\mathrm{f}}}{\tau} e^{-t / \tau}
\end{aligned}
$$

Substitute for $d I / d t$ in equation (1) and simplify to obtain:

$$
I_{\mathrm{f}}\left(1-e^{-t / \tau}\right)=\tau\left(\frac{I_{\mathrm{f}}}{\tau} e^{-t / \tau}\right)
$$

$$
\begin{aligned}
& \text { or } \\
& 1-e^{-t / \tau}=e^{-t / \tau} \Rightarrow 1=2 e^{-t / \tau}
\end{aligned}
$$

Solving for $t$ and substituting for $\tau$ yields:

$$
t=-\tau \ln \frac{1}{2}=-\frac{L}{R} \ln \frac{1}{2}
$$

Substitute numerical values and evaluate $t$ :

$$
t=-\frac{5.00 \mathrm{mH}}{15.0 \Omega} \ln \frac{1}{2}=231 \mu \mathrm{~s}
$$

69 •••• [SSM] In the circuit shown in Figure 28-54, let $\boldsymbol{\varepsilon}_{0}=12.0 \mathrm{~V}$,
$R=3.00 \Omega$, and $L=0.600 \mathrm{H}$. The switch is closed at time $t=0$. During the time from $t=0$ to $t=L / R$, find (a) the amount of energy supplied by the battery, (b) the amount of energy dissipated in the resistor, and (c) the amount of energy delivered to the inductor. Hint: Find the energy transfer rates as functions of time and integrate.

Picture the Problem We can integrate $d E / d t=\boldsymbol{E}_{0} I$, where $I=I_{\mathrm{f}}\left(1-e^{-t / \tau}\right)$, to find the energy supplied by the battery, $d E_{\mathrm{J}} / d t=I^{2} R$ to find the energy dissipated in the resistor, and $U_{L}(\tau)=\frac{1}{2} L(I(\tau))^{2}$ to express the energy that has been stored in the inductor when $t=L / R$.
$\begin{aligned} & \text { (a) Express the rate at which energy } \\ & \text { is supplied by the battery: }\end{aligned} \quad \frac{d E}{d t}=\varepsilon_{0} I$
$\begin{aligned} & \text { Express the current in the circuit } \\ & \text { as a function of time: }\end{aligned} \quad I=\frac{\boldsymbol{\varepsilon}_{0}}{R}\left(1-e^{-t / \tau}\right)$

Substitute for $I$ to obtain:

$$
\frac{d E}{d t}=\frac{\varepsilon_{0}^{2}}{R}\left(1-e^{-t / \tau}\right)
$$

Separate variables and integrate from $t=0$ to $t=\tau$ to obtain:

$$
\begin{aligned}
E & =\frac{\boldsymbol{\varepsilon}_{0}^{2}}{R} \int_{0}^{\tau}\left(1-e^{-t / \tau}\right) d t \\
& =\frac{\boldsymbol{\varepsilon}_{0}^{2}}{R}\left[\tau-\left(-\tau e^{-1}+\tau\right)\right] \\
& =\frac{\boldsymbol{\varepsilon}_{0}^{2}}{R} \frac{\tau}{e}=\frac{\boldsymbol{\varepsilon}_{0}^{2} L}{R^{2} e}
\end{aligned}
$$

Substitute numerical values and evaluate $E$ :

$$
E=\frac{(12.0 \mathrm{~V})^{2}(0.600 \mathrm{H})}{(3.00 \Omega)^{2} e}=3.53 \mathrm{~J}
$$

(b) Express the rate at which energy is being dissipated in the resistor:

$$
\begin{aligned}
\frac{d E_{\mathrm{J}}}{d t} & =I^{2} R=\left[\frac{\varepsilon_{0}}{R}\left(1-e^{-t / \tau}\right)\right]^{2} R \\
& =\frac{\varepsilon_{0}^{2}}{R}\left(1-2 e^{-t / \tau}+e^{-2 t / \tau}\right)
\end{aligned}
$$

Separate variables and integrate from $t=0$ to $t=L / R$ to obtain:

$$
\begin{aligned}
E_{\mathrm{J}} & =\frac{\varepsilon_{0}^{2}}{R} \int_{0}^{L / R}\left(1-2 e^{-t / \tau}+e^{-2 t / \tau}\right) d t \\
& =\frac{\varepsilon_{0}^{2}}{R}\left(\frac{2 \frac{L}{R}}{e}-\frac{\frac{L}{R}}{2}-\frac{\frac{L}{R}}{2 e^{2}}\right) \\
& =\frac{\varepsilon_{0}^{2} L}{R^{2}}\left(\frac{2}{e}-\frac{1}{2}-\frac{1}{2 e^{2}}\right)
\end{aligned}
$$

Substitute numerical values and evaluate $E_{\mathrm{J}}$ :

$$
\begin{aligned}
E_{\mathrm{J}} & =\frac{(12.0 \mathrm{~V})^{2}(0.600 \mathrm{H})}{(3.00 \Omega)^{2}}\left(\frac{2}{e}-\frac{1}{2}-\frac{1}{2 e^{2}}\right) \\
& =1.61 \mathrm{~J}
\end{aligned}
$$

(c) Express the energy stored in the inductor when $t=\frac{L}{R}$ :

$$
\begin{aligned}
U_{L}\left(\frac{L}{R}\right) & =\frac{1}{2} L\left(I\left(\frac{L}{R}\right)\right) \\
& =\frac{1}{2} L\left(\frac{\varepsilon_{0}}{R}\left(1-e^{-1}\right)\right)^{2} \\
& =\frac{L \varepsilon_{0}^{2}}{2 R^{2}}\left(1-e^{-1}\right)^{2}
\end{aligned}
$$

Substitute numerical values and evaluate $U_{\mathrm{L}}$ :

$$
\begin{aligned}
U_{L}\left(\frac{L}{R}\right) & =\frac{(0.600 \mathrm{H})(12.0 \mathrm{~V})^{2}}{2(3.00 \Omega)^{2}}\left(1-e^{-1}\right)^{2} \\
& =1.92 \mathrm{~J}
\end{aligned}
$$

## Remarks: Note that, as we would expect from energy conservation,

 $E=E_{J}+E_{\text {L }}$.
## General Problems

70 - A 100-turn coil has a radius of 4.00 cm and a resistance of $25.0 \Omega$.
(a) The coil is in a uniform magnetic field that is perpendicular to the plane of the coil. What rate of change of the magnetic field strength will induce a current of 4.00 A in the coil? (b) What rate of change of the magnetic field strength is
required if the magnetic field makes an angle of $20^{\circ}$ with the normal to the plane of the coil?

Picture the Problem We can apply Faraday's and Ohm's laws to obtain expressions for the induced emf that we can equate and solve for the rate at which the perpendicular magnetic field must change to induce a current of 4.00 A in the coil.
(a) Using Faraday's law, relate the induced emf in the coil to the

$$
|\boldsymbol{E}|=\frac{d \phi_{\mathrm{m}}}{d t}=N A \frac{d B}{d t}
$$ changing magnetic flux:

Using Ohm's law, relate the induced emf to the resistance of the coil and the current in it:

Equate these expressions and solve for $d B / d t$ :

$$
N A \frac{d B}{d t}=I R \Rightarrow \frac{d B}{d t}=\frac{I R}{N A}=\frac{I R}{N \pi r^{2}}
$$

Substitute numerical values and evaluate $d B / d t$ :

$$
\frac{d B}{d t}=\frac{(4.00 \mathrm{~A})(25.0 \Omega)}{(100) \pi(0.0400 \mathrm{~m})^{2}}=199 \mathrm{~T} / \mathrm{s}
$$

(b) Using Faraday's law, relate the induced emf in the coil to the changing magnetic flux when the field makes an angle $\theta$ with respect

$$
|\varepsilon|=I R
$$

to the normal to the coil area:

Proceed as in (a) to obtain:

$$
\begin{aligned}
|\boldsymbol{\mathcal { E }}| & =\frac{d \phi_{\mathrm{m}}}{d t}=\frac{d}{d t}(N A \vec{B} \cdot \hat{n}) \\
& =N A \cos \theta \frac{d B}{d t}
\end{aligned}
$$

$$
\frac{d B}{d t}=\frac{I R}{N \pi r^{2} \cos \theta}
$$

Substitute numerical values and evaluate $d B / d t$ :

$$
\begin{aligned}
\frac{d B}{d t} & =\frac{(4.00 \mathrm{~A})(25.0 \Omega)}{(100) \pi(0.0400 \mathrm{~m})^{2} \cos 20^{\circ}} \\
& =212 \mathrm{~T} / \mathrm{s}
\end{aligned}
$$

71 •• [SSM] Figure 28-59 shows a schematic drawing of an ac generator. The basic generator consists of a rectangular loop of dimensions $a$ and $b$ and has $N$ turns connected to slip rings. The loop rotates (driven by a gasoline engine) at an angular speed of $\omega$ in a uniform magnetic field $\overrightarrow{\boldsymbol{B}}$. (a) Show that the induced potential difference between the two slip rings is given by $\mathcal{E}=N B a b \omega \sin \omega t$.
(b) If $a=2.00 \mathrm{~cm}, b=4.00 \mathrm{~cm}, N=250$, and $B=0.200 \mathrm{~T}$, at what angular frequency $\omega$ must the coil rotate to generate an emf whose maximum value is 100V?

Picture the Problem (a) We can apply Faraday's law and the definition of magnetic flux to derive an expression for the induced emf in the coil (potential difference between the slip rings). In Part (b) we can solve the equation derived in Part (a) for $\omega$ and evaluate this expression under the given conditions.
(a) Use Faraday's law to express the induced emf:

Using the definition of magnetic

$$
\begin{aligned}
& \varepsilon=-\frac{d \phi_{\mathrm{m}}(t)}{d t} \\
& \phi_{\mathrm{m}}(t)=N B A \cos \omega t
\end{aligned}
$$

flux, relate the magnetic flux through the loop to its angular velocity:

Substitute for $\phi_{\mathrm{m}}(t)$ to obtain:

$$
\begin{aligned}
\mathcal{E} & =-\frac{d}{d t}[N B A \cos \omega t] \\
& =-N B a b \omega(-\sin \omega t) \\
& =N B a b \omega \sin \omega t
\end{aligned}
$$

(b) Express the condition under which $\mathcal{E}=\mathcal{E}_{\text {max }}$ :
$\sin \omega t=1$
and

$$
\varepsilon_{\max }=N B a b \omega \Rightarrow \omega=\frac{\varepsilon_{\max }}{N B a b}
$$

Substitute numerical values and evaluate $\omega$ :

$$
\omega=\frac{100 \mathrm{~V}}{(250)(0.200 \mathrm{~T})(0.0200 \mathrm{~m})(0.0400 \mathrm{~m})}=2.50 \mathrm{krad} / \mathrm{s}
$$

72 •• Prior to 1960, magnetic field strengths were usually measured by a rotating coil gaussmeter. This device used a small multi-turn coil rotating at a high speed on an axis perpendicular to the magnetic field. This coil was connected to an ac voltmeter by means of slip rings, like those shown in Figure 28-61. In one specific design, the rotating coil has 400 turns and an area of $1.40 \mathrm{~cm}^{2}$. The coil rotates at $180 \mathrm{rev} / \mathrm{min}$. If the magnetic field strength is 0.450 T , find the maximum induced emf in the coil and the orientation of the normal to the plane of the coil relative to the field for which this maximum induced emf occurs.

Picture the Problem We can apply Faraday's law and the definition of magnetic flux to derive an expression for the induced emf in the rotating coil gaussmeter.

Use Faraday's law to express the induced emf:

$$
\varepsilon=-\frac{d \phi_{\mathrm{m}}}{d t}
$$

Using the definition of magnetic flux, relate the magnetic flux through the loop to its angular velocity:

Substitute for $\boldsymbol{\phi}_{\mathrm{m}}(\boldsymbol{t})$ and simplify to obtain:

$$
\phi_{\mathrm{m}}(t)=N B A \cos \omega t
$$

$$
\begin{aligned}
\mathcal{E} & =-\frac{d}{d t}[N B A \cos \omega t] \\
& =-N B A \omega(-\sin \omega t) \\
& =N B A \omega \sin \omega t=\varepsilon_{\max } \sin \omega t
\end{aligned}
$$

where

$$
\mathcal{E}_{\max }=N B A \omega
$$

Substitute numerical values and evaluate $\mathcal{E}_{\text {max }}$ :

$$
\mathcal{E}_{\max }=(400)(0.450 \mathrm{~T})\left(1.40 \times 10^{-4} \mathrm{~m}^{2}\right)\left(180 \frac{\mathrm{rev}}{\mathrm{~min}} \times \frac{2 \pi \mathrm{rad}}{\mathrm{rev}} \times \frac{1 \mathrm{~min}}{60 \mathrm{~s}}\right)=0.475 \mathrm{~V}
$$

The maximum induced emf occurs at the instant the normal to the plane of the coil is perpendicular to the magnetic field $\overrightarrow{\boldsymbol{B}}$. At this instant, $\phi_{\mathrm{m}}$ is zero, but $\varepsilon$ is a maximum.

73 •• Show that the equivalent self-inductance for two inductors that have self-inductances $L_{1}$ and $L_{2}$, and are connected in series is given by $L_{\text {eq }}=L_{1}+L_{2}$ if there is no flux linkage between the two inductors. (Saying there is no flux linkage between them is equivalent to saying that the mutual inductance between them is zero.)

Picture the Problem We can use the equality of the currents in the inductors connected in series and the additive nature of the total induced emf across the inductors to show that the self-inductances are additive.

Relate the total induced emf $\varepsilon$ to the effective self-inductance $L_{\text {eq }}$ and the rate at which the current is changing in the inductors:

Because the inductors $L_{1}$ and $L_{2}$ are in series:

$$
\varepsilon=L_{\mathrm{eq}} \frac{d I}{d t}
$$

$$
I_{1}=I_{2}=I \Rightarrow \frac{d I_{1}}{d t}=\frac{d I_{2}}{d t}=\frac{d I}{d t}
$$

Express the total induced emf:

$$
\begin{aligned}
\mathcal{E} & =\varepsilon_{1}+\varepsilon_{2}=L_{1} \frac{d I}{d t}+L_{2} \frac{d I}{d t} \\
& =\left(L_{1}+L_{2}\right) \frac{d I}{d t}
\end{aligned}
$$

Substitute in equation (1) and simplify to obtain:

$$
L_{\mathrm{eq}}=L_{1}+L_{2}
$$

74 •• Show that the equivalent self-inductance for two inductors that have self-inductances $L_{1}$ and $L_{2}$, and are connected in parallel is given by $\frac{1}{L_{\mathrm{eq}}}=\frac{1}{L_{1}}+\frac{1}{L_{2}}$ if there is no flux linkage between the two inductors. (Saying there is no flux linkage between them is equivalent to saying that the mutual inductance between them is equal to zero.)

Picture the Problem We can use the common potential difference across the parallel combination of inductors and the fact that the current into the parallel combination is the sum of the currents through each inductor to find an expression of the equivalent self-inductance.

Define $L_{\text {eq }}$ by:

$$
\begin{equation*}
L_{\mathrm{eq}}=\frac{\varepsilon}{d I / d t} \Rightarrow \frac{d I}{d t}=\varepsilon \frac{1}{L_{\mathrm{eq}}} \tag{1}
\end{equation*}
$$

Relate the common potential difference across the inductors to their self-inductances and the rate at which the current is changing in each:

$$
\begin{equation*}
\varepsilon_{1}=L_{1} \frac{d I_{1}}{d t} \tag{2}
\end{equation*}
$$

and

$$
\begin{equation*}
\varepsilon_{2}=L_{2} \frac{d I_{2}}{d t} \tag{3}
\end{equation*}
$$

Because the current divides at the parallel junction:

$$
I=I_{1}+I_{2} \Rightarrow \frac{d I}{d t}=\frac{d I_{1}}{d t}+\frac{d I_{2}}{d t}
$$

Solve equations (2) and (3) for $d I_{1} / d t$ and $d I_{2} / d t$ and substitute to

$$
\frac{d I}{d t}=\frac{\varepsilon_{1}}{L_{1}}+\frac{\varepsilon_{2}}{L_{2}}
$$

obtain:

Express the relationship between an

$$
\mathcal{E}=\mathcal{E}_{1}=\mathcal{E}_{2}
$$

emf $\boldsymbol{E}$ applied across the parallel combination of inductors and the emfs $\mathcal{E}_{1}$ and $\mathcal{E}_{2}$ across the individual inductors:

Substituting yields:

$$
\frac{d I}{d t}=\frac{\varepsilon}{L_{1}}+\frac{\varepsilon}{L_{2}}=\varepsilon\left(\frac{1}{L_{1}}+\frac{1}{L_{2}}\right)
$$

Substitute in equation (1) and solve for $1 / L_{\mathrm{eq}}$ :

$$
\frac{1}{L_{\mathrm{eq}}}=\frac{1}{L_{1}}+\frac{1}{L_{2}}
$$

75 •• A circuit consists of a 12 V battery, a switch, and a light bulb-all connected in series. It is known that the light bulb requires a minimum current of 0.10 A in order to produce a visible glow. In this circuit, this particular bulb draws 2.0 W when the switch has been closed for a long time. Now, an inductor is put in series with the bulb and the rest of the circuit. If the light bulb begins to glow 3.5 ms after the switch is closed, how large is the self-inductance of the inductor? Ignore any heating time of the filament and assume the glow is observed as soon as the current in the filament reaches the 0.10 A threshold.

Picture the Problem We can use Equation 28-25 to express the current in the circuit as a function of time and the expression $P=\mathcal{E}^{2} / R$ for the rate at which energy is dissipated in the light bulb to express the resistance of the circuit.

Use Equation 28-25 to express the current in the $R L$ circuit:

$$
\begin{aligned}
& I(t)=I_{0}\left(1-e^{-t / \tau}\right)=\frac{\varepsilon}{R}\left(1-e^{-R t / L}\right) \\
& P=\frac{\varepsilon^{2}}{R} \Rightarrow R=\frac{\varepsilon^{2}}{P}
\end{aligned}
$$

related to the rate
dissipates energy:
Substitute for $R$ and simplify to obtain:

$$
\begin{aligned}
& I(t)=\frac{\varepsilon}{\frac{\varepsilon}{\varepsilon^{2}}}\left(1-e^{-\frac{V^{2}}{P} t / L}\right)=\frac{P}{\varepsilon}\left(1-e^{-V^{2} t / P L}\right) \\
& \text { or, for } I(t)=I_{\min }, \\
& I_{\min }=\frac{P}{\varepsilon}\left(1-e^{-V^{2} t / P L}\right)
\end{aligned}
$$

Solving for $L$ yields:

$$
L=\frac{-\varepsilon^{2} t}{P \ln \left(1-\frac{\varepsilon I_{\min }}{P}\right)}
$$

Substitute numerical values and evaluate $L$ :

$$
\begin{aligned}
L & =\frac{-(12 \mathrm{~V})^{2}(3.5 \mathrm{~ms})}{(2.0 \mathrm{~W}) \ln \left(1-\frac{(12 \mathrm{~V})(0.10 \mathrm{~A})}{2.0 \mathrm{~W}}\right)} \\
& =0.28 \mathrm{H}
\end{aligned}
$$

76 •• Your friend decides to generate electrical power by rotating a 100 000turn coil of wire around an axis in the plane of the coil and through its center The coil is perpendicular to Earth' magnetic field in a region where the field strength is equal to 0.300 G . The loops of the coil have a radius 25.0 cm , and the coil has negligible resistance. (a) If your friend turns the coil at a rate of $150 \mathrm{rev} / \mathrm{s}$, what peak current will exist in a $1500-\Omega$ resistor that is connected across the terminals the coil? (b) The average of the square of the current will equal half of the square of the peak current. What will be the average power delivered to the resistor? Is this an economical way to generate power? HINT: Energy has to be expended to keep the coil rotating.

Picture the Problem We can use Ohm's law to express the peak current in terms of the peak induced emf in the coil and the resistance of the resistor attached to the coil and Faraday's law to find the peak induced emf in the coil.
(a) Apply Ohm's law to the coil and external resistor to obtain:

$$
I_{\text {peak }}=\frac{\mathcal{E}_{\max }}{R}
$$

Appling Faraday's law yields:

$$
\begin{aligned}
\varepsilon & =-\frac{d \phi_{\mathrm{m}}}{d t}=-\frac{d}{d t}(N B A \cos \omega t) \\
& =N B A \omega \sin \omega t=\varepsilon_{\max } \sin \omega t
\end{aligned}
$$

where $\boldsymbol{\varepsilon}_{\text {max }}=\boldsymbol{N B A} \boldsymbol{\omega}$.
Substitute for $\varepsilon_{\text {max }}$ to obtain:

$$
\begin{aligned}
I_{\text {peak }} & =\frac{N B A \omega}{R}=\frac{N B \pi r^{2}(2 \pi f)}{R} \\
& =\frac{2 \pi^{2} N B r^{2} f}{R}
\end{aligned}
$$

Substitute numerical values and evaluate $I_{\text {peak }}$ :

$$
\begin{aligned}
I_{\text {peak }} & =\frac{2 \pi^{2}\left(10^{5}\right)\left(0.300 \mathrm{G} \times \frac{1 \mathrm{~T}}{10^{4} \mathrm{G}}\right)(0.250 \mathrm{~m})^{2}\left(150 \frac{\mathrm{rev}}{\mathrm{~s}}\right)}{1500 \Omega}=370.11 \mathrm{~mA} \\
& =370 \mathrm{~mA}
\end{aligned}
$$

(b) The average power supplied is the power dissipated in Joule heat in the resistor:

Substitute numerical values and evaluate $P_{\mathrm{av}}$ :

$$
\begin{aligned}
P_{\mathrm{av}} & =\frac{1}{2}(370.11 \mathrm{~mA})^{2}(1500 \Omega) \\
& =103 \mathrm{~W}
\end{aligned}
$$

A power of 103 W is equal to 0.137 hp . Your friend is likely to tire. This is probably not an efficient way to generate power.

77 •• [SSM] Figure 28-60a shows an experiment designed to measure the acceleration due to gravity. A large plastic tube is encircled by a wire, which is arranged in single loops separated by a distance of 10 cm . A strong magnet is dropped through the top of the loop. As the magnet falls through each loop the voltage rises and then the voltage rapidly falls through zero to a large negative value and then returns to zero. The shape of the voltage signal is shown in Figure 28-62b. (a) Explain the basic physics behind the generation of this voltage pulse.
(b) Explain why the tube cannot be made of a conductive material.
(c) Qualitatively explain the shape of the voltage signal in Figure 28-60b. (d) The times at which the voltage crosses zero as the magnet falls through each loop in succession are given in the table in the next column. Use these data to calculate a value for $g$.

## Picture the Problem

(a) As the magnet passes through a loop it induces an emf because of the changing flux through the loop. This allows the coil to "sense" when the magnet is passing through it.
(b) One cannot use a cylinder made of conductive material because eddy currents induced in it by a falling magnet would slow the magnet.
(c) As the magnet approaches the loop the flux increases, resulting in the negative voltage signal of increasing magnitude. When the magnet is passing a loop, the flux reaches a maximum value and then decreases, so the induced emf becomes zero and then positive. The instant at which the induced emf is zero is the instant at which the magnet is at the center of the loop.
(d) Each time represents a point when the distance has increased by 10 cm . The following graph of distance versus time was plotted using a spreadsheet program. The regression curve, obtained using Excel's "Add Trendline" feature, is shown as a dashed line.


The coefficient of the second-degree term is $\frac{1}{2} g$. Consequently,

$$
g=2\left(4.9257 \mathrm{~m} / \mathrm{s}^{2}\right)=9.85 \mathrm{~m} / \mathrm{s}^{2}
$$

78 •• The rectangular coil shown in Figure 28-61 has 80 turns, is 25 cm wide, is 30 cm long, and is located in a magnetic field of 0.14 T directed out of the page, as shown. Only half of the coil is in the region of the magnetic field. The resistance of the coil is $24 \Omega$. Find the magnitude and the direction of the induced current if the coil is moved with a speed of $2.0 \mathrm{~m} / \mathrm{s}(a)$ to the right, (b) up the page, $(c)$ to the left, and $(d)$ down the page.

Picture the Problem The current equals the induced emf divided by the resistance. We can calculate the emf induced in the circuit as the coil moves by calculating the rate of change of the flux through the coil. The flux is proportional to the area of the coil in the magnetic field. We can find the direction of the current from Lenz's law.
(a) and (c) Express the magnitude of the induced current:

$$
\begin{equation*}
I=\frac{|\mathcal{E}|}{R} \tag{1}
\end{equation*}
$$

Using Faraday's law, express the magnitude of the induced emf:

$$
|\varepsilon|=\frac{d \phi_{\mathrm{m}}}{d t}
$$

When the coil is moving to the right (or to the left), the flux does not

$$
|\boldsymbol{\varepsilon}|=\frac{d \phi_{\mathrm{m}}}{d t}=0 \Rightarrow I=\frac{|\boldsymbol{\varepsilon}|}{R}=0
$$

change (until the coil leaves the region of magnetic field). Thus:
(b) Letting $x$ represent the length of $\quad \phi_{\mathrm{m}}=N B w x$ the side of the rectangular coil that is in the magnetic field, express the magnetic flux through the coil:

Compute the rate of change of the flux when the coil is moving up or down:

$$
\begin{aligned}
\frac{d \phi_{\mathrm{m}}}{d t} & =N B w \frac{d x}{d t} \\
& =(80)(0.14 \mathrm{~T})(0.25 \mathrm{~m})(2.0 \mathrm{~m} / \mathrm{s}) \\
& =5.60 \mathrm{~V}
\end{aligned}
$$

Substitute in equation (1) to obtain:

$$
I=\frac{5.60 \mathrm{~V}}{24 \Omega}=0.23 \mathrm{~A} \text { clockwise }
$$

(d) When the coil is moving downward, the outward flux decreases and the induced current will be in such a direction as to produce outward flux. The magnitude of the current is the same as in Part (b) and

$$
I=0.23 \mathrm{~A} \text { counterclockwise } .
$$

79 •• [SSM] A long solenoid has $n$ turns per unit length and carries a current that varies with time according to $I=I_{0} \sin \omega t$. The solenoid has a circular cross section of radius $R$. Find the induced electric field, at points near the plane equidistant from the ends of the solenoid, as a function of both the time $t$ and the perpendicular distance $r$ from the axis of the solenoid for (a) $r<R$ and (b) $r>R$.

Picture the Problem We can apply Faraday's law to relate the induced electric field $E$ to the rates at which the magnetic flux is changing at distances $r<R$ and $r>R$ from the axis of the solenoid.
(a) Apply Faraday's law to relate the induced electric field to the magnetic flux in the solenoid within a
cylindrical region of radius $r<R$ :

$$
\int_{\mathrm{C}} \overrightarrow{\boldsymbol{E}} \cdot \overrightarrow{d \ell}=-\frac{d \phi_{\mathrm{m}}}{d t}
$$

or

$$
\begin{equation*}
E(2 \pi r)=-\frac{d \phi_{\mathrm{m}}}{d t} \tag{1}
\end{equation*}
$$

Express the field within the solenoid: $\quad B=\mu_{0} n I$
Express the magnetic flux through $\quad \phi_{\mathrm{m}}=B A=\pi r^{2} \mu_{0} n I$
an area for which $r<R$ :

Substitute in equation (1) to obtain:

$$
\begin{aligned}
E(2 \pi r) & =-\frac{d}{d t}\left[\pi r^{2} \mu_{0} n I\right] \\
& =-\pi r^{2} \mu_{0} n \frac{d I}{d t}
\end{aligned}
$$

Because $I=I_{0} \sin \omega t$ :

$$
\begin{aligned}
E_{r<R} & =-\frac{1}{2} r \mu_{0} n \frac{d}{d t}\left[I_{0} \sin \omega t\right] \\
& =-\frac{1}{2} r \mu_{0} n I_{0} \omega \cos \omega t
\end{aligned}
$$

(b) Proceed as in (a) with $r>R$ to obtain:

$$
\begin{aligned}
E(2 \pi r) & =-\frac{d}{d t}\left[\pi R^{2} \mu_{0} n I\right] \\
& =-\pi R^{2} \mu_{0} n \frac{d I}{d t} \\
& =-\pi R^{2} \mu_{0} n I_{0} \omega \cos \omega t
\end{aligned}
$$

Solving for $E_{r>R}$ yields:

$$
E_{r>R}=-\frac{\mu_{0} n R^{2} I_{0} \omega}{2 r} \cos \omega t
$$

80 ••• A coaxial cable consists of two very thin-walled conducting cylinders of radii $r_{1}$ and $r_{2}$ (Figure 28-62). The currents in the inner and outer cylinders are equal in magnitude but opposite in direction. (a) Use Ampère's law to find the magnetic field as a function of the perpendicular distance $r$ from the central axis of the cable for (1) $0<r<r_{1}$ (2) $r_{1}<r<r_{2}$ and (3) $r>r_{2}$. (b) Show that the magnetic energy density in the region between the cylinders is given by $\mu_{\mathrm{m}}=\frac{1}{2}\left(\mu_{0} / 4 \pi\right) I^{2} /\left(\pi r^{2}\right)$. (c) Show that the total magnetic energy in a cable volume of length $\ell$ is given by $U_{\mathrm{m}}=\left(\mu_{0} / 4 \pi\right) I^{2} \ell \ln \left(r_{2} / r_{1}\right)$. (d) Use the result in Part (c) and the relationship between magnetic energy, current and inductance to show that the self-inductance per unit length of this cable arrangement is given by $L / \ell=\left(\mu_{0} / 2 \pi\right) \ln \left(r_{2} / r_{1}\right)$.

Picture the Problem The system exhibits cylindrical symmetry, so one can use Ampère's law to determine $B$ inside the inner cylinder, between the cylinders, and outside the outer cylinder. We can use $u_{\mathrm{m}}=B^{2} / 2 \mu_{0}$ and the expression for $B$ from Part (a) to express the magnetic energy density in the region between the cylinders. We can integrate this expression for $u_{\mathrm{m}}$ over the volume between the cylinders to find the total magnetic energy in a volume of length $\ell$. Finally, we can use our result in Part (c) and $U_{\mathrm{m}}=\frac{1}{2} L I^{2}$ to find the self-inductance of the cylinders per unit length.
(a) For $r<r_{1}$ and for $r>r_{2}$ the net enclosed current is zero; consequently, in these regions:

For $r_{1}<r<r_{2}$ :
(b) Express the magnetic energy density in the region between the cylinders:

Substitute for $B$ and simplify to obtain:
(c) Express the magnetic energy $d U_{\mathrm{m}}$ in the cylindrical element of volume $d V$ :

Integrate this expression from $r=r_{1}$ to $r=r_{2}$ to obtain:
(d) Express the energy in the magnetic field in terms of $L$ and $I$ :

From our result in (c):

$$
\frac{U_{\mathrm{m}}}{I^{2}}=\frac{\mu_{0}}{4 \pi} \ell \ln \left(\frac{r_{2}}{r_{1}}\right)
$$

Substitute to obtain:

$$
L=2\left[\frac{\mu_{0}}{4 \pi} \ell \ln \left(\frac{r_{2}}{r_{1}}\right)\right]=\frac{\mu_{0}}{2 \pi} \ell \ln \left(\frac{r_{2}}{r_{1}}\right)
$$

Express the ratio $L / \ell$ :

$$
\frac{L}{\ell}=\frac{\mu_{0}}{2 \pi} \ln \left(\frac{r_{2}}{r_{1}}\right)
$$

$81 \quad \cdots$ A coaxial cable consists of two very thin-walled conducting cylinders of radii $r_{1}$ and $r_{2}$ (Figure 28-63). The currents in the inner and outer cylinders are equal in magnitude but opposite in direction. Compute the flux through a rectangular area of sides $\ell$ and $r_{2}-r_{1}$ between the conductors shown in Figure 28-P95. Use the relationship between flux and current ( $\phi_{\mathrm{m}}=L I$ ) to show the selfinductance per unit length of the cable by is given by $L / \ell=\left(\mu_{0} / 2 \pi\right) \ln \left(r_{2} / r_{1}\right)$.

Picture the Problem We can use its definition to express the magnetic flux through a rectangular element of area $d A$ and then integrate from $r=r_{1}$ to $r=r_{2}$ to express the total flux through the region. Substituting in $L=\phi_{\mathrm{m}} / I$ will yield the same result found in Part (d) of Problem 80.

Use the definition of self-inductance
to relate the magnetic flux through the region of interest to the current $I$ :

Consider a strip of unit length $\ell$ and

$$
\begin{equation*}
L=\frac{\phi_{\mathrm{m}}}{I} \tag{1}
\end{equation*}
$$

width $d r$ at a distance $r$ from the axis. The flux through this area is given by:

Apply Ampere's law to express the magnetic field at a distance $r$ from the axis:

Substituting for $B$ yields:

$$
d \phi_{\mathrm{m}}=B d A=B \ell d r
$$

$$
d \phi_{\mathrm{m}}=\frac{\mu_{0} I \ell}{2 \pi} \frac{d r}{r}
$$

Integrate from $r=r_{1}$ to $r=r_{2}$ to obtain:

$$
\phi_{\mathrm{m}}=\frac{\mu_{0} I \ell}{2 \pi} \int_{r_{1}}^{r_{2}} \frac{d r}{r} \Rightarrow \phi_{\mathrm{m}}=\frac{\mu_{0} I \ell}{2 \pi} \ln \left(\frac{r_{2}}{r_{1}}\right)
$$

Substituting for $\phi_{\mathrm{m}}$ in equation (1) and dividing both sides by $\ell$ yields:

$$
2 \pi r B=\mu_{0} I \Rightarrow B=\frac{\mu_{0} I}{2 \pi r}
$$

$$
\frac{L}{\ell}=\frac{\mu_{0}}{2 \pi} \ln \left(\frac{r_{2}}{r_{1}}\right)
$$

82 ••• Figure 28-64 shows a rectangular loop of wire that is 0.300 m wide, is 1.50 m long, and lies in the vertical plane which is perpendicular to a region that has a uniform magnetic field. The magnitude of the uniform magnetic field is 0.400 T and the direction of the magnetic field is into the page. The portion of the loop not in the magnetic field is 0.100 m long. The resistance of the loop is $0.200 \Omega$ and its mass is 50.0 g . The loop is released from rest at $t=0$. (a) What is the magnitude and direction of the induced current when the loop has a downward speed $v$ ? (b) What is the force that acts on the loop as a result of this current?
(c) What is the net force acting on the loop? (d) Write out Newton's second law for the loop. (e) Obtain an expression for the speed of the loop as a function of time. (f) Integrate the expression obtained in Part (e) to find the distance the loop falls as a function of time. $(\mathrm{g})$ Using a spreadsheet program, make a graph of the position of the loop as a function of time (letting $t=0$ at the start) for values of $y$ between 0 and 1.40 m (i.e., when the loop leaves the magnetic field). ( $h$ ) At what time does the loop completely leave the field region? Compare this to the time it would have taken if there were no field.

Picture the Problem We can use $I=\boldsymbol{\mathcal { E }} / R$ and $\boldsymbol{\mathcal { E }}=B \boldsymbol{v} \ell$ to find the current induced in the loop and Lenz's law to determine its direction. We can apply the equation for the force on a current-carrying wire to find the net magnetic force acting on the loop and then sum the forces to find the net force on the loop. Separating the variables in the differential equation and integrating will lead us to an expression for $v(t)$ and a second integration to an expression for $y(t)$. We can solve the latter equation for $y=1.40 \mathrm{~m}$ to find the time it takes the loop to exit the magnetic field and our expression for $v(t)$ to find its exit speed. Finally, we can use a constantacceleration equation to find its exit speed in the absence of the magnetic field.
$\begin{aligned} & \text { (a) Relate the magnitude of the } \\ & \text { induced current to the induced emf }\end{aligned} \quad I=\frac{\varepsilon}{R}$ and the resistance of the loop:

Relate the induced emf $\boldsymbol{\varepsilon}$ to the motion of the loop:

$$
\mathcal{E}=B v \ell
$$

where $\ell$ is the length of the horizontal portion of the loop.

Substitute for $\varepsilon$ to obtain:

$$
I=\frac{B \ell}{R} v
$$

As the loop falls, the flux into it (the loop) decreases. The direction of the induced current is such that its magnetic field opposes this decrease; i.e., clockwise.
(b) Express the velocity-dependent $\quad F_{v}=B I \ell$ force that acts on the loop in terms of the current in the loop:

Substitute for $I$ to obtain:

$$
F_{v}=B\left(\frac{B \ell}{R}\right) v \ell=\frac{B^{2} \ell^{2}}{R} v
$$

Apply $\boldsymbol{d} \overrightarrow{\boldsymbol{F}}=\boldsymbol{I} \overrightarrow{\boldsymbol{\ell} \ell} \times \overrightarrow{\boldsymbol{B}}$ to the horizontal portion of the loop that is in the magnetic field to conclude that the net magnetic force is upward. Note that the magnetic force on the left side of the loop is to the left and the magnetic force on the right side of the loop is to the right.
(c) The net force acting on the loop is the difference between the downward gravitational force and the upward magnetic force:
(d) Apply Newton's $2^{\text {nd }}$ law of motion to the loop to obtain its equation of motion:

Factor $g$ to obtain an alternate form of the equation of motion:
(e) Separating the variables yields:

Integrate $v$ from 0 to $v$ and $t$ from 0 to $t$ :

Transforming from logarithmic to exponential form and solving for $v$ yields:

Noting that $v_{\mathrm{t}}=\frac{a}{b}$, we have:
(f) Write $v$ as $d y / d t$ and separate variables to obtain:

$$
\begin{aligned}
F_{\text {net }} & =m g-F_{v} \\
& =m g-\frac{B^{2} \ell^{2}}{R} v
\end{aligned}
$$

$$
m g-\frac{B^{2} \ell^{2}}{R} v=m \frac{d v}{d t} \Rightarrow \frac{d v}{d t}=g-\frac{B^{2} \ell^{2}}{m R} v
$$

$$
\frac{d v}{d t}=g\left(1-\frac{B^{2} \ell^{2}}{m g R} v\right)=g\left(1-\frac{v}{v_{\mathrm{t}}}\right)
$$

$$
\text { where } v_{\mathrm{t}}=\frac{m g R}{B^{2} \ell^{2}}
$$

$$
\frac{d v}{g-\frac{B^{2} \ell^{2}}{m R} v}=d t \text { or } \frac{d v}{a-b v}=d t
$$

$$
\text { where } a=g \text { and } b=\frac{B^{2} \ell^{2}}{m R}
$$

$$
\int_{0}^{v} \frac{d v}{a-b v}=\int_{0}^{t} d t \Rightarrow-\frac{1}{b} \ln \left(\frac{a-b v}{a}\right)=t
$$

$$
v(t)=\frac{a}{b}\left(1-e^{-b t}\right)
$$

$v(t)=v_{\mathrm{t}}\left(1-e^{-t / \tau}\right)$
where $v_{\mathrm{t}}=\frac{m g R}{B^{2} \ell^{2}}$ and $\tau=\frac{v_{\mathrm{t}}}{a}=\frac{\nu_{\mathrm{t}}}{g}$.

$$
d y=v_{\mathrm{t}}\left(1-e^{-t / \tau}\right) d t
$$

Integrate $y$ from 0 to $y$ and $t$ from 0 to $t$ :
$\int_{0}^{y} d y=v_{\mathrm{t}} \int_{0}^{t}\left(1-e^{-t / \tau}\right) d t$
and

$$
y(t)=v_{\mathrm{t}}\left[t-\tau\left(1-e^{-t / \tau}\right)\right]
$$

$(g)$ A spreadsheet program to generate the data for graphs of position $y$ as a function of time $t$ is shown below. The formulas used to calculate the quantities in the columns are as follows:

| Cell | Formula/Content | Algebraic Form |
| :---: | :---: | :---: |
| B1 | 0.0500 | $m$ |
| B2 | 0.200 | $R$ |
| B3 | 0.400 | $B$ |
| B4 | 0.300 | $L$ |
| B5 | $\$ \mathrm{~B} \$ 1 * \$ \mathrm{~B} \$ 7 * \$ 32 /\left(\$ \mathrm{~B} \$ 3 \wedge 2 * \$ \mathrm{~B} \$ 4^{\wedge} 2\right)$ | $v_{\mathrm{t}}$ |
| B6 | $\$ \mathrm{~B} \$ 5 / \$ \mathrm{~B} \$ 7$ | $\tau$ |
| B7 | 9.81 | $g$ |
| A10 | 0.00 | $t$ |
| B10 | $\$ \mathrm{~B} \$ 5^{*}\left(\mathrm{~A} 10-\$ \mathrm{~B} \$ 6^{*}(1-\mathrm{EXP}(-\mathrm{A} 10 / \$ \mathrm{~B} \$ 6))\right)$ | $y$ |
| C10 | $0.5^{*} \$ \mathrm{~B} \$ 7^{*} \mathrm{~A} 10^{\wedge} 2$ | $\frac{1}{2} g t^{2}$ |


|  | A | B | C |
| :--- | :---: | :--- | :--- |
| 1 | $m=$ | 0.0500 | kg |
| 2 | $R=$ | 0.200 | ohms |
| 3 | $B=$ | 0.400 | T |
| 4 | $L=$ | 0.300 | m |
| 5 | $v_{t}=$ | 6.813 | $\mathrm{~m} / \mathrm{s}$ |
| 6 | $\tau=$ | 0.694 | s |
| 7 | $g=$ | 9.81 | $\mathrm{~m} / \mathrm{s}^{2}$ |
| 8 |  |  |  |
| 9 | $t$ | $y$ | $y(\mathrm{no} B)$ |
| 10 | 0.00 | 0.000 | 0.000 |
| 11 | 0.01 | 0.000 | 0.000 |
| 12 | 0.02 | 0.002 | 0.002 |
| 13 | 0.03 | 0.004 | 0.004 |
| 14 | 0.04 | 0.008 | 0.008 |
| 15 | 0.05 | 0.012 | 0.012 |
|  |  |  |  |
| 62 | 0.52 | 1.049 | 1.326 |
| 63 | 0.53 | 1.085 | 1.378 |
| 64 | 0.54 | 1.122 | 1.430 |
| 65 | 0.55 | 1.159 | 1.484 |
| 66 | 0.56 | 1.196 | 1.538 |


| 67 | 0.57 | 1.234 | 1.594 |
| :---: | :---: | :---: | :---: |
| 68 | 0.58 | 1.273 | 1.650 |
| 69 | 0.59 | 1.311 | 1.707 |
| 70 | 0.60 | 1.351 | 1.766 |
| 71 | 0.61 | 1.390 | 1.825 |
| 72 | 0.62 | 1.430 | 1.885 |
| 73 | 0.63 | 1.471 | 1.947 |
| 74 | 0.64 | 1.511 | 2.009 |
|  |  |  |  |

Examination of either the table or the following graph shows that, when the loop falls in the magnetic field, $y=1.4 \mathrm{~m}$ when $t \approx 0.61 \mathrm{~s}$. In the absence of the magnetic field, $y=1.4 \mathrm{~m}$ when $t \approx 0.53 \mathrm{~s}$.

The following graph shows $y$ as a function of $t$ for $B \neq 0$ (solid curve) and $B=0$ (dashed curve).


In the absence of the magnetic field, the loop fall a distance of 1.40 m in about 0.08 s less time than it takes to fall the same distance in the presence of the magnetic field.
$83 \quad$ ••• A coil of $N$ turns and area $A$ suspended from the ceiling by a wire that provides a linear restoring torque with torsion constant $\kappa$. The two ends of the coil are connected to each other, the coil has resistance $R$, and the moment of inertia of the coil is $I$. The plane of the coil is vertical, and parallel to a uniform horizontal magnetic field $\overrightarrow{\boldsymbol{B}}$ when the wire is not twisted (i.e., $\theta=0$ ). The coil is rotated about a vertical axis through its center by a small angle $\theta_{0}$ and released. The coil then undergoes damped harmonic oscillation. Show that its angle with its
equilibrium position will vary with time according to $\theta(t)=\theta_{0} e^{-t / 2 \tau} \cos \omega^{\prime} t$, where $\tau=R I /(N B A)^{2}, \omega_{0}=\sqrt{\kappa / I}$ and $\omega^{\prime}=\omega_{0} \sqrt{1-\left(2 \omega_{0} \tau\right)^{-2}}$.

Picture the Problem Picture the Problem If the coil is rotated through an angle $\theta$, the wire exerts a restoring torque equal to $-\kappa \theta$ acts on it returning it to its equilibrium position. However, when it rotates with angular speed $d \theta / d t$, there will be an emf induced in the coil. The direction of the current resulting from this induced emf will be such that its magnetic field will oppose the change in flux resulting from the rotation of the coil. The net effect is that the magnetic field exerts a torque on the coil in a direction opposite to the direction of the angular velocity of the coil. We can show that $\theta$ will vary with time according to $\theta(t)=\theta_{0} e^{-t / 2 \tau} \cos \omega^{\prime} t$, where $\tau=R I /(N B A)^{2}, \omega_{0}=\sqrt{\kappa / I}$ and $\omega^{\prime}=\omega_{0} \sqrt{1-\left(2 \omega_{0} \tau\right)^{-2}}$ by demonstrating that $\theta(t)=\theta_{0} e^{-t / 2 \tau} \cos \omega^{\prime} t$ satisfies the differential equation obtained in our solution for Part (a).

Apply $\sum \tau=I \alpha$ to the rotating coil to obtain:

The magnitude of the retarding (damping) torque is given by:

Substitute for $\tau_{\text {restoring }}$ and $\tau_{\text {retarding }}$ to obtain:

Apply Faraday's law to express the emf induced in the coil:

From Ohm's law, the magnitude of the induced current $i$ in the coil is:

Substitute for the induced current $i$ in equation (1) to obtain:

For small displacements from equilibrium, $\cos \theta \approx 1$ and:

$$
\tau_{\text {restoring }}-\tau_{\text {retarding }}=I \frac{d^{2} \theta}{d t^{2}}
$$

$\tau_{\text {retarding }}=N i B A \cos \theta$
where $i$ is the current induced in the coil whose cross-sectional area is $A$.

$$
\begin{equation*}
-\kappa \theta-N i B A \cos \theta=I \frac{d^{2} \theta}{d t^{2}} \tag{1}
\end{equation*}
$$

$$
\varepsilon=-\frac{d}{d t}(N B A \sin \theta)=-(N B A \cos \theta) \frac{d \theta}{d t}
$$

$$
i=\frac{\mathcal{E}}{R}=\frac{N B A \cos \theta}{R} \frac{d \theta}{d t}
$$

$$
-\kappa \theta-\frac{(N B A)^{2} \cos ^{2} \theta}{R} \frac{d \theta}{d t}=I \frac{d^{2} \theta}{d t^{2}}
$$

$$
-\kappa \theta-\frac{(N B A)^{2}}{R} \frac{d \theta}{d t} \approx I \frac{d^{2} \theta}{d t^{2}}
$$

Rearrange to express the differential equation in standard form, then substitute using $\omega_{0}=\sqrt{\kappa / I}$ and

$$
\frac{d^{2} \theta}{d t^{2}}+\frac{(N B A)^{2}}{R I} \frac{d \theta}{d t}+\frac{\kappa}{I} \theta \approx 0
$$

$\tau=R I /(N B A)^{2}:$
or

$$
\begin{equation*}
\frac{d^{2} \theta}{d t^{2}}+\frac{1}{\tau} \frac{d \theta}{d t}+\omega_{0}^{2} \theta=0 \tag{2}
\end{equation*}
$$

Assume that the solution to equation (2) is given by $\theta(t)=\theta_{0} e^{-t / 2 \tau} \cos \omega^{\prime} t$ and evaluate its first and second derivatives with respect to time:

$$
\begin{aligned}
\frac{d \theta}{d t} & =\theta_{0} \frac{d}{d t}\left[e^{-t / 2 \tau} \cos \omega^{\prime} t\right]=\theta_{0}\left[e^{-t / 2 \tau} \frac{d}{d t}\left(\cos \omega^{\prime} t\right)+\cos \omega^{\prime} t \frac{d}{d t}\left(e^{-t / 2 \tau}\right)\right] \\
& =\theta_{0}\left(-\omega^{\prime} e^{-t / 2 \tau} \sin \omega^{\prime} t-\frac{1}{2 \tau} e^{-t / 2 \tau} \theta_{0} \cos \omega^{\prime} t\right) \\
& =-\theta_{0} e^{-t / 2 \tau}\left(\omega^{\prime} \sin \omega^{\prime} t+\frac{1}{2 \tau} \cos \omega^{\prime} t\right)
\end{aligned}
$$

and

$$
\begin{aligned}
& \frac{d^{2} \theta}{d t^{2}}= \frac{d}{d t}\left[-\theta_{0} \omega^{\prime} e^{-t / 2 \tau}\right. \\
&\left.=-\theta_{0} e^{-t / 2 \tau} \omega^{\prime} t-\theta_{0} \frac{1}{2 \tau} e^{-t / 2 \tau} \cos \omega^{\prime} t\right] \\
&\left(\omega^{\prime} \sin \omega^{\prime} t+\frac{1}{2 \tau} \cos \omega^{\prime} t\right) \\
&=-\theta_{0}\left(\omega_{0} e^{\prime} \sin \omega^{\prime} t+\frac{1}{2 \tau} \cos \omega^{\prime} t\right) \frac{d}{d t} e^{-t / 2 \tau} \\
&+\omega_{0} \frac{1}{2 \tau}\left(\omega^{\prime 2} \cos \omega^{\prime} t-\frac{\omega^{\prime}}{2 \tau} \sin \omega^{\prime} t\right) \\
&=-\theta_{0} e^{-t / 2 \tau}\left(\omega^{\prime 2} \cos \omega^{\prime} t-\frac{1}{2 \tau} \cos \omega^{\prime} t\right) e^{-t / 2 \tau} \\
&\left.\sin \omega^{\prime} t-\frac{1}{4 \tau^{2}} \cos \omega^{\prime} t\right)
\end{aligned}
$$

Substitute these derivatives in equation (2) to obtain:

$$
\begin{aligned}
-\theta_{0} e^{-t / 2 \tau} & \left(\omega^{\prime 2} \cos \omega^{\prime} t-\frac{\omega^{\prime}}{\tau} \sin \omega^{\prime} t-\frac{1}{4 \tau^{2}} \cos \omega^{\prime} t\right) \\
& -\frac{1}{\tau} \theta_{0} e^{-t / 2 \tau}\left(\omega^{\prime} \sin \omega^{\prime} t+\frac{1}{2 \tau} \cos \omega^{\prime} t\right)+\omega_{0}^{2} \theta_{0} e^{-t / 2 \tau} \cos \omega^{\prime} t=0
\end{aligned}
$$

Because $\theta_{0}$ and $e^{-t / \tau}$ are never zero, we can divide them out of the equation and simplify to obtain:

$$
-\omega^{\prime 2} \cos \omega^{\prime} t-\frac{1}{4 \tau^{2}} \cos \omega^{\prime} t+\omega_{0}^{2} \cos \omega^{\prime} t=0
$$

or

$$
\left(-\omega^{\prime 2}-\frac{1}{4 \tau^{2}}+\omega_{0}^{2}\right) \cos \omega^{\prime} t=0
$$

This equation is satisfied provided:

$$
-\omega^{12}-\frac{1}{4 \tau^{2}}+\omega_{0}^{2}=0
$$

Solving for $\omega$ ' yields:

$$
\omega^{\prime}=\omega_{0} \sqrt{1-\left(2 \omega_{0} \tau\right)^{-2}}
$$

Thus we've shown that the angular position of the oscillating coil, relative to its equilibrium position, varies with time according to $\theta(t)=\theta_{0} e^{-t / 2 \tau} \cos \omega^{\prime} t$, where $\tau=R I /(N B A)^{2}, \omega_{0}=\sqrt{\kappa / I}$ and $\omega^{\prime}=\omega_{0} \sqrt{1-\left(2 \omega_{0} \tau\right)^{-2}}$.

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