Chapter 24
Capacitance

Conceptual Problems

1 • If the voltage across a parallel-plate capacitor is doubled, its capacitance (*a*) doubles (*b*) drops by half (*c*) remains the same.

Determine the Concept The capacitance of a parallel-plate capacitor is a function of the surface area of its plates, the separation of these plates, and the electrical properties of the matter between them. The capacitance is, therefore, independent of the voltage across the capacitor. $\boxed{(c)}$ is correct.

2 • If the charge on an isolated spherical conductor is doubled, its self-capacitance (*a*) doubles (*b*) drops by half (*c*) remains the same.

Determine the Concept The capacitance of an isolated spherical capacitor is given by $C = 4\pi\epsilon_0 R$, where *R* is its radius. The capacitance is, therefore, independent of the charge of the capacitor. $\boxed{(c)}$ is correct.

3 • True or false: The electrostatic energy density is uniformly distributed in the region between the conductors of a cylindrical capacitor.

Determine the Concept False. The electrostatic energy density is not uniformly distributed because the magnitude of the electric field strength is not uniformly distributed,

4 • If the distance between the plates of a charged and isolated parallel-plate capacitor is doubled, what is the ratio of the final stored energy to the initial stored energy?

Determine the Concept The energy stored in the electric field of a parallel-plate capacitor is related to the potential difference across the capacitor by $U = \frac{1}{2}QV$. If *Q* is constant, *U* is directly proportional to *V* and doubling *V* doubles *U*. Hence the ratio of the initial stored energy to the final stored energy is $\boxed{2}$.

5 • **[SSM]** A parallel-plate capacitor is connected to a battery. The space between the two plates is empty. If the separation between the capacitor plates is tripled while the capacitor remains connected to the battery, what is the ratio of the final stored energy to the initial stored energy?

Determine the Concept The energy stored in a capacitor is given by $U = \frac{1}{2}QV$ and the capacitance of a parallel-plate capacitor by $C = \epsilon_0 A/d$. We can

combine these relationships, using the definition of capacitance and the condition that the potential difference across the capacitor is constant, to express U as a function of d.

Express the energy stored in the capacitor:

$$U = \tfrac{1}{2}QV \qquad (1)$$

Use the definition of capacitance to express the charge of the capacitor:

$$Q = CV$$

Express the capacitance of a parallel-plate capacitor in terms of the separation d of its plates:

$$C = \frac{\epsilon_0 A}{d}$$

where A is the area of one plate.

Substituting for Q and C in equation (1) yields:

$$U = \frac{\epsilon_0 AV^2}{2d}$$

Because $U \propto \frac{1}{d}$, tripling the separation of the plates will reduce the energy stored in the capacitor to one-third its previous value. Hence the ratio of the final stored energy to the initial stored energy is $\boxed{1/3}$.

6 • If the capacitor of Problem 5 is disconnected from the battery before the separation between the plates is tripled, what is the ratio of the final stored energy to the initial stored energy?

Picture the Problem Let V represent the initial potential difference between the plates, U the energy stored in the capacitor initially, d the initial separation of the plates, and V', U', and d' these physical quantities when the plate separation has been tripled. We can use $U = \tfrac{1}{2}QV$ to relate the energy stored in the capacitor to the potential difference across it and $V = Ed$ to relate the potential difference to the separation of the plates.

Express the energy stored in the capacitor before the tripling of the separation of the plates:

$$U = \tfrac{1}{2}QV$$

Express the energy stored in the capacitor after the tripling of the separation of the plates:

$U' = \frac{1}{2}QV'$

because the charge on the plates does not change.

Express the ratio of U' to U and simplify to obtain:

$$\frac{U'}{U} = \frac{\frac{1}{2}QV'}{\frac{1}{2}QV} = \frac{V'}{V}$$

The potential differences across the capacitor plates before and after the plate separation, in terms of the electric field E between the plates, are given by:

$V = Ed$

and

$V' = Ed'$

because E depends solely on the charge on the plates and, as observed above, the charge does not change during the separation process.

Substituting for V and V' to obtain:

$$\frac{U'}{U} = \frac{Ed'}{Ed} = \frac{d'}{d}$$

For $d' = 3d$:

$\frac{U'}{U} = \frac{3d}{d} = 3 \Rightarrow$ The ratio of the final stored energy to the initial stored energy is $\boxed{3}$.

7 • True or false:

(*a*) The equivalent capacitance of two capacitors in parallel is always greater than the larger of the two capacitance values.

(*b*) The equivalent capacitance of two capacitors in series is always less than the least of the two capacitance values if the charges on the two plates that are connected by an otherwise isolated conductor sum to zero.

(*a*) True. The equivalent capacitance of two capacitors in parallel is the sum of the individual capacitances.

(*b*) True. The equivalent capacitance of two capacitors in series is the reciprocal of the sum of the reciprocals of the individual capacitances.

8 • Two uncharged capacitors have capacitances C_0 and $2C_0$, respectively, and are connected in series. This series combination is then connected across the terminals a battery. Which of the following is true?

(*a*) The capacitor $2C_0$ has twice the charge of the other capacitor.
(*b*) The voltage across each capacitor is the same.
(*c*) The energy stored by each capacitor is the same.
(*d*) The equivalent capacitance is $3C_0$.
(*e*) The equivalent capacitance is $2C_0/3$.

(*a*) False. Capacitors connected in series carry the same charge Q.

(*b*) False. The voltage V across a capacitor whose capacitance is C_0 is Q/C_0 and the voltage across the second capacitor is $Q/(2C_0)$.

(*c*) False. The energy stored in a capacitor is given by $\frac{1}{2}QV$.

(*d*) False. This would be the equivalent capacitance if they were connected in parallel.

(*e*) True. Taking the reciprocal of the sum of the reciprocals of C_0 and $2C_0$ yields $C_{eq} = 2C_0/3$.

9 • **[SSM]** A dielectric is inserted between the plates of a parallel-plate capacitor, completely filling the region between the plates. Air initially filled the region between the two plates. The capacitor was connected to a battery during the entire process. True or false:

(*a*) The capacitance value of the capacitor increases as the dielectric is inserted between the plates.
(*b*) The charge on the capacitor plates decreases as the dielectric is inserted between the plates.
(*c*) The electric field between the plates does not change as the dielectric is inserted between the plates.
(*d*) The energy storage of the capacitor decreases as the dielectric is inserted between the plates.

Determine the Concept The capacitance of the capacitor is given by $C = \frac{\kappa \epsilon_0 A}{d}$, the charge on the capacitor is given by $Q = CV$, and the energy stored in the capacitor is given by $U = \frac{1}{2}CV^2$.

(*a*) True. As the dielectric material is inserted, κ increases from 1 (air) to its value for the given dielectric material.

(*b*) False. Because $Q = CV$, and C increases, Q must increase.

(*c*) True. $E = V/d$, where d is the plate separation.

(*d*) False. The energy storage of a capacitor is independent of the presence of dielectric and is given by $U = \frac{1}{2}QV$.

10 •• Capacitors A and B (Figure 24-33) have identical plate areas and gap separations. The space between the plates of each capacitor is half-filled with a dielectric as shown. Which has the larger capacitance, capacitor A or capacitor B? Explain your answer.

Picture the Problem We can treat configuration A as two capacitors in parallel and configuration B as two capacitors in series. Finding the equivalent capacitance of each configuration and examining their ratio will allow us to decide whether A or B has the greater capacitance. In both cases, we'll let C_1 be the capacitance of the dielectric-filled capacitor and C_2 be the capacitance of the air capacitor.

In configuration A we have:

$$\boldsymbol{C}_{\mathrm{A}} = \boldsymbol{C}_1 + \boldsymbol{C}_2$$

Express C_1 and C_2:

$$C_1 = \frac{\kappa\,\epsilon_0\,A_1}{d_1} = \frac{\kappa\,\epsilon_0\,\frac{1}{2}A}{d} = \frac{\kappa\,\epsilon_0\,A}{2d}$$

and

$$C_2 = \frac{\epsilon_0\,A_2}{d_2} = \frac{\epsilon_0\,\frac{1}{2}A}{d} = \frac{\epsilon_0\,A}{2d}$$

Substitute for C_1 and C_2 and simplify to obtain:

$$C_{\mathrm{A}} = \frac{\kappa\,\epsilon_0\,A}{2d} + \frac{\epsilon_0\,A}{2d} = \frac{\epsilon_0\,A}{2d}(\kappa + 1)$$

In configuration B we have:

$$\frac{1}{C_{\mathrm{B}}} = \frac{1}{C_1} + \frac{1}{C_2} \Rightarrow C_b = \frac{C_1C_2}{C_1 + C_2}$$

Express C_1 and C_2:

$$C_1 = \frac{\epsilon_0\,A_1}{d_1} = \frac{\epsilon_0\,A}{\frac{1}{2}d} = \frac{2\,\epsilon_0\,A}{d}$$

and

$$C_2 = \frac{\kappa\,\epsilon_0\,A_2}{d_2} = \frac{\kappa\,\epsilon_0\,A}{\frac{1}{2}d} = \frac{2\kappa\,\epsilon_0\,A}{d}$$

Substitute for C_1 and C_2 and simplify to obtain:

$$C_B = \frac{\left(\frac{2\epsilon_0 A}{d}\right)\left(\frac{2\kappa\epsilon_0 A}{d}\right)}{\frac{2\epsilon_0 A}{d} + \frac{2\kappa\epsilon_0 A}{d}} = \frac{\left(\frac{2\epsilon_0 A}{d}\right)\left(\frac{2\kappa\epsilon_0 A}{d}\right)}{\frac{2\epsilon_0 A}{d}(\kappa+1)} = \frac{2\epsilon_0 A}{d}\left(\frac{\kappa}{\kappa+1}\right)$$

Divide $\boldsymbol{C}_B$ by $\boldsymbol{C}_A$

$$\frac{C_B}{C_A} = \frac{\frac{2\epsilon_0 A}{d}\left(\frac{\kappa}{\kappa+1}\right)}{\frac{\epsilon_0 A}{2d}(\kappa+1)} = \frac{4\kappa}{(\kappa+1)^2}$$

Because $\frac{4\kappa}{(\kappa+1)^2} < 1$ for $\kappa > 1$:

$$\boxed{C_A > C_B}$$

11 •• **[SSM]** (*a*) Two identical capacitors are connected in parallel. This combination is then connected across the terminals of a battery. How does the total energy stored in the parallel combination of the two capacitors compare to the total energy stored if just one of the capacitors were connected across the terminals of the same battery? (*b*) Two identical capacitors that have been discharged are connected in series. This combination is then connected across the terminals of a battery. How does the total energy stored in the series combination of the two capacitors compare to the total energy stored if just one of the capacitors were connected across the terminals of the same battery?

Picture the Problem The energy stored in a capacitor whose capacitance is C and across which there is a potential difference V is given by $U = \frac{1}{2}CV^2$. Let C_0 represent the capacitance of the each of the two identical capacitors.

(*a*) The energy stored in the parallel system is given by:

$$U_{\text{parallel}} = \tfrac{1}{2}C_{\text{eq}}V^2$$

When the capacitors are connected in parallel, their equivalent capacitance is:

$$C_{\text{parallel}} = C_0 + C_0 = 2C_0$$

Substituting for C_{eq} and simplifying yields:

$$U_{\text{parallel}} = \tfrac{1}{2}(2C_0)V^2 = C_0V^2 \qquad (1)$$

If just one capacitor is connected to the same battery the stored energy is:

$$U_{1\,\text{capacitor}} = \tfrac{1}{2}C_0V^2 \qquad (2)$$

Dividing equation (1) by equation (2) and simplifying yields:

$$\frac{U_{\text{parallel}}}{U_{1\,\text{capacitor}}} = \frac{C_0V^2}{\tfrac{1}{2}C_0V^2} = 2$$

or

$$U_{\text{parallel}} = \boxed{2U_{1\,\text{capacitor}}}$$

(*b*) The energy stored in the series system is given by:

$$U_{\text{series}} = \tfrac{1}{2}C_{eq}V^2$$

When the capacitors are connected in series, their equivalent capacitance is:

$$C_{\text{series}} = \tfrac{1}{2}C_0$$

Substituting for C_{eq} and simplifying yields:

$$U_{\text{series}} = \tfrac{1}{2}\left(\tfrac{1}{2}C_0\right)V^2 = \tfrac{1}{4}C_0V^2 \qquad (3)$$

Dividing equation (3) by equation (2) and simplifying yields:

$$\frac{U_{\text{series}}}{U_{1\,\text{capacitor}}} = \frac{\tfrac{1}{4}C_0V^2}{\tfrac{1}{2}C_0V^2} = \tfrac{1}{2}$$

or

$$U_{\text{series}} = \boxed{\tfrac{1}{2}U_{1\,\text{capacitor}}}$$

12 •• Two identical capacitors that have been discharged are connected in series across the terminals of a 100-V battery. When only one of the capacitors is connected across the terminals of this battery, the energy stored is U_0. What is the total energy stored in the two capacitors when the series combination is connected to the battery? (*a*) $4U_0$, (*b*) $2U_0$, (*c*) U_0, (*d*) $U_0/2$, (*e*) $U_0/4$

Picture the Problem We can use the expression $U = \tfrac{1}{2}CV^2$ to express the ratio of the energy stored in the single capacitor and in the identical-capacitors-in-series combination.

Express the energy stored in the capacitors when they are connected to the 100-V battery:

$$U = \tfrac{1}{2}C_{eq}V^2$$

Express the equivalent capacitance of the two identical capacitors connected in series:

$$\frac{1}{\boldsymbol{C}_{\text{eq}}}=\frac{1}{\boldsymbol{C}}+\frac{1}{\boldsymbol{C}}\Rightarrow\boldsymbol{C}_{\text{eq}}=\tfrac{1}{2}\boldsymbol{C}$$

Substitute for C_{eq} to obtain:

$$U=\tfrac{1}{2}\left(\tfrac{1}{2}C\right)V^2=\tfrac{1}{4}CV^2$$

Express the energy stored in one capacitor when it is connected to the 100-V battery:

$$U_0=\tfrac{1}{2}CV^2$$

Express the ratio of U to U_0:

$$\frac{U}{U_0}=\frac{\tfrac{1}{4}CV^2}{\tfrac{1}{2}CV^2}=\frac{1}{2}\Rightarrow U=\tfrac{1}{2}U_0$$

$\boxed{(d)}$ is correct.

Estimation and Approximation

13 •• [SSM] Disconnect the coaxial cable from a television or other device and estimate the diameter of the inner conductor and the diameter of the shield. Assume a plausible value (see Table 24–1) for the dielectric constant of the dielectric separating the two conductors and estimate the capacitance per unit length of the cable.

Picture the Problem The outer diameter of a "typical" coaxial cable is about 5 mm, while the inner diameter is about 1 mm. From Table 24-1 we see that a reasonable range of values for κ is 3-5. We can use the expression for the capacitance of a cylindrical capacitor to estimate the capacitance per unit length of a coaxial cable.

The capacitance of a cylindrical dielectric-filled capacitor is given by:

$$C=\frac{2\pi\kappa\,\epsilon_0\,L}{\ln\left(\dfrac{R_2}{R_1}\right)}$$

where L is the length of the capacitor, R_1 is the radius of the inner conductor, and R_2 is the radius of the second (outer) conductor.

Divide both sides by L to obtain an expression for the capacitance per unit length of the cable:

$$\frac{C}{L}=\frac{2\pi\kappa\,\epsilon_0}{\ln\left(\dfrac{R_2}{R_1}\right)}=\frac{\kappa}{2k\ln\left(\dfrac{R_2}{R_1}\right)}$$

If $\kappa = 3$:

$$\frac{C}{L} = \frac{3}{2\left(8.988\times10^{9}\ \mathrm{N\cdot m^2/C^2}\right)\ln\left(\frac{2.5\,\mathrm{mm}}{0.5\,\mathrm{mm}}\right)} \approx 0.1\,\mathrm{nF/m}$$

If $\kappa = 5$:

$$\frac{C}{L} = \frac{5}{2\left(8.988\times10^{9}\ \mathrm{N\cdot m^2/C^2}\right)\ln\left(\frac{2.5\,\mathrm{mm}}{0.5\,\mathrm{mm}}\right)} \approx 0.2\,\mathrm{nF/m}$$

A reasonable range of values for C/L, corresponding to $3 \le \kappa \le 5$, is:

$$\boxed{0.1\,\mathrm{nF/m} \le \frac{C}{L} \le 0.2\,\mathrm{nF/m}}$$

14 •• You are part of an engineering research team that is designing a pulsed nitrogen laser. To create the high-energy densities needed to operate such a laser, the electrical discharge from a high-voltage capacitor is used. Typically, the energy requirement per pulse (i.e., per discharge) is 100 J. Estimate the capacitance required if the discharge is to creates a spark across a gap of about 1.0 cm. Assume that the dielectric breakdown of nitrogen is the same as the value for normal air.

Picture the Problem The energy stored in a capacitor is given by $U = \frac{1}{2}CV^2$.

Relate the energy stored in a capacitor to its capacitance and the potential difference across it:

$$U = \tfrac{1}{2}CV^2 \Rightarrow C = \frac{2U}{V^2}$$

The potential difference across the spark gap is related to the width of the gap d and the electric field E in the gap:

$$V = Ed$$

Substitute for V in the expression for C to obtain:

$$C = \frac{2U}{E^2d^2}$$

Substitute numerical values and evaluate C:

$$C = \frac{2(100\,\mathrm{J})}{\left(3\times10^{6}\ \mathrm{V/m}\right)^2(1.0\,\mathrm{cm})^2} = \boxed{22\,\mu\mathrm{F}}$$

15 •• **[SSM]** Estimate the capacitance of the Leyden jar shown in the Figure 24-34. The figure of a man is one-tenth the height of an average man.

Picture the Problem Modeling the Leyden jar as a parallel-plate capacitor, we can use the equation for the capacitance of a dielectric-filled parallel-plate capacitor that relates its capacitance to the area A of its plates and their separation (the thickness of the glass) d to estimate the capacitance of the jar. See Table 24-1 for the dielectric constants of various materials.

The capacitance of a dielectric-filled parallel-plate capacitor is given by:

$$C = \frac{\kappa \epsilon_0 A}{d}$$

where κ is the dielectric constant.

Let the plate area be the sum of the area of the lateral surface of the jar and its base:

$$A = A_{\substack{\text{lateral}\\ \text{area}}} + A_{\text{base}} = 2\pi Rh + \pi R^2$$

where h is the height of the jar and R is its inside radius.

Substitute for A and simplify to obtain:

$$C = \frac{\kappa \epsilon_0 \left(2\pi Rh + \pi R^2\right)}{d} = \frac{\pi\kappa \epsilon_0 R(2h + R)}{d}$$

If the glass of the Leyden jar is Bakelite of thickness 2.0 mm and the radius and height of the jar are 4.0 cm and 40 cm, respectively, then:

$$C = \frac{\pi(4.9)(4.0\text{ cm})\left(8.854\times10^{-12}\,\frac{\text{C}^2}{\text{N}\cdot\text{m}^2}\right)[2(40\text{ cm}) + 4.0\text{ cm}]}{2.0\text{ mm}} = \boxed{2.3\text{ nF}}$$

Capacitance

16 • An isolated conducting sphere that has a 10.0 cm radius has an electric potential of 2.00 kV (the potential far from the sphere is zero). (*a*) How much charge is on the sphere? (*b*) What is the self-capacitance of the sphere? (*c*) By how much does the self-capacitance change if the sphere's electric potential is increased to 6.00 kV?

Picture the Problem The charge on the spherical conductor is related to its radius and potential according to $V = kQ/r$ and we can use the definition of capacitance to find the self-capacitance of the sphere.

(*a*) Relate the potential V of the spherical conductor to the charge on it and to its radius:

$$V = \frac{kQ}{r} \Rightarrow Q = \frac{rV}{k}$$

Substitute numerical values and evaluate Q:

$$Q = \frac{(10.0\,\text{cm})(2.00\,\text{kV})}{8.988\times10^{9}\,\frac{\text{N}\cdot\text{m}^2}{\text{C}^2}} = 22.252\,\text{nC}$$

$$= \boxed{22.3\,\text{nC}}$$

(*b*) Use the definition of capacitance to relate the self-capacitance of the sphere to its charge and potential:

$$C = \frac{Q}{V} = \frac{22.252\,\text{nC}}{2.00\,\text{kV}} = \boxed{11.1\,\text{pF}}$$

(*c*) It doesn't. The self-capacitance of a sphere is a function of its radius.

17 • The charge on one plate of a capacitor is +30.0 μC and the charge on the other plate is –30.0 μC. The potential difference between the plates is 400 V. What is the capacitance of the capacitor?

Picture the Problem We can use its definition to find the capacitance of this capacitor.

Use the definition of capacitance to obtain:

$$C = \frac{Q}{V} = \frac{30.0\,\mu\text{C}}{400\,\text{V}} = \boxed{75.0\,\text{nF}}$$

18 •• Two isolated conducting spheres of equal radius R have charges $+Q$ and $-Q$, respectively. Their centers are separated by a distance d that is large compared to their radius. Estimate the capacitance of this unusual capacitor.

Picture the Problem Let the separation of the spheres be d and their radii be R. Outside the two spheres the electric field is approximately the field due to point charges of $+Q$ and $-Q$, each located at the centers of spheres, separated by distance d. We can derive an expression for the potential at the surface of each sphere and then use the potential difference between the spheres and the definition of capacitance and to estimate the capacitance of the two-sphere system.

The capacitance of the two-sphere system is given by:

$$C = \frac{Q}{\Delta V}$$

where ΔV is the potential difference between the spheres.

The potential at any point outside the two spheres is:

$$V = \frac{k(+Q)}{r_1} + \frac{k(-Q)}{r_2}$$

where r_1 and r_2 are the distances from the given point to the centers of the spheres.

For a point on the surface of the sphere with charge $+Q$:

$r_1 = R$ and $r_2 = d + \delta$ where $|\delta| < R$

Substitute to obtain:

$$V_{+Q} = \frac{k(+Q)}{R} + \frac{k(-Q)}{d+\delta}$$

For $\delta << d$:

$$V_{+Q} = \frac{kQ}{R} - \frac{kQ}{d}$$

and

$$V_{-Q} = -\frac{kQ}{R} + \frac{kQ}{d}$$

The potential difference between the spheres is:

$$\Delta V = V_Q - V_{-Q} = \frac{kQ}{R} - \frac{kQ}{d} - \left(\frac{-kQ}{R} + \frac{kQ}{d}\right) = 2kQ\left(\frac{1}{R} - \frac{1}{d}\right)$$

Substitute for ΔV in the expression for C to obtain:

$$C = \frac{Q}{2kQ\left(\frac{1}{R} - \frac{1}{d}\right)} = \frac{2\pi \epsilon_0}{\left(\frac{1}{R} - \frac{1}{d}\right)} = \frac{2\pi \epsilon_0 R}{1 - \frac{R}{d}}$$

For $d >> R$:

$$C = \boxed{2\pi \epsilon_0 R}$$

The Storage of Electrical Energy

19 • [SSM] (*a*) The potential difference between the plates of a 3.00-μF capacitor is 100 V. How much energy is stored in the capacitor? (*b*) How much additional energy is required to increase the potential difference between the plates from 100 V to 200 V?

Picture the Problem Of the three equivalent expressions for the energy stored in a charged capacitor, the one that relates U to C and V is $U = \frac{1}{2}CV^2$.

(*a*) Express the energy stored in the capacitor as a function of *C* and *V*:

$$U = \tfrac{1}{2}CV^2$$

Substitute numerical values and evaluate *U*:

$$U = \tfrac{1}{2}(3.00\,\mu\text{F})(100\,\text{V})^2 = \boxed{15.0\,\text{mJ}}$$

(*b*) Express the additional energy required as the difference between the energy stored in the capacitor at 200 V and the energy stored at 100 V:

$$\begin{aligned}\Delta U &= U(200\,\text{V}) - U(100\,\text{V})\\ &= \tfrac{1}{2}(3.00\,\mu\text{F})(200\,\text{V})^2 - 15.0\,\text{mJ}\\ &= \boxed{45.0\,\text{mJ}}\end{aligned}$$

20 • The charges on the plates of a 10-μF capacitor are ±4.0 μC. (*a*) How much energy is stored in the capacitor? (*b*) If charge is transferred until the charges on the plates are equal to ±2.0 μC, how much stored energy remains?

Picture the Problem Of the three equivalent expressions for the energy stored in a charged capacitor, the one that relates *U* to *Q* and *C* is $U = \frac{1}{2}\frac{Q^2}{C}$.

(*a*) Express the energy stored in the capacitor as a function of *C* and *Q*:

$$U = \frac{1}{2}\frac{Q^2}{C}$$

Substitute numerical values and evaluate *U*:

$$U = \frac{1}{2}\frac{(4.0\,\mu\text{C})^2}{10\,\mu\text{F}} = \boxed{0.80\,\mu\text{J}}$$

(*b*) Express the energy remaining when half the charge is removed:

$$U\left(\tfrac{1}{2}Q\right) = \frac{1}{2}\frac{(2.0\,\mu\text{C})^2}{10\,\mu\text{F}} = \boxed{0.20\,\mu\text{J}}$$

21 • (*a*) Find the energy stored in a 20.0-nF capacitor when to the charges on the plates are ±5.00 μC. (*b*) How much additional energy is stored if charges are increased from ±5.00 μC to ±10.0 μC?

Picture the Problem Of the three equivalent expressions for the energy stored in a charged capacitor, the one that relates *U* to *Q* and *C* is $U = \frac{1}{2}\frac{Q^2}{C}$.

(*a*) Express the energy stored in the capacitor as a function of *C* and *Q*:

$$U = \frac{1}{2}\frac{Q^2}{C}$$

Substitute numerical values and evaluate U:

$$U(5.00\,\mu\text{C}) = \frac{1}{2}\frac{(5.00\,\mu\text{C})^2}{20.0\,\text{nF}} = \boxed{0.625\,\text{mJ}}$$

(*b*) Express the additional energy required as the difference between the energy stored in the capacitor when its charge is 5 μC and when its charge is 10 μC:

$$\Delta U = U(10.0\,\mu\text{C}) - U(5.00\,\mu\text{C})$$
$$= \frac{1}{2}\frac{(10.0\,\mu\text{C})^2}{20.0\,\text{nF}} - 0.625\,\text{mJ}$$
$$= 2.50\text{ mJ} - 0.625\text{ mJ} = \boxed{1.88\text{ mJ}}$$

22 • What is the maximum electric energy density in a region containing dry air at standard conditions?

Picture the Problem The energy per unit volume in an electric field varies with the square of the electric field according to $u = \frac{1}{2}\epsilon_0 E^2$. Under standard conditions, dielectric breakdown occurs at approximately E = 3.0 MV/m.

Express the energy per unit volume in an electric field:

$$u = \tfrac{1}{2}\epsilon_0 E^2$$

Substitute numerical values and evaluate u:

$$\boldsymbol{u} = \tfrac{1}{2}\left(8.854\times10^{-12}\,\frac{\text{C}^2}{\text{N}\cdot\text{m}^2}\right)\left(3.0\,\frac{\text{MV}}{\text{m}}\right)^2$$
$$\approx \boxed{40\,\text{J/m}^3}$$

23 •• **[SSM]** An air-gap parallel-plate capacitor that has a plate area of 2.00 m^2 and a separation of 1.00 mm is charged to 100 V. (*a*) What is the electric field between the plates? (*b*) What is the electric energy density between the plates? (*c*) Find the total energy by multiplying your answer from Part (*b*) by the volume between the plates. (*d*) Determine the capacitance of this arrangement. (*e*) Calculate the total energy from $U = \frac{1}{2}CV^2$, and compare your answer with your result from Part (*c*).

Picture the Problem Knowing the potential difference between the plates, we can use $E = V/d$ to find the electric field between them. The energy per unit volume is given by $u = \frac{1}{2}\epsilon_0 E^2$ and we can find the capacitance of the parallel-plate capacitor using $C = \epsilon_0 A/d$.

(*a*) Express the electric field between the plates in terms of their separation and the potential difference between them:

$$E = \frac{V}{d} = \frac{100\,\text{V}}{1.00\,\text{mm}} = \boxed{100\,\text{kV/m}}$$

(*b*) Express the energy per unit volume in an electric field:

$$u = \tfrac{1}{2}\epsilon_0 E^2$$

Substitute numerical values and evaluate *u*:

$$u = \tfrac{1}{2}\left(8.854\times10^{-12}\,\frac{\text{C}^2}{\text{N}\cdot\text{m}^2}\right)(100\,\text{kV/m})^2$$
$$= 44.27\ \text{mJ/m}^3 = \boxed{44.3\,\text{mJ/m}^3}$$

(*c*) The total energy is given by:

$$U = uV = uAd$$
$$= (44.27\,\text{mJ/m}^3)(2.00\,\text{m}^2)(1.00\,\text{mm})$$
$$= \boxed{88.5\,\mu J}$$

(*d*) The capacitance of a parallel-plate capacitor is given by:

$$C = \frac{\epsilon_0 A}{d}$$

Substitute numerical values and evaluate *C*:

$$C = \frac{\left(8.854\times10^{-12}\,\frac{\text{C}^2}{\text{N}\cdot\text{m}^2}\right)(2.00\,\text{m}^2)}{1.00\,\text{mm}}$$
$$= 17.71\,\text{nF} = \boxed{17.7\,\text{nF}}$$

(*e*) The total energy is given by:

$$U = \tfrac{1}{2}CV^2$$

Substitute numerical values and evaluate *U*:

$$\boldsymbol{U} = \tfrac{1}{2}(17.71\,\text{nF})(100\,\text{V})^2$$
$$= \boxed{88.5\,\boldsymbol{\mu}\text{J, in agreement with (}\boldsymbol{c}\text{).}}$$

24 •• A solid metal sphere has radius of 10.0 cm and a concentric metal spherical shell has an inside radius of 10.5 cm. The solid sphere has a charge 5.00 nC. (*a*) Estimate the energy stored in the electric field in the region between the spheres. *Hint: You can treat the spheres essentially as parallel flat slabs separated by 0.5 cm*. (*b*) Estimate the capacitance of this two-sphere system. (*c*) Estimate the total energy stored in the electric field from $\tfrac{1}{2}Q^2/C$ and compare it to your answer in Part (*a*).

Picture the Problem The total energy stored in the electric field is the product of the energy density in the space between the spheres and the volume of this space.

(*a*) The total energy *U* stored in the electric field is given by:

$$U = uV$$

where *u* is the energy density and *V* is the volume between the spheres.

The energy density of the field is:

$$u = \tfrac{1}{2}\epsilon_0 E^2$$

where E is the field between the spheres.

The volume between the spheres is approximately:

$$V \approx 4\pi r_1^2 (r_2 - r_1)$$

Substitute for u and V to obtain:

$$U = 2\pi \epsilon_0 E^2 r_1^2 (r_2 - r_1) \qquad (1)$$

The magnitude of the electric field between the concentric spheres is the sum of the electric fields due to each charge distribution:

$$E = E_Q + E_{-Q}$$

Because the two surfaces are so close together, the electric field between them is approximately the sum of the fields due to two plane charge distributions:

$$E = \frac{\sigma_Q}{2\epsilon_0} + \frac{\sigma_{-Q}}{2\epsilon_0} = \frac{\sigma_Q}{\epsilon_0}$$

Substitute for σ_Q to obtain:

$$E \approx \frac{Q}{4\pi r_1^2 \epsilon_0}$$

Substitute for E in equation (1) and simplify:

$$U = 2\pi \epsilon_0 \left(\frac{Q}{4\pi r_1^2 \epsilon_0}\right)^2 r_1^2 (r_2 - r_1) = \frac{Q^2}{8\pi \epsilon_0}\frac{r_2 - r_1}{r_1^2}$$

Substitute numerical values and evaluate U:

$$\boldsymbol{U} = \frac{(5.00\,\text{nC})^2(10.5\,\text{cm} - 10.0\,\text{cm})}{8\pi\left(8.854\times10^{-12}\,\dfrac{\text{C}^2}{\text{N}\cdot\text{m}^2}\right)(10.0\,\text{cm})^2} = 56.2\times10^{-9}\ \text{J} = \boxed{0.06\,\mu\text{J}}$$

(*b*) The capacitance of the two-sphere system is given by:

$$C = \frac{Q}{\Delta V}$$

where ΔV is the potential difference between the two spheres.

The electric potentials at the surfaces of the spheres are:

$$V_1 = \frac{Q}{4\pi \epsilon_0 r_1} \text{ and } V_2 = \frac{Q}{4\pi \epsilon_0 r_2}$$

Substitute for ΔV and simplify to obtain:

$$C = \frac{Q}{\dfrac{Q}{4\pi \epsilon_0 r_1} - \dfrac{Q}{4\pi \epsilon_0 r_2}} = 4\pi \epsilon_0 \frac{r_1 r_2}{r_2 - r_1}$$

Substitute numerical values and evaluate C:

$$C = 4\pi\left(8.854\times10^{-12}\,\frac{\text{C}^2}{\text{N}\cdot\text{m}^2}\right)\frac{(10.0\,\text{cm})(10.5\,\text{cm})}{10.5\,\text{cm}-10.0\,\text{cm}} = 0.2337\,\text{nF} = \boxed{0.2\,\text{nF}}$$

(*c*) Use $\frac{1}{2}Q^2/C$ to find the total energy stored in the electric field between the spheres:

$$U = \frac{1}{2}\left[\frac{(5.00\,\text{nC})^2}{0.2337\,\text{nF}}\right] = \boxed{0.05\,\mu\text{J}}$$

a result that agrees to within 5% with the exact result obtained in (*a*).

25 •• A parallel-plate capacitor has plates of area 500 cm^2 and is connected across the terminals of a battery. After some time has passed, the capacitor is disconnected from the battery. When the plates are then moved 0.40 cm farther apart, the charge on each plate remains constant but the potential difference between the plates increases by 100 V. (*a*) What is the magnitude of the charge on each plate? (*b*) Do you expect the energy stored in the capacitor to increase, decrease, or remain constant as the plates are moved this way? Explain your answer. (*c*) Support your answer to Part (*b*), by determining the change in stored energy in the capacitor due to the movement of the plates.

Picture the Problem (*a*) We can relate the charge Q on the positive plate of the capacitor to the charge density of the plate σ using its definition. The charge density, in turn, is related to the electric field between the plates according to $\sigma = \epsilon_0 E$ and the electric field can be found from $E = \Delta V/\Delta d$. We can use $\Delta U = \frac{1}{2}Q\Delta V$ in Part (*c*) to find the increase in the energy stored due to the movement of the plates.

(*a*) Express the charge Q on the positive plate of the capacitor in terms of the plate's charge density σ and surface area A:

$$Q = \sigma A$$

Relate σ to the electric field E between the plates of the capacitor:

$$\sigma = \epsilon_0 E$$

Express E in terms of the change in V as the plates are separated a distance Δd:

$$E = \frac{\Delta V}{\Delta d}$$

Substitute for σ and E to obtain:

$$Q = \epsilon_0 EA = \epsilon_0 A\frac{\Delta V}{\Delta d}$$

Substitute numerical values and evaluate Q:

$$\boldsymbol{Q} = \left(8.854\times10^{-12}\,\mathrm{C^2/N\cdot m^2}\right)\left(500\,\mathrm{cm^2}\right)\frac{100\,\mathrm{V}}{0.40\,\mathrm{cm}} = 11.1\,\mathrm{nC} = \boxed{11\,\mathrm{nC}}$$

(*b*) Because work has to be done to pull the plates farther apart, you would expect the energy stored in the capacitor to increase.

(*c*) Express the change in the electrostatic energy in terms of the change in the potential difference:

$$\Delta U = \tfrac{1}{2}Q\Delta V$$

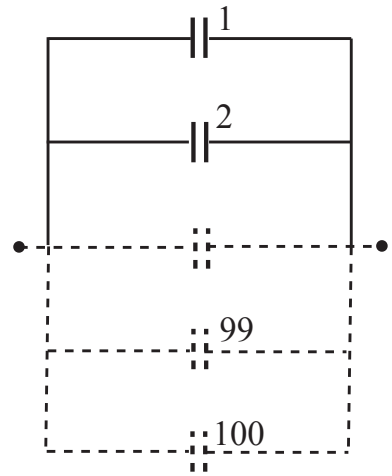
Substitute numerical values and evaluate ΔU:

$$\Delta \boldsymbol{U} = \tfrac{1}{2}\left(11.1\,\mathrm{nC}\right)\left(100\,\mathrm{V}\right) = \boxed{0.55\,\mu\mathrm{J}}$$

Combinations of Capacitors

26 • (*a*) How many 1.00-μF capacitors connected in parallel would it take to store a total charge of 1.00 mC if the potential difference of across each capacitor is 10.0 V? Diagram the parallel combination. (*b*) What would be the potential difference across this parallel combination? (*c*) If the capacitors in Part (*a*) are discharged, connected in series, and then energized until the potential difference across each is equal to 10.0 V, find the charge on each capacitor and the potential difference across the connection.

Picture the Problem We can apply the properties of capacitors connected in parallel to determine the number of 1.00-μF capacitors connected in parallel it would take to store a total charge of 1.00 mC with a potential difference of 10.0 V across each capacitor. Knowing that the capacitors are connected in parallel (Parts (*a*) and (*b*)) we determine the potential difference across the combination. In Part (*c*) we can use our knowledge of how potential differences add in a series circuit to find the potential difference across the combination and the definition of capacitance to find the charge on each capacitor.



(*a*) Express the number of capacitors *n* in terms of the charge *q* on each and the total charge *Q*:

$$n = \frac{Q}{q}$$

Relate the charge *q* on one capacitor to its capacitance *C* and the potential difference across it:

$$q = CV$$

Substitute for *q* to obtain:

$$n = \frac{Q}{CV}$$

Substitute numerical values and evaluate *n*:

$$n = \frac{1.00\,\text{mC}}{(1.00\,\mu\text{F})(10.0\,\text{V})} = \boxed{100}$$

(*b*) Because the capacitors are connected in parallel the potential difference across the combination is the same as the potential difference across each of them:

$$V_{\substack{\text{parallel}\\\text{combination}}} = V = \boxed{10.0\,\text{V}}$$

(*c*) With the capacitors connected in series, the potential difference across the combination will be the sum of the potential differences across the 100 capacitors:

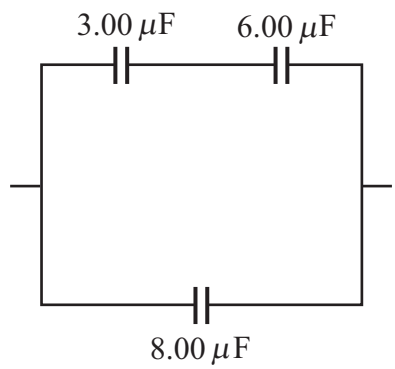
$$V_{\substack{\text{series}\\ \text{combination}}} = 100(10.0\,\text{V}) = \boxed{1.00\,\text{kV}}$$

Use the definition of capacitance to find the charge on each capacitor:

$$q = CV = (1\,\mu\text{F})(10\,\text{V}) = \boxed{10.0\,\mu\text{C}}$$

27 • A 3.00-μF capacitor and a 6.00-μF capacitor are discharged and then connected in series, and the series combination is then connected in parallel with an 8.00-μF capacitor. Diagram this combination. What is the equivalent capacitance of this combination?

Picture the Problem The capacitor array is shown in the diagram. We can find the equivalent capacitance of this combination by first finding the equivalent capacitance of the 3.00-μF and 6.00-μF capacitors in series and then the equivalent capacitance of this capacitor with the 8.00-μF capacitor in parallel.



Express the equivalent capacitance for the 3.00-μF and 6.00-μF capacitors in series:

$$\frac{1}{C_{3+6}} = \frac{1}{3.00\,\mu\text{F}} + \frac{1}{6.00\,\mu\text{F}}$$

Solve for C_{3+6}:

$$C_{3+6} = 2.00\,\mu\text{F}$$

Find the equivalent capacitance of a 2.00-μF capacitor in parallel with an 8.00-μF capacitor:

$$C_{2+8} = 2.00\,\mu\text{F} + 8.00\,\mu\text{F} = \boxed{10.00\,\mu\text{F}}$$

28 • Three capacitors are connected in a triangle as shown in Figure 24-35. Find an expression for the equivalent capacitance between points *a* and *c* in terms of the three capacitance values.

Picture the Problem Because we're interested in the equivalent capacitance across terminals *a* and *c*, we need to recognize that capacitors C_1 and C_3 are in series with each other and in parallel with capacitor C_2.

Find the equivalent capacitance of C_1 and C_3 in series:

$$\frac{1}{C_{1+3}} = \frac{1}{C_1} + \frac{1}{C_3} \Rightarrow C_{1+3} = \frac{C_1 C_3}{C_1 + C_3}$$

Find the equivalent capacitance of C_{1+3} and C_2 in parallel:

$$C_{eq} = C_2 + C_{1+3} = \boxed{C_2 + \frac{C_1 C_3}{C_1 + C_3}}$$

29 •• A 10.0-μF capacitor and a 20.0-μF capacitor are connected in parallel across the terminals of a 6.00-V battery. (*a*) What is the equivalent capacitance of this combination? (*b*) What is the potential difference across each capacitor? (*c*) Find the charge on each capacitor. (*d*) Find the energy stored in each capacitor.

Picture the Problem Because the capacitors are connected in parallel we can add their capacitances to find the equivalent capacitance of the combination. Also, because they are in parallel, they have a common potential difference across them. We can use the definition of capacitance to find the charge on each capacitor.

(*a*) Find the equivalent capacitance of the two capacitors in parallel:

$$C_{eq} = 10.0\,\mu\text{F} + 20.0\,\mu\text{F} = \boxed{30.0\,\mu\text{F}}$$

(*b*) Because capacitors in parallel have a common potential difference across them:

$$V = V_{10} = V_{20} = \boxed{6.00\,\text{V}}$$

(*c*) Use the definition of capacitance to find the charge on each capacitor:

$$Q_{10} = C_{10}V = (10.0\,\mu\text{F})(6.00\,\text{V}) = \boxed{60.0\,\mu\text{C}}$$

and

$$Q_{20} = C_{20}V = (20.0\,\mu\text{F})(6.00\,\text{V}) = \boxed{120\,\mu\text{C}}$$

(*d*) Use $\boldsymbol{U} = \frac{1}{2}\boldsymbol{QV}$ to find the energy stored in each capacitor:

$$U_{10} = \tfrac{1}{2}Q_{10}V_{10} = \tfrac{1}{2}(60.0\,\mu\text{C})(6.00\,\text{V}) = \boxed{180\,\mu\text{J}}$$

and

$$U_{20} = \tfrac{1}{2}Q_{20}V_{20} = \tfrac{1}{2}(120.0\,\mu\text{C})(6.00\,\text{V}) = \boxed{360\,\mu\text{J}}$$

30 •• A 10.0-μF capacitor and a 20.0-μF capacitor are connected in parallel across the terminals of a 6.00-V battery. (*a*) What is the equivalent capacitance of this combination? (*b*) What is the potential difference across each capacitor?

(*c*) Find the charge on each capacitor. (*d*) Find the energy stored in each capacitor.

Picture the Problem We can use the properties of capacitors in series to find the equivalent capacitance and the charge on each capacitor. We can then apply the definition of capacitance to find the potential difference across each capacitor.

(*a*) Because the capacitors are connected in series they have equal charges:

$$Q_{10} = Q_{20} = C_{eq}V$$

Express the equivalent capacitance of the two capacitors in series:

$$\frac{1}{C_{eq}} = \frac{1}{10.0\,\mu\text{F}} + \frac{1}{20.0\,\mu\text{F}}$$

Solve for C_{eq} to obtain:

$$C_{eq} = \frac{(10.0\,\mu\text{F})(20.0\,\mu\text{F})}{10.0\,\mu\text{F} + 20.0\,\mu\text{F}} = \boxed{6.67\,\mu\text{F}}$$

(*b*) Because the capacitors are in series, they have the same charge. Substitute numerical values to obtain:

$$Q_{10} = Q_{20} = (6.67\,\mu\text{F})(6.00\,\text{V}) = \boxed{40.0\,\mu\text{C}}$$

(*c*) Apply the definition of capacitance to find the potential difference across each capacitor:

$$V_{10} = \frac{Q_{10}}{C_{10}} = \frac{40.0\,\mu\text{C}}{10.0\,\mu\text{F}} = \boxed{4.00\,\text{V}}$$

and

$$V_{20} = \frac{Q_{20}}{C_{20}} = \frac{40.0\,\mu\text{C}}{20.0\,\mu\text{F}} = \boxed{2.00\,\text{V}}$$

(*d*) Use $U = \frac{1}{2}QV$ to find the energy stored in each capacitor:

$$U_{10} = \tfrac{1}{2}Q_{10}V_{10} = \tfrac{1}{2}(40.0\,\mu\text{C})(4.00\,\text{V}) = \boxed{80.0\,\mu\text{J}}$$

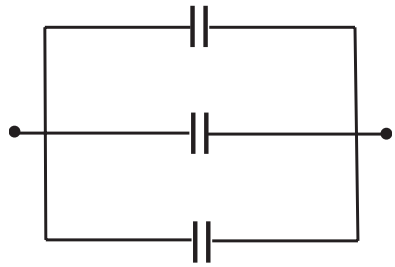
and

$$U_{20} = \tfrac{1}{2}Q_{20}V_{20} = \tfrac{1}{2}(40.0\,\mu\text{C})(2.00\,\text{V}) = \boxed{40.0\,\mu\text{J}}$$

31 •• Three identical capacitors are connected so that their maximum equivalent capacitance, which is 15.0 μF, is obtained. (*a*) Determine how the capacitors are connected and diagram the combination. (*b*) There are three additional ways to connect all three capacitors. Diagram these three ways and determine the equivalent capacitances for each arrangement.

Picture the Problem We can use the properties of capacitors connected in series and in parallel to find the equivalent capacitances for various connection combinations.

(*a*) If their capacitance is to be a maximum, the capacitors must be connected in parallel:



Find the capacitance of each capacitor:

$$C_{eq} = 3C = 15.0\,\mu\text{F} \Rightarrow C = 5.00\,\mu\text{F}$$

(*b*) (1) Connect the three capacitors in series:

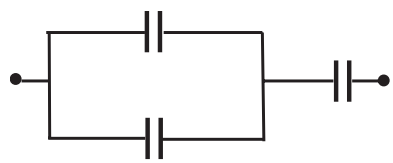
Because the capacitors are in series, their equivalent capacitance is the reciprocal of the sum of their reciprocals:

$$\frac{1}{C_{eq}} = \frac{1}{5.00\,\mu\text{F}} + \frac{1}{5.00\,\mu\text{F}} + \frac{1}{5.00\,\mu\text{F}}$$

and

$$C_{eq} = \boxed{1.67\,\mu\text{F}}$$

(2) Connect two in parallel, with the third in series with that combination:



Find the equivalent capacitance of the two capacitors that are in parallel and then the equivalent capacitance of the network of three capacitors:

$$C_{\substack{\text{eq, two}\\ \text{in parallel}}} = 2(5.00\,\mu\text{F}) = 10.0\,\mu\text{F}$$

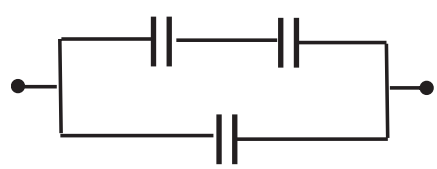
and

$$\frac{1}{C_{eq}} = \frac{1}{10.0\,\mu\text{F}} + \frac{1}{5.00\,\mu\text{F}}$$

Solving for C_{eq} yields:

$$C_{eq} = \boxed{3.33\,\mu\text{F}}$$

(3) Connect two in series, with the third in parallel with that combination:



Find the equivalent capacitance of the two capacitors connected in series:

$$\frac{1}{C_{\text{eq, two in series}}} = \frac{1}{5.00\,\mu\text{F}} + \frac{1}{5.00\,\mu\text{F}}$$

or

$$C_{\text{eq, two in series}} = 2.50\,\mu\text{F}$$

Find the capacitance equivalent to 2.50 μF and 5.00 μF in parallel:

$$C_{\text{eq}} = 2.50\,\mu\text{F} + 5.00\,\mu\text{F} = \boxed{7.50\,\mu\text{F}}$$

32 •• For the circuit shown in Figure 24-36, the capacitors were each discharged before being connected to the voltage source. Find (*a*) the equivalent capacitance of the combination, (*b*) the charge stored on the positively charge plate of each capacitor, (*c*) the voltage across each capacitor, and (*d*) the energy stored in each capacitor.

Picture the Problem We can use the properties of capacitors connected in series and in parallel to find the equivalent capacitance between the terminals and these properties and the definition of capacitance to find the charge on each capacitor.

(*a*) Relate the equivalent capacitance of the two capacitors in series to their individual capacitances:

$$\frac{1}{C_{4+15}} = \frac{1}{4.00\,\mu\text{F}} + \frac{1}{15.0\,\mu\text{F}}$$

Solving for C_{4+15} yields:

$$C_{4+15} = 3.158\,\mu\text{F}$$

Find the equivalent capacitance of C_{4+15} in parallel with the 12.0-μF capacitor:

$$C_{\text{eq}} = 3.158\,\mu\text{F} + 12.0\,\mu\text{F} = 15.16\,\mu\text{F} = \boxed{15.2\,\mu\text{F}}$$

(*b*) Use the definition of capacitance to find the charge stored on the 12-μF capacitor:

$$Q_{12} = C_{12}V_{12} = C_{12}V = (12.0\,\mu\text{F})(200\,\text{V}) = \boxed{2.40\,\text{mC}}$$

Because the capacitors in series have the same charge:

$$Q_4 = Q_{15} = C_{4+15}V = (3.158\,\mu\text{F})(200\,\text{V}) = 0.6316\,\text{mC} = \boxed{0.632\,\text{mC}}$$

(*c*) Because the 12.0-μF capacitor is connected directly across the source, the voltage across it is:

$$\boldsymbol{V}_{12} = \boxed{200\,\text{V}}$$

Use the definition of capacitance to find V_4 and V_{15}:

$$V_4 = \frac{Q_4}{C_4} = \frac{0.6316\text{ mC}}{4.00\ \mu\text{F}} = \boxed{158\text{ V}}$$

and

$$V_{15} = \frac{Q_{15}}{C_{15}} = \frac{0.6316\text{ mC}}{15.0\ \mu\text{F}} = \boxed{42\text{ V}}$$

(*d*) Use $U = \frac{1}{2}QV$ to find the energy stored in each capacitor:

$$U_4 = \tfrac{1}{2}Q_4V_4 = \tfrac{1}{2}(0.6316\text{ mC})(158\text{ V}) = \boxed{49.9\text{ mJ}}$$

$$U_{15} = \tfrac{1}{2}Q_{15}V_{15} = \tfrac{1}{2}(0.6316\text{ mC})(42\text{ V}) = \boxed{13.3\text{ mJ}}$$

and

$$U_{12} = \tfrac{1}{2}Q_{12}V_{12} = \tfrac{1}{2}(2.40\text{ mC})(200\text{ V}) = \boxed{240\text{ mJ}}$$

33 •• (*a*) Show that the equivalent capacitance of two capacitors in series can be written

$$C_{\text{eq}} = \frac{C_1C_2}{C_1 + C_2}$$

(*b*) Using only this formula and some algebra, show that C_{eq} must always be less than C_1 and C_2, and hence must be less than the smaller of the two values.
(*c*) Show that the equivalent capacitance of three capacitors in series is can be written

$$C_{\text{eq}} = \frac{C_1C_2C_3}{C_1C_2 + C_2C_3 + C_1C_3}$$

(*d*) Using only this formula and some algebra, show that C_{eq} must always be less than each of C_1, C_2 and C_3, and hence must be less than the least of the three values.

Picture the Problem We can use the properties of capacitors in series to establish the results called for in this problem.

(*a*) Express the equivalent capacitance of two capacitors in series:

$$\frac{1}{C_{\text{eq}}} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{C_2 + C_1}{C_1C_2}$$

Solve for C_{eq} by taking the reciprocal of both sides of the equation to obtain:

$$C_{\text{eq}} = \boxed{\frac{C_1C_2}{C_1 + C_2}}$$

(*b*) Divide numerator and denominator of this expression by C_1 to obtain:

$$C_{eq} = \frac{C_2}{1+\frac{C_2}{C_1}}$$

Because $1+\frac{C_2}{C_1} > 1$:

$$C_{eq} < C_2$$

Divide numerator and denominator of this expression by C_2 to obtain:

$$C_{eq} = \frac{C_1}{1+\frac{C_1}{C_2}}$$

Because $1+\frac{C_1}{C_2} > 1$:

$\boldsymbol{C}_{eq} < \boldsymbol{C}_1$, showing that C_{eq} must always be less than both C_1 and C_2, and hence must be less than the smaller of the two values.

(*c*) Using our result from part (*a*) for two of the capacitors, add a third capacitor C_3 in series to obtain:

$$\frac{1}{C_{eq}} = \frac{C_1+C_2}{C_1C_2} + \frac{1}{C_3} = \frac{C_1C_3+C_2C_3+C_1C_2}{C_1C_2C_3}$$

Take the reciprocal of both sides of the equation to obtain:

$$C_{eq} = \boxed{\frac{C_1C_2C_3}{C_1C_2+C_2C_3+C_1C_3}}$$

(*d*) Rewrite the result of Part (*c*) as follows:

$$C_{eq} = \left(\frac{C_1C_2}{C_1C_2+C_2C_3+C_1C_3}\right)C_3$$

Divide numerator and denominator of this expression by C_1C_2 to obtain:

$$C_{eq} = \left(\frac{1}{\frac{C_1C_2}{C_1C_2}+\frac{C_2C_3}{C_1C_2}+\frac{C_1C_3}{C_1C_2}}\right)C_3 = \left(\frac{1}{1+\frac{C_3}{C_1}+\frac{C_3}{C_2}}\right)C_3$$

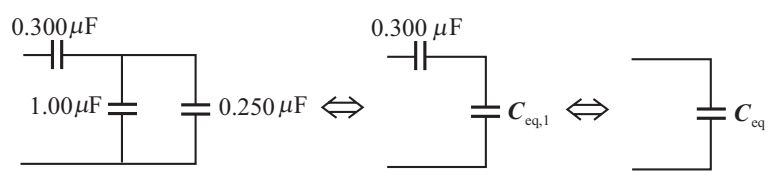
Because $1+\frac{\boldsymbol{C}_3}{\boldsymbol{C}_1}+\frac{\boldsymbol{C}_3}{\boldsymbol{C}_2} > 1$:

$$\boldsymbol{C}_{eq} < \boldsymbol{C}_3$$

Proceed similarly to show that: $\boldsymbol{C}_{eq} < \boldsymbol{C}_1$ and $\boldsymbol{C}_{eq} < \boldsymbol{C}_2$, showing that C_{eq} must always be less than C_1, C_2 and C_3, and hence must be less than the smaller of the three values.

34 •• For the circuit shown in Figure 24-37 find (*a*) the equivalent capacitance between the terminals, (*b*) the charge stored on the positively charge plate of each capacitor, (c) the voltage across each capacitor, and (*d*) the total stored energy.

Picture the Problem Let C_{eq1} represent the equivalent capacitance of the parallel combination and C_{eq} the total equivalent capacitance between the terminals. We can use the equations for capacitors in parallel and then in series to find C_{eq}. Because the charge on C_{eq} is the same as on the 0.300-μF capacitor and C_{eq1}, we'll know the charge on the 0.300-μF capacitor when we have found the total charge Q_{eq} stored by the circuit. We can find the charges on the 1.00-μF and 0.250-μF capacitors by first finding the potential difference across them and then using the definition of capacitance.



(*a*) Find the equivalent capacitance for the parallel combination:

$$C_{eq1} = 1.00\,\mu\text{F} + 0.250\,\mu\text{F} = 1.25\,\mu\text{F}$$

The 0.300-μF capacitor is in series with C_{eq1}. Their equivalent capacitance C_{eq} is given by:

$$\frac{1}{C_{eq}} = \frac{1}{0.300\,\mu\text{F}} + \frac{1}{1.25\,\mu\text{F}}$$

and

$$C_{eq} = 0.24194\,\mu\text{F} = \boxed{0.242\,\mu\text{F}}$$

(*b*) Express the total charge stored by the circuit Q_{eq}:

$$\begin{aligned} Q_{eq} &= Q_{0.300} = Q_{1.25} = C_{eq}V \\ &= (0.24194\,\mu\text{F})(10.0\,\text{V}) \\ &= 2.4194\,\mu\text{C} \end{aligned}$$

The 1.00-μF and 0.250-μF capacitors, being in parallel, have a common potential difference. Express this potential difference in terms of the 10.0 V across the system and the potential difference across the 0.300-μF capacitor:

$$V_{1.25} = 10.0\,\text{V} - V_{0.3} = 10.0\,\text{V} - \frac{Q_{0.3}}{C_{0.3}} = 10.0\,\text{V} - \frac{2.4194\,\mu\text{C}}{0.300\,\mu\text{F}} = 1.935\,\text{V}$$

Using the definition of capacitance, find the charge on the 1.00-μF and 0.250-μF capacitors:

$$Q_{1.00} = C_{1.00}V_{1.00} = (1.00\,\mu\text{F})(1.935\,\text{V}) = \boxed{1.9\,\mu\text{C}}$$

and

$$Q_{0.250} = C_{0.250}V_{0.250} = (0.250\,\mu\text{F})(1.935\,\text{V}) = \boxed{0.48\,\mu\text{C}}$$

Because the voltage across the parallel combination of the 1.00-μF and 0.250-μF capacitors is 1.935 V, the voltage across the 0.300-μF capacitor is 8.065 V and:

$$Q_{0.300} = C_{0.300}V_{0.300} = (0.300\,\mu\text{F})(8.065\,\text{V}) = \boxed{2.42\,\mu\text{C}}$$

(*c*) From (*b*), the voltages across the 0.300-μF capacitor and the parallel combination of the 1.00-μF and 0.250-μF capacitors is:

$$V_{0.300} = \boxed{8.06\,\text{V}}$$

and

$$V_{1.00} = V_{0.250} = 10.0\,\text{V} - 8.06\,\text{V} = \boxed{1.9\,\text{V}}$$

(*d*) The total stored energy is given by:

$$U = \tfrac{1}{2}C_{\text{eq}}V^2$$

Substitute numerical values and evaluate *U*:

$$U = \tfrac{1}{2}(0.2419\,\mu\text{F})(10.0\,\text{V})^2 = \boxed{12.1\,\mu\text{J}}$$

35 •• Five identical capacitors of capacitance C_0 are connected in a so-called "bridge" network, as shown in Figure 24-38. (*a*) What is the equivalent capacitance between points *a* and *b*? (*b*) Find the equivalent capacitance between points *a* and *b* if the capacitor at the center is replaced by a capacitor that has a capacitance of 10 C_0.

Picture the Problem Note that there are three parallel paths between *a* and *b*. We can find the equivalent capacitance of the capacitors connected in series in the upper and lower branches and then find the equivalent capacitance of three

capacitors in parallel.

(*a*) Find the equivalent capacitance of the series combination of capacitors in the upper and lower branch:

$$\frac{1}{C_{eq}} = \frac{1}{C_0} + \frac{1}{C_0} \Rightarrow C_{eq} = \tfrac{1}{2}C_0$$

Now we have two capacitors with capacitance $\frac{1}{2}C_0$ in parallel with a capacitor whose capacitance is C_0. Find their equivalent capacitance:

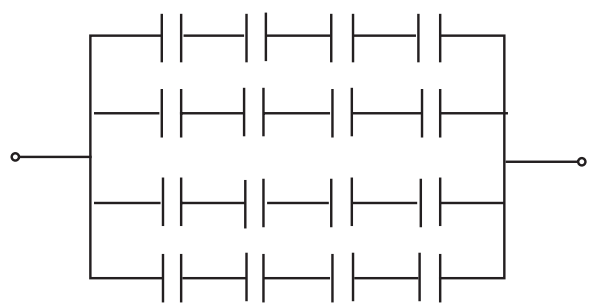
$$C'_{eq} = \tfrac{1}{2}C_0 + C_0 + \tfrac{1}{2}C_0 = \boxed{2C_0}$$

(*b*) If the central capacitance is $10C_0$, then:

$$C'_{eq} = \tfrac{1}{2}C_0 + 10C_0 + \tfrac{1}{2}C_0 = \boxed{11C_0}$$

36 •• You and your laboratory team have been given a project by your electrical engineering professor. Your team must design a network of capacitors that has a equivalent capacitance of 2.00 μF and breakdown voltage of 400 V. The restriction is that your team must use only 2.00-μF capacitors that have individual breakdown voltages of 100 V. Diagram the combination.

Picture the Problem Place four of the capacitors in series. Then the potential across each is 100 V when the potential across the combination is 400 V. The equivalent capacitance of the series combination is 0.500 μF. If we place four such series combinations in parallel, as shown in the circuit diagram, the total capacitance between the terminals is 2.00 μF.



37 •• Find the different equivalent capacitances that can be obtained by using two or three of the following capacitors: a 1.00-μF capacitor, a 2.00-μF capacitor, and a 4.00-μF capacitor.

Picture the Problem We can connect two capacitors in parallel, all three in parallel, two in series, three in series, two in parallel in series with the third, and two in series in parallel with the third.

Connect 2 in parallel to obtain:

$$C_{eq} = 1.00\,\mu\text{F} + 2.00\,\mu\text{F} = \boxed{3.00\,\mu\text{F}}$$

or

$$C_{eq} = 1.00\,\mu\text{F} + 4.00\,\mu\text{F} = \boxed{5.00\,\mu\text{F}}$$

or

$$C_{eq} = 2.00\,\mu\text{F} + 4.00\,\mu\text{F} = \boxed{6.00\,\mu\text{F}}$$

Connect all three in parallel to obtain:

$$C_{eq} = 1.00\,\mu\text{F} + 2.00\,\mu\text{F} + 4.00\,\mu\text{F} = \boxed{7.00\,\mu\text{F}}$$

Connect two in series:

$$C_{eq} = \frac{(1.00\,\mu\text{F})(2.00\,\mu\text{F})}{1.00\,\mu\text{F} + 2.00\,\mu\text{F}} = \boxed{0.667\,\mu\text{F}}$$

or

$$C_{eq} = \frac{(1.00\,\mu\text{F})(4.00\,\mu\text{F})}{1.00\,\mu\text{F} + 4.00\,\mu\text{F}} = \boxed{0.800\,\mu\text{F}}$$

or

$$C_{eq} = \frac{(2.00\,\mu\text{F})(4.00\,\mu\text{F})}{2.00\,\mu\text{F} + 4.00\,\mu\text{F}} = \boxed{1.33\,\mu\text{F}}$$

Connecting all three in series yields:

$$C_{eq} = \frac{(1.00\,\mu\text{F})(2.00\,\mu\text{F})(4.00\,\mu\text{F})}{(1.00\,\mu\text{F})(2.00\,\mu\text{F}) + (2.00\,\mu\text{F})(4.00\,\mu\text{F}) + (1.00\,\mu\text{F})(4.00\,\mu\text{F})} = \boxed{0.571\,\mu\text{F}}$$

Connect two in parallel, and the parallel combination in series with the third:

$$C_{eq} = \frac{(4.00\,\mu\text{F})(1.00\,\mu\text{F} + 2.00\,\mu\text{F})}{1.00\,\mu\text{F} + 2.00\,\mu\text{F} + 4.00\,\mu\text{F}} = \boxed{1.71\,\mu\text{F}}$$

or

$$C_{eq} = \frac{(1.00\,\mu\text{F})(4.00\,\mu\text{F} + 2.00\,\mu\text{F})}{1.00\,\mu\text{F} + 2.00\,\mu\text{F} + 4.00\,\mu\text{F}} = \boxed{0.857\,\mu\text{F}}$$

or

$$C_{eq} = \frac{(2.00\,\mu\text{F})(4.00\,\mu\text{F} + 1.00\,\mu\text{F})}{1.00\,\mu\text{F} + 2.00\,\mu\text{F} + 4.00\,\mu\text{F}} = \boxed{1.43\,\mu\text{F}}$$

Connect two in series, and the series combination in parallel with the third:

$$C_{eq} = \frac{(1.00\,\mu F)(2.00\,\mu F)}{1.00\,\mu F + 2.00\,\mu F} + 4.00\,\mu F = \boxed{4.67\,\mu F}$$

or

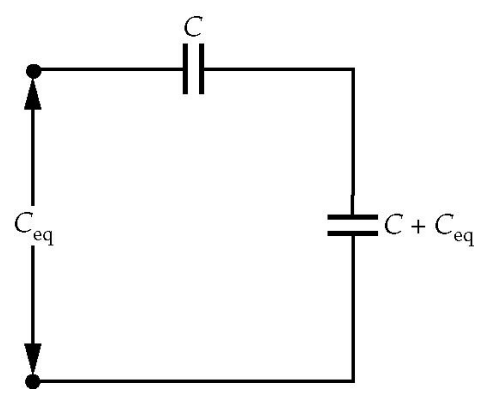
$$C_{eq} = \frac{(4.00\,\mu F)(2.00\,\mu F)}{4.00\,\mu F + 2.00\,\mu F} + 1.00\,\mu F = \boxed{2.33\,\mu F}$$

or

$$C_{eq} = \frac{(1.00\,\mu F)(4.00\,\mu F)}{1.00\,\mu F + 4.00\,\mu F} + 2.00\,\mu F = \boxed{2.80\,\mu F}$$

38 ••• What is the equivalent capacitance (in terms of C which is the capacitance of one of the capacitors) of the infinite ladder of capacitors shown in Figure 24-39?

Picture the Problem Let C be the capacitance of each capacitor in the ladder and let C_{eq} be the equivalent capacitance of the infinite ladder less the series capacitor in the first rung. Because the capacitance is finite and non-zero, adding one more stage to the ladder will not change the capacitance of the network. The capacitance of the two capacitor combination shown to the right is the equivalent of the infinite ladder, so it has capacitance C_{eq} also.



The equivalent capacitance of the parallel combination of C and C_{eq} is:

$$C + C_{eq}$$

The equivalent capacitance of the series combination of C and $(C + C_{eq})$ is C_{eq}, so:

$$\frac{1}{C_{eq}} = \frac{1}{C} + \frac{1}{C + C_{eq}}$$

Simply this expression to obtain a quadratic equation in C_{eq}:

$$C_{eq}^2 + CC_{eq} - C^2 = 0$$

Solving for the positive value of C_{eq} yields:

$$C_{eq} = \left(\frac{\sqrt{5}-1}{2}\right)C = \boxed{0.618C}$$

Parallel-Plate Capacitors

39 • A parallel-plate capacitor has a capacitance of 2.00 μF and a plate separation of 1.60 mm. (*a*) What is the maximum potential difference between the plates, so that dielectric breakdown of the air between the plates does not occur? (*b*) How much charge is stored at this potential difference?

Picture the Problem The potential difference V across a parallel-plate capacitor, the electric field E between its plates, and the separation d of the plates are related according to $V = Ed$. We can use this relationship to find V_{max} corresponding to dielectric breakdown and the definition of capacitance to find the maximum charge on the capacitor.

(*a*) Express the potential difference V across the plates of the capacitor in terms of the electric field between the plates E and their separation d:

$$V = Ed$$

V_{max} corresponds to E_{max}:

$$V_{max} = (3.00\,\text{MV/m})(1.60\,\text{mm}) = \boxed{4.80\,\text{kV}}$$

(*b*) Using the definition of capacitance, find the charge Q stored at this maximum potential difference:

$$Q = CV_{max} = (2.00\,\mu\text{F})(4.80\,\text{kV}) = \boxed{9.60\,\text{mC}}$$

40 • An electric field of 2.00×10^4 V/m exists between the circular plates of a parallel-plate capacitor that has a plate separation of 2.00 mm. (*a*) What is the potential difference across the capacitor plates? (*b*) What plate radius is required if the positively charged plate is to have a charge of 10.0 μC?

Picture the Problem The potential difference V across a parallel-plate capacitor, the electric field E between its plates, and the separation d of the plates are related according to $V = Ed$. In Part (*b*) we can use the definition of capacitance and the expression for the capacitance of a parallel-plate capacitor to find the required plate radius.

(*a*) Express the potential difference V across the plates of the capacitor in terms of the electric field between the plates E and their separation d:

$$V = Ed$$

Substitute numerical values and evaluate V:

$$V = \left(2.00\times10^{4}\ \text{V/m}\right)\left(2.00\,\text{mm}\right) = \boxed{40.0\,\text{V}}$$

(*b*) Use the definition of capacitance to relate the capacitance of the capacitor to its charge and the potential difference across it:

$$C = \frac{Q}{V}$$

The capacitance of a parallel-plate capacitor is given by:

$$C = \frac{\epsilon_0 A}{d} = \frac{\epsilon_0 \pi R^2}{d}$$

where R is the radius of the circular plates.

Equate these two expressions for C:

$$\frac{\epsilon_0 \pi R^2}{d} = \frac{Q}{V} \Rightarrow R = \sqrt{\frac{Qd}{\epsilon_0 \pi V}}$$

Substitute numerical values and evaluate R:

$$R = \sqrt{\frac{\left(10.0\,\mu\text{C}\right)\left(2.00\,\text{mm}\right)}{\pi\left(8.854\times10^{-12}\,\frac{\text{C}^2}{\text{N}\cdot\text{m}^2}\right)\left(40.0\,\text{V}\right)}} = \boxed{4.24\,\text{m}}$$

41 •• A parallel-plate, air-gap capacitor has a capacitance of 0.14 μF. The plates are 0.50 mm apart. (*a*) What is the area of each plate? (*b*) What is the potential difference between the plates if the positively charged plate has a charge of 3.2 μC? (*c*) What is the stored energy? (*d*) What is the maximum energy this capacitor can store before dielectric breakdown of the air between the plates occurs?

Picture the Problem We can use the expression for the capacitance of a parallel-plate capacitor to find the area of each plate and the definition of capacitance to find the potential difference when the capacitor is charged to 3.2 μC. We can find the stored energy using $U = \frac{1}{2}CV^2$ and the definition of capacitance and the relationship between the potential difference across a parallel-plate capacitor and the electric field between its plates to find the charge at which dielectric breakdown occurs. Recall that $E_{\text{max, air}}$ = 3.00 MV/m.

(*a*) The capacitance of a parallel-plate capacitor is given by:

$$C = \frac{\epsilon_0 A}{d} \Rightarrow A = \frac{Cd}{\epsilon_0}$$

Substitute numerical values and evaluate *A*:

$$A = \frac{(0.14\,\mu\text{F})(0.50\,\text{mm})}{8.854\times10^{-12}\,\frac{\text{C}^2}{\text{N}\cdot\text{m}^2}} = 7.906\,\text{m}^2 = \boxed{7.9\,\text{m}^2}$$

(*b*) Using the definition of capacitance, find the potential difference across the capacitor when it is charged to 3.2 μC:

$$V = \frac{Q}{C} = \frac{3.2\,\mu\text{C}}{0.14\,\mu\text{F}} = 22.9\,\text{V} = \boxed{23\,\text{V}}$$

(*c*) Express the stored energy as a function of the capacitor's capacitance and the potential difference across it:

$$U = \tfrac{1}{2}CV^2$$

Substitute numerical values and evaluate *U*:

$$U = \tfrac{1}{2}(0.14\,\mu\text{F})(22.9\,\text{V})^2 = 36.7\,\mu\text{J} = \boxed{37\,\mu\text{J}}$$

(*d*) The maximum energy this capacitor can store before dielectric breakdown of the air between the plates occurs is given by:

$$U_{\max} = \tfrac{1}{2}CV_{\max}^2$$

Relate the maximum potential difference to the maximum electric field between the plates:

$$V_{\max} = E_{\max}d$$

Substituting for $V_{\max}$ yields:

$$U_{\max} = \tfrac{1}{2}Cd^2E_{\max}^2$$

Substitute numerical values and evaluate $U_{\max}$:

$$U_{\max} = \tfrac{1}{2}(0.14\,\mu\text{F})(0.50\,\text{mm})^2(3.00\,\text{MV/m})^2 = \boxed{0.16\,\text{J}}$$

42 •• Design a 0.100-μF parallel-plate capacitor that has air between its plates and that can be charged to a maximum potential difference of 1000 V before dielectric breakdown occurs. (*a*) What is the minimum possible separation

between the plates? (*b*) What minimum area must each plate of the capacitor have?

Picture the Problem The potential difference across the capacitor plates V is related to their separation d and the electric field between them according to $V = Ed$. We can use this equation with $E_{max} = 3.00$ MV/m to find d_{min}. In Part (*b*) we can use the expression for the capacitance of a parallel-plate capacitor to find the required area of the plates.

(*a*) Use the relationship between the potential difference across the plates and the electric field between them to find the minimum separation of the plates:

$$d_{min} = \frac{V}{E_{max}} = \frac{1000\,\text{V}}{3.00\,\text{MV/m}} = \boxed{0.333\,\text{mm}}$$

(*b*) The capacitance of a parallel-plate capacitor is given by:

$$C = \frac{\epsilon_0 A}{d} \Rightarrow A = \frac{Cd}{\epsilon_0}$$

Substitute numerical values and evaluate A:

$$A = \frac{(0.100\,\mu\text{F})(0.333\,\text{mm})}{8.854\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2} = \boxed{3.76\,\text{m}^2}$$

Cylindrical Capacitors

43 • In preparation for an experiment that you will do in your introductory nuclear physics lab, you are shown the inside of a Geiger tube. You measure the radius and the length of the central wire of the Geiger tube to be 0.200 mm and 12.0 cm, respectively. The outer surface of the tube is a conducing cylindrical shell that has an inner radius of 1.50 cm. The shell is coaxial with the wire and has the same length (12.0 cm). Calculate (*a*) the capacitance of your tube, assuming that the gas in the tube has a dielectric constant of 1.00, and (*b*) the value of the linear charge density on the wire when the potential difference between the wire and shell is 1.20 kV?

Picture the Problem The capacitance of a cylindrical capacitor is given by $C = 2\pi\kappa\,\epsilon_0\,L/\ln(R_2/R_1)$ where L is its length and R_1 and R_2 the radii of the inner and outer conductors.

(*a*) The capacitance of the coaxial cylindrical shell is given by:

$$C = \frac{2\pi\kappa\,\epsilon_0\,L}{\ln\left(\frac{R_2}{R_1}\right)}$$

Substitute numerical values and evaluate C:

$$C = \frac{2\pi(1.00)\left(8.854\times10^{-12}\ \mathrm{C^2/N\cdot m^2}\right)(0.120\,\mathrm{m})}{\ln\left(\dfrac{1.50\,\mathrm{cm}}{0.200\,\mathrm{mm}}\right)} = 1.546\ \mathrm{pF} = \boxed{1.55\,\mathrm{pF}}$$

(*b*) Use the definition of capacitance to express the charge per unit length:

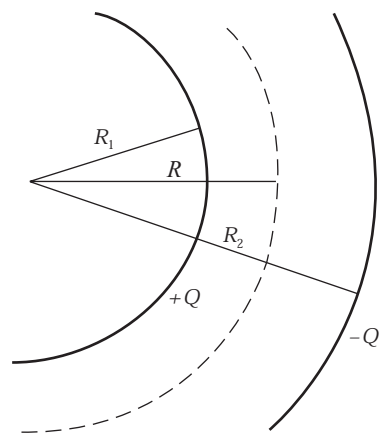
$$\lambda = \frac{Q}{L} = \frac{CV}{L}$$

Substitute numerical values and evaluate λ:

$$\lambda = \frac{(1.546\,\mathrm{pF})(1.20\,\mathrm{kV})}{0.120\,\mathrm{m}} = \boxed{15.5\,\mathrm{nC/m}}$$

44 •• A cylindrical capacitor consists of a long wire that has a radius R_1, a length L and a charge $+Q$. The wire is enclosed by a coaxial outer cylindrical shell that has a inner radius R_2, length L, and charge $-Q$. (*a*) Find expressions for the electric field and energy density as a function of the distance R from the axis. (*b*) How much energy resides in a region between the conductors that has a radius R, a thickness dR, and a volume $2\pi rL\,dR$? (*c*) Integrate your expression from Part (*b*) to find the total energy stored in the capacitor. Compare your result with that obtained by using the formula $U = Q^2/(2C)$ in conjunction with the known expression for the capacitance of a cylindrical capacitor.

Picture the Problem The diagram shows a partial cross-sectional view of the inner wire and the outer cylindrical shell. By symmetry, the electric field is radial in the space between the wire and the concentric cylindrical shell. We can apply Gauss's law to cylindrical surfaces of radii $R < R_1$, $R_1 < R < R_2$, and $R > R_2$ to find the electric field and, hence, the energy density in these regions.



(*a*) Apply Gauss's law to a cylindrical surface of radius $R < R_1$ and length L to obtain:

$$E_{R<R_1}(2\pi RL) = \frac{Q_{\text{inside}}}{\epsilon_0} = 0$$

and

$$E_{R<R_1} = \boxed{0}$$

Because $E = 0$ for $R < R_1$:

$$u_{R<R_1} = \boxed{0}$$

Apply Gauss's law to a cylindrical surface of radius $R_1 < R < R_2$ and length L to obtain:

$$E_{R_1<R<R_2}(2\pi RL) = \frac{Q_{\text{inside}}}{\epsilon_0} = \frac{\lambda L}{\epsilon_0}$$

where λ is the linear charge density.

Solve for $E_{R_1<R<R_2}$ to obtain:

$$E_{R_1<R<R_2} = \frac{\lambda}{2\pi\,\epsilon_0\,R} = \boxed{\frac{2kQ}{RL}}$$

The energy density in the region $R_1 < R < R_2$ is given by:

$$u_{R_1<R<R_2} = \tfrac{1}{2}\epsilon_0\,E^2_{R_1<R<R_2}$$

Substituting for $E_{R_1<R<R_2}$ and simplifying yields:

$$u_{R_1<R<R_2} = \tfrac{1}{2}\epsilon_0\left(\frac{2k\lambda}{R}\right)^2 = \tfrac{1}{2}\epsilon_0\left(\frac{2kQ}{RL}\right)^2 = \boxed{\frac{2k^2\,\epsilon_0\,Q^2}{R^2L^2}}$$

Apply Gauss's law to a cylindrical surface of radius $R > R_2$ and length L to obtain:

$$E_{R>R_2}(2\pi RL) = \frac{Q_{\text{inside}}}{\epsilon_0} = 0$$

and

$$E_{R>R_2} = \boxed{0}$$

Because $E = 0$ for $R > R_2$:

$$u_{R>R_2} = \boxed{0}$$

(*b*) Express the energy residing in a cylindrical shell between the conductors of radius R, thickness dR, and volume $2\pi RL\,dR$:

$$dU = 2\pi RLu(R)dR = 2\pi RL\left(\frac{2k^2\,\epsilon_0\,Q^2}{R^2L^2}\right)dr = \boxed{\frac{kQ^2}{RL}dR}$$

(*c*) Integrate dU from $R = R_1$ to $R = R_2$ to obtain:

$$U = \frac{kQ^2}{L}\int_{R_1}^{R_2}\frac{dR}{R} = \boxed{\frac{kQ^2}{L}\ln\left(\frac{R_2}{R_1}\right)}$$

Use $U = \frac{1}{2}CV^2$ and the expression for the capacitance of a cylindrical capacitor to obtain:

$$U = \tfrac{1}{2}CV^2 = \tfrac{1}{2}\frac{Q^2}{C} = \frac{Q^2}{2\left(\dfrac{2\pi \epsilon_0 L}{\ln\left(\dfrac{R_2}{R_1}\right)}\right)} = \boxed{\frac{kQ^2}{L}\ln\left(\frac{R_2}{R_1}\right)}$$

in agreement with the result from Part (*b*).

45 ••• Three concentric, thin long conducting cylindrical shells have radii of 2.00 mm, 5.00 mm, and 8.00 mm. The space between the shells is filled with air. The innermost and outermost shells are connected at one end by a conducting wire. Find the capacitance per unit length of this configuration.

Picture the Problem Note that with the innermost and outermost cylinders connected together the system corresponds to two cylindrical capacitors connected in parallel. We can use $C = \dfrac{2\pi \epsilon_0 \kappa L}{\ln(R_2/R_1)}$ to express the capacitance per unit length and then calculate and add the capacitances per unit length of each of the cylindrical shell capacitors.

Relate the capacitance of a cylindrical capacitor to its length L and inner and outer radii R_1 and R_2:

$$C = \frac{2\pi \epsilon_0 \kappa L}{\ln(R_2/R_1)}$$

Divide both sides of the equation by L to express the capacitance per unit length:

$$\frac{C}{L} = \frac{2\pi \epsilon_0 \kappa}{\ln(R_2/R_1)}$$

Express the capacitance per unit length of the cylindrical system:

$$\frac{C}{L} = \left(\frac{C}{L}\right)_{\text{outer}} + \left(\frac{C}{L}\right)_{\text{inner}} \qquad (1)$$

Substitute numerical values and evaluate the capacitance per unit length of the outer cylindrical shell combination:

$$\left(\frac{C}{L}\right)_{\text{outer}} = \frac{2\pi\left(8.854\times10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2\right)(1.00)}{\ln(0.800\,\text{cm}/0.500\,\text{cm})} = 118.4\,\text{pF/m}$$

Substitute numerical values and evaluate capacitance per unit length of the inner cylindrical shell combination:

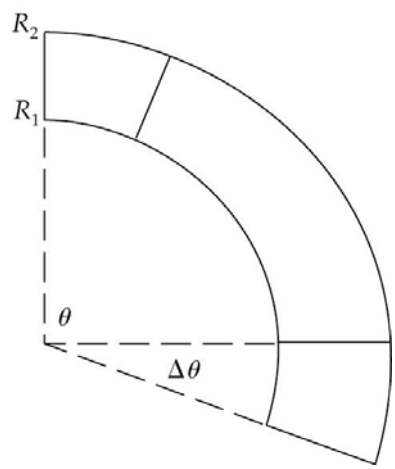
$$\left(\frac{C}{L}\right)_{\text{inner}} = \frac{2\pi\left(8.854\times10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2\right)(1.00)}{\ln(0.500\,\text{cm}/0.200\,\text{cm})} = 60.7\,\text{pF/m}$$

Substituting numerical results in equation (1) yields:

$$\frac{C}{L} = 118.4\,\text{pF/m} + 60.7\,\text{pF/m} = \boxed{179\,\text{pF/m}}$$

46 ••• A *goniometer* is a precise instrument for measuring angles. A *capacitive goniometer* is shown in Figure 24-40*a*. Each plate of the variable capacitor (Figure 24-40*b*) consists of a flat metal semicircle that has an inner radius R_1 and an outer radius R_2. The plates share a common rotation axis, and the width of the air gap separating the plates is *d*. Calculate the capacitance as a function of the angle θ and the parameters given.

Picture the Problem We can use the expression for the capacitance of a parallel-plate capacitor of variable area and the geometry of the figure to express the capacitance of the goniometer.



The capacitance of the parallel-plate capacitor is given by:

$$C = \frac{\epsilon_0\,(A - \Delta A)}{d}$$

The area of the plates is:

$$A = \pi\left(R_2^2 - R_1^2\right)\frac{\theta}{2\pi} = \left(R_2^2 - R_1^2\right)\frac{\theta}{2}$$

If the top plate rotates through an angle $\Delta\theta$, then the area is reduced by:

$$\Delta A = \pi\left(R_2^2 - R_1^2\right)\frac{\Delta\theta}{2\pi} = \left(R_2^2 - R_1^2\right)\frac{\Delta\theta}{2}$$

Substitute for A and ΔA in the expression for C and simplify to obtain:

$$C = \frac{\epsilon_0}{d}\left[\left(R_2^2 - R_1^2\right)\frac{\theta}{2} - \left(R_2^2 - R_1^2\right)\frac{\Delta\theta}{2}\right]$$
$$= \boxed{\frac{\epsilon_0\left(R_2^2 - R_1^2\right)}{2d}\left(\theta - \Delta\theta\right)}$$

47 ••• A *capacitive pressure gauge* is shown in Figure 24-41. Each plate has an area A. The plates are separated by a material that has a dielectric constant κ, a thickness d, and a Young's modulus Y. If a pressure increase of ΔP is applied to the plates, derive an expression for the change in capacitance.

Picture the Problem Let C be the capacitance of the capacitor when the pressure is P and C' be the capacitance when the pressure is $P + \Delta P$. We'll assume that (*a*) the change in the thickness of the plates is small, and (*b*) the total volume of material between the plates is conserved. We can use the expression for the capacitance of a dielectric-filled parallel-plate capacitor and the definition of Young's modulus to express the change in the capacitance ΔC of the given capacitor when the pressure on its plates is increased by ΔP.

Express the change in capacitance resulting from the decrease in separation of the capacitor plates by Δd:

$$\Delta C = C' - C = \frac{\kappa\,\epsilon_0\,A'}{d - \Delta d} - \frac{\kappa\,\epsilon_0\,A}{d}$$

Because the volume is constant:

$$A'd' = Ad$$

or

$$A' = \left(\frac{d}{d'}\right)A = \left(\frac{d}{d - \Delta d}\right)A$$

Substitute for A' in the expression for ΔC and simplify to obtain:

$$\Delta C = \frac{\kappa\,\epsilon_0\,A}{d - \Delta d}\left(\frac{d}{d - \Delta d}\right) - \frac{\kappa\,\epsilon_0\,A}{d}$$
$$= \frac{\kappa\,\epsilon_0\,A}{d\left(d - \Delta d\right)^2}d^2 - \frac{\kappa\,\epsilon_0\,A}{d}$$
$$= \frac{\kappa\,\epsilon_0\,A}{d}\left[\frac{d^2}{\left(d - \Delta d\right)^2} - 1\right]$$
$$= C\left[\frac{d^2}{\left(d - \Delta d\right)^2} - 1\right]$$

From the definition of Young's modulus:

$$\frac{\Delta d}{d} = -\frac{P}{Y} \Rightarrow \Delta d = -\left(\frac{P}{Y}\right)d$$

Substitute for Δd in the expression for ΔC to obtain:

$$\Delta C = \frac{\kappa \epsilon_0 A}{d}\left[\frac{d^2}{\left\{d+\left(\frac{P}{Y}\right)d\right\}^2}-1\right]$$

$$= C\left[\left\{1+\left(\frac{P}{Y}\right)\right\}^{-2}-1\right]$$

Expand $\left(1-\frac{P}{Y}\right)^{-2}$ binomially to obtain:

$$\left(1-\frac{P}{Y}\right)^{-2} = 1-2\frac{P}{Y}+3\left(\frac{P}{Y}\right)^2+\ldots$$

Provided $P << Y$:

$$\left(1-\frac{P}{Y}\right)^{-2} \approx 1-2\frac{P}{Y}$$

Substitute in the expression for ΔC and simplify to obtain:

$$\Delta C = C\left[1-2\frac{P}{Y}-1\right] = \boxed{-2\frac{P}{Y}C}$$

Spherical Capacitors

48 •• Model Earth as a conducting sphere. (*a*) What is its self-capacitance? (*b*) Assume the magnitude of the electric field at Earth's surface is 150 V/m. What charge density does this correspond to? Express this value in fundamental charge units *e* per square centimeter

Picture the Problem (*a*) We can use the definition of capacitance and the expression for the electric potential at the surface of Earth to find Earth's self-capacitance. In Part (*b*) we can use $E = \sigma/\epsilon_0$ to find Earth's surface charge density.

(*a*) The self-capacitance of Earth is given by:

$C = \frac{Q}{V}$ where Q is the charge on Earth and V is the potential at its surface.

Because $V = \frac{kQ}{R}$ where R is the radius of Earth:

$$C = \frac{Q}{\frac{kQ}{R}} = \frac{R}{k}$$

Substitute numerical values and evaluate C:

$$C = \frac{6370\ \text{km}}{8.988\times10^{9}\ \frac{\text{N}\cdot\text{m}^2}{\text{C}^2}} = 0.7087\ \text{mF}$$
$$= \boxed{0.709\ \text{mF}}$$

(*b*) The electric field at the surface of Earth is related to Earth's charge density:

$$E = \frac{\sigma}{\epsilon_0} \Rightarrow \sigma = \epsilon_0\ E$$

Substitute numerical values and evaluate σ:

$$\sigma = \left(8.854\times10^{-12}\ \frac{\text{C}^2}{\text{N}\cdot\text{m}^2}\right)\left(150\ \frac{\text{V}}{\text{m}}\right) = 1.328\ \frac{\text{nC}}{\text{m}^2}\times\left(\frac{1\ \text{m}}{10^2\ \text{cm}}\right)^2\times\frac{1e}{1.602\times10^{-19}\ \text{C}}$$
$$= \boxed{829\times10^{3}\ \frac{e}{\text{cm}^2}}$$

49 •• **[SSM]** A spherical capacitor consists of a thin spherical shell that has a radius R_1 and a thin, concentric spherical shell that has a radius R_2, where $R_2 > R_1$. (*a*) Show that the capacitance is given by $C = 4\pi\epsilon_0 R_1R_2/(R_2 - R_1)$. (*b*) Show that when the radii of the shells are nearly equal, the capacitance approximately is given by the expression for the capacitance of a parallel-plate capacitor, $C = \epsilon_0 A/d$, where A is the area of the sphere and $d = R_2 - R_1$.

Picture the Problem We can use the definition of capacitance and the expression for the potential difference between charged concentric spherical shells to show that $C = 4\pi\,\epsilon_0\ R_1R_2/(R_2 - R_1)$.

(*a*) Using its definition, relate the capacitance of the concentric spherical shells to their charge Q and the potential difference V between their surfaces:

$$C = \frac{Q}{V}$$

Express the potential difference between the conductors:

$$V = kQ\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = kQ\frac{R_2 - R_1}{R_1R_2}$$

Substitute for V and simplify to obtain:

$$C = \frac{Q}{kQ\dfrac{R_2 - R_1}{R_1R_2}} = \frac{R_1R_2}{k(R_2 - R_1)} = \boxed{\frac{4\pi\,\epsilon_0\,R_1R_2}{R_2 - R_1}}$$

(*b*) Because $R_2 = R_1 + d$:

$$R_1R_2 = R_1(R_1 + d) = R_1^2 + R_1d \approx R_1^2 = R^2$$

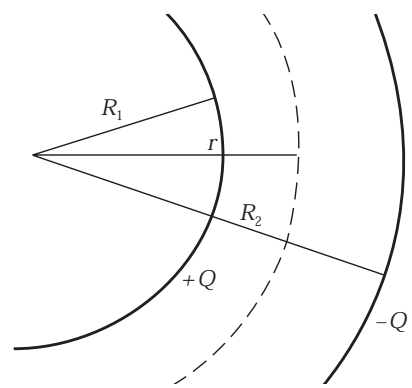
because d is small.

Substitute to obtain:

$$C \approx \frac{4\pi\,\epsilon_0\,R^2}{d} = \boxed{\frac{\epsilon_0\,A}{d}}$$

50 •• A spherical capacitor is composed of an inner sphere which has a radius R_1 and a charge $+Q$ and an outer concentric spherical thin shell which has a radius R_2 and a charge $-Q$. (*a*) Find the electric field and the energy density as a function of r, where r is the distance from the center of the sphere, for $0 \le r < \infty$. (*b*) Calculate the energy associated with the electrostatic field in a spherical shell between the conductors that has a radius r, a thickness dr, and a volume $4\pi r^2\,dr$? (*c*) Integrate your expression from Part (*b*) to find the total energy and compare your result with the result obtained using $U = \frac{1}{2}QV$.

Picture the Problem The diagram shows a partial cross-sectional view of the inner and outer spherical shells. By symmetry, the electric field is radial. We can apply Gauss's law to spherical surfaces of radii $r < R_1$, $R_1 < r < R_2$, and $r > R_2$ to find the electric field and, hence, the energy density in these regions.



(*a*) Apply Gauss's law to a spherical surface of radius $r < R_1$ to obtain:

$$E_r\left(4\pi r^2\right) = \frac{Q_{\text{inside}}}{\epsilon_0} = 0$$

and, because $Q_{\text{inside}} = 0$,

$$E_r = \boxed{0}$$

Because $E = 0$ for $r < R_1$:

$$u = \boxed{0}$$

Apply Gauss's law to a spherical surface of radius $R_1 < r < R_2$ to obtain:

$$E_r\left(4\pi r^2\right) = \frac{Q_{\text{inside}}}{\epsilon_0} = \frac{Q}{\epsilon_0}$$

Solve for E_r to obtain:

$$E_{r_2} = \frac{Q}{4\pi\,\epsilon_0\,r^2} = \boxed{\frac{kQ}{r^2}} \quad R_1 \le r \le R_2$$

Express the energy density in the region $R_1 < r < R_2$:

$$u = \tfrac{1}{2}\epsilon_0\,E^2_{R_1<r<R_2} = \tfrac{1}{2}\epsilon_0\left(\frac{kQ}{r^2}\right)^2$$

$$= \boxed{\frac{k^2\,\epsilon_0\,Q^2}{2r^4}} \quad R_1 \le r \le R_2$$

Apply Gauss's law to a cylindrical surface of radius $r > R_2$ to obtain:

$$E_r\left(4\pi r^2\right) = \frac{Q_{\text{inside}}}{\epsilon_0} = 0$$

and, because $Q_{\text{inside}} = 0$,

$$E_r = \boxed{0}$$

Because $E = 0$ for $r > R_2$:

$$u_r = \boxed{0}$$

(*b*) Express the energy in the electrostatic field in a spherical shell of radius r, thickness dr, and volume $4\pi r^2 dr$ between the conductors:

$$dU = 4\pi r^2 u(r)dr = 4\pi r^2\left(\frac{k^2\,\epsilon_0\,Q^2}{2r^4}\right)dr$$

$$= \boxed{\frac{kQ^2}{2r^2}dr}$$

(*c*) Integrate dU from $r = R_1$ to R_2 to obtain:

$$U = \frac{kQ^2}{2}\int_{R_1}^{R_2}\frac{dr}{r^2} = \frac{kQ^2\left(R_2 - R_1\right)}{2R_1R_2}$$

$$= \boxed{\frac{1}{2}Q^2\left(\frac{R_2 - R_1}{4\pi\,\epsilon_0\,R_1R_2}\right)}$$

Note that the quantity in parentheses is $1/C$, so we have $U = \tfrac{1}{2}Q^2/C$.

51 ••• An isolated conducting sphere of radius R has a charge Q distributed uniformly over its surface. Find the distance R' from the center of the sphere such that half the total electrostatic energy of the system is associated with the electric field beyond that distance.

Picture the Problem We know, from Gauss's law, that the field inside the shell is zero. We can then express the energy in the electrostatic field in a spherical shell

of radius r, thickness dr, and volume $4\pi r^2 dr$ outside the given spherical shell and find the total energy in the electric field by integrating from R to ∞. If we then integrate the same expression from R to R' we can find the distance R' from the center of the sphere such that half the total electrostatic field energy of the system is within that distance.

Apply Gauss's law to a spherical shell of radius $r > R$ to obtain:

$$E_{r>R}\left(4\pi r^2\right) = \frac{Q_{\text{inside}}}{\epsilon_0} = \frac{Q}{\epsilon_0}$$

Solve for $E_{r>R}$ outside the spherical shell:

$$E_{r>R} = \frac{kQ}{r^2}$$

Express the energy density in the region $r > R$:

$$u = \tfrac{1}{2}\epsilon_0 E_{r>R}^2 = \tfrac{1}{2}\epsilon_0\left(\frac{kQ}{r^2}\right)^2 = \frac{k^2\epsilon_0 Q^2}{2r^4}$$

Express the energy in the electrostatic field in a spherical shell of radius r, thickness dr, and volume $4\pi r^2 dr$ outside the spherical shell:

$$\begin{aligned} dU &= 4\pi r^2 u(r)dr \\ &= 4\pi r^2\left(\frac{k^2\epsilon_0 Q^2}{2r^4}\right)dr \\ &= \frac{kQ^2}{2r^2}dr \end{aligned}$$

Integrate dU from R to ∞ to obtain:

$$U_{\text{tot}} = \frac{kQ^2}{2}\int_R^\infty \frac{dr}{r^2} = \frac{kQ^2}{2R}$$

Integrate dU from R to R' to obtain:

$$U = \frac{kQ^2}{2}\int_R^{R'} \frac{dr}{r^2} = \frac{kQ^2}{2}\left(\frac{1}{R} - \frac{1}{R'}\right)$$

Set $U = \frac{1}{2}U_{\text{tot}}$ to obtain:

$$\frac{kQ^2}{2}\left(\frac{1}{R} - \frac{1}{R'}\right) = \frac{kQ^2}{4R} \Rightarrow R' = \boxed{2R}$$

Disconnected and Reconnected Capacitors

52 •• A 2.00-μF capacitor is energized to a potential difference of 12.0 V. The wires connecting the capacitor to the battery are then disconnected from the battery and connected across a second, initially uncharged, capacitor. The potential difference across the 2.00-μF capacitor then drops to 4.00 V. What is the capacitance of the second capacitor?

Picture the Problem Let C_1 represent the capacitance of the 2.00-μF capacitor and C_2 the capacitance of the 2nd capacitor. Note that when they are connected as

described in the problem statement they are in parallel and, hence, share a common potential difference. We can use the equation for the equivalent capacitance of two capacitors in parallel and the definition of capacitance to relate C_2 to C_1 and to the charge stored in and the potential difference across the equivalent capacitor.

Using the definition of capacitance, find the charge on capacitor C_1:

$$Q_1 = C_1V = (2.00\,\mu\text{F})(12.0\,\text{V}) = 24.0\,\mu\text{C}$$

Express the equivalent capacitance of the two-capacitor system and solve for C_2:

$$C_{\text{eq}} = C_1 + C_2 \Rightarrow C_2 = C_{\text{eq}} - C_1$$

Using the definition of capacitance, express C_{eq} in terms of Q_2 and V_2:

$$C_{\text{eq}} = \frac{Q_2}{V_2} = \frac{Q_1}{V_2}$$

where V_2 is the common potential difference (they are in parallel) across the two capacitors and Q_1 and Q_2 are the (equal) charges on the two capacitors.

Substitute for C_{eq} to obtain:

$$C_2 = \frac{Q_1}{V_2} - C_1$$

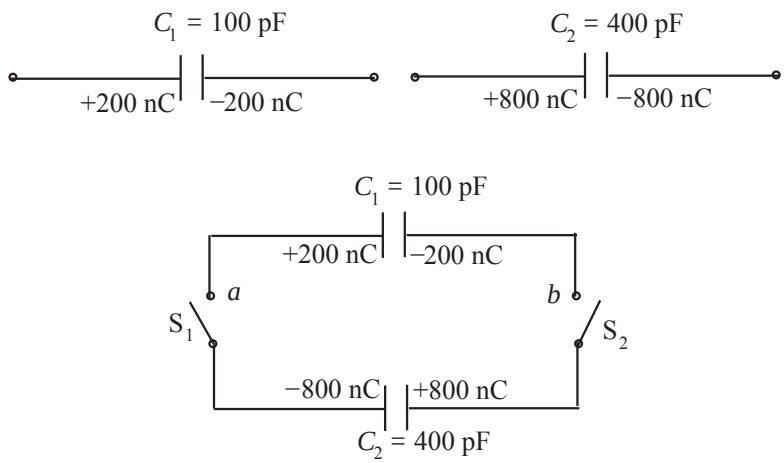
Substitute numerical values and evaluate C_2:

$$C_2 = \frac{24.0\,\mu\text{C}}{4.00\,\text{V}} - 2.00\,\mu\text{F} = \boxed{4.00\,\mu\text{F}}$$

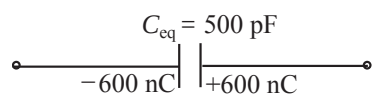
53 •• [SSM] A 100-pF capacitor and a 400-pF capacitor are both charged to 2.00 kV. They are then disconnected from the voltage source and are connected together, positive plate to negative plate and negative plate to positive plate. (*a*) Find the resulting potential difference across each capacitor. (*b*) Find the energy dissipated when the connections are made.

Picture the Problem (*a*) Just after the two capacitors are disconnected from the voltage source, the 100-pF capacitor carries a charge of 200 nC and the 400-pF capacitor carries a charge of 800 nC. After switches S_1 and S_2 in the circuit are closed, the capacitors are in parallel between points *a* and *b,* and the equivalent capacitance of the system is $C_{\text{eq}} = C_{100} + C_{400}$. The plates to the right in the diagram below form a single conductor with a charge of 600 nC, and the plates to the left form a conductor with charge $-Q$ = –600 nC. The potential difference across each capacitor is $V = Q/C_{\text{eq}}$. In Part (*b*) we can find the energy dissipated

when the connections are made by subtracting the energy stored in the system after they are connected from the energy stored in the system before they are connected.



(*a*) When the switches are closed and the capacitors are connected together, their initial charges redistribute and the final charge on the two-capacitor system is 600 nC and the equivalent capacitance is 500 pF:



The potential difference across each capacitor is the potential difference across the equivalent capacitor:

$$V = \frac{Q}{C_{eq}} = \frac{600\text{ nC}}{500\text{ pF}} = \boxed{1.20\text{ kV}}$$

(*b*) The energy dissipated when the capacitors are connected is the difference between the energy stored after they are connected and the energy stored before they were connected:

$$U_{\text{dissipated}} = U_{\text{before}} - U_{\text{after}} \qquad (1)$$

U_{before} is given by:

$$\begin{aligned} U_{\text{before}} &= U_1 + U_2 \\ &= \tfrac{1}{2}Q_1V_1 + \tfrac{1}{2}Q_2V_2 \\ &= \tfrac{1}{2}(Q_1 + Q_2)V_i \end{aligned}$$

where V_i is the charging voltage.

U_{after} is given by:

$$U_{\text{after}} = \tfrac{1}{2}QV$$

where Q is the total charge stored after the capacitors have been connected and V is the voltage found in Part (*a*).

Substitute for U_{before} and U_{after} in equation (1) and simplify to obtain:

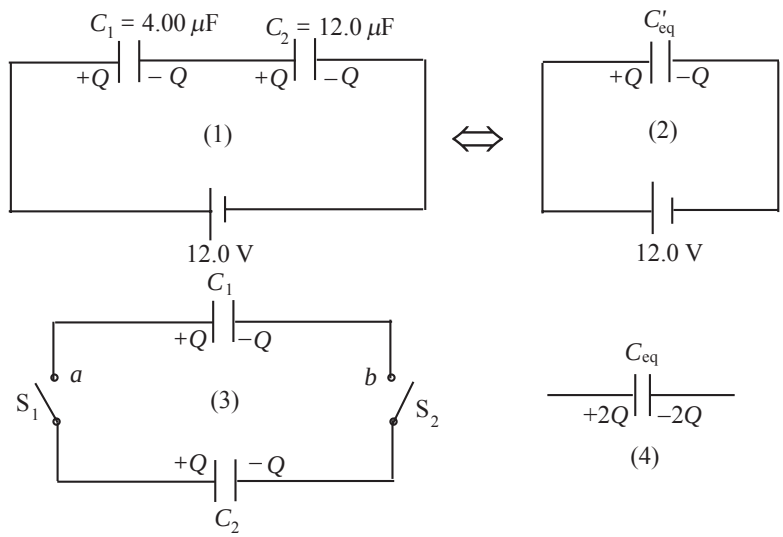
$$U_{\text{dissipated}} = \tfrac{1}{2}(Q_1 + Q_2)V_i - \tfrac{1}{2}QV$$

Substitute numerical values and evaluate $U_{\text{dissipated}}$:

$$U_{\text{dissipated}} = \tfrac{1}{2}(200\text{ nC} + 800\text{ nC})(2.00\text{ kV}) - \tfrac{1}{2}(600\text{ nC})(1.20\text{ kV}) = \boxed{640\ \mu\text{J}}$$

54 •• Two capacitors, one that has a capacitance of 4.00 μF and one that has a capacitance of 12.0 μF, are first discharged and then are connected in series. The series combination is then connected across the terminals of a 12.0-V battery. Next they are carefully disconnected so that they are not discharged and they are then reconnected to each other—positive plate to positive plate and negative plate to negative plate. (*a*) Find the potential difference across each capacitor after they are reconnected. (*b*) Find the energy stored in the capacitors before they are disconnected from the battery, and find the energy stored after they are reconnected.

Picture the Problem Let C_1 represent the capacitance of the 4.00-μF capacitor and C_2 the capacitance of the 12.0-μF capacitor. (*a*) Just after the two capacitors are disconnected from the battery, they both carry charge Q. After switches S_1 and S_2 in the circuit are closed, the capacitors are in parallel between points *a* and *b,* and each will have the charge it acquired while they were connected in series across the battery. We can use the definition of capacitance and the equivalent capacitance of the two capacitors to find the common potential difference across them. In Part (*b*) we can use $U = \tfrac{1}{2}CV^2$ to find the initial and final energy stored in the capacitors.



(*a*) From diagram (4):

$$V_{4.00} = V_{12.0} = \frac{2Q}{C_{eq}} \qquad (1)$$

where Q is the charge on each capacitor *before* they are disconnected.

Find the equivalent capacitance of the two capacitors when they are connected in parallel:

$$C_{eq} = C_1 + C_2 = 4.00\,\mu\text{F} + 12.0\,\mu\text{F} = 16.0\,\mu\text{F}$$

Express the charge Q on each capacitor before they are disconnected:

$$Q = C'_{eq}V$$

Express the equivalent capacitance of the two capacitors connected in series:

$$C'_{eq} = \frac{C_1C_2}{C_1 + C_2} = \frac{(4.00\,\mu\text{F})(12.0\,\mu\text{F})}{4.00\,\mu\text{F} + 12.0\,\mu\text{F}} = 3.00\,\mu\text{F}$$

Substitute to find Q:

$$Q = (3.00\,\mu\text{F})(12.0\,\text{V}) = 36.0\,\mu\text{C}$$

Substitute numerical values for Q and C_{eq} in equation (1) and evaluate $V_{4.00} = V_{12.0}$:

$$V_{4.00} = V_{12.0} = \frac{2(36.0\,\mu\text{C})}{16.0\,\mu\text{F}} = \boxed{4.50\,\text{V}}$$

(*b*) The energy stored in the capacitors initially is:

$$U_i = \tfrac{1}{2}C'_{eq}V_i^2 = \tfrac{1}{2}(3.00\,\mu\text{F})(12.0\,\text{V})^2 = \boxed{216\,\mu\text{J}}$$

The energy stored in the capacitors when they have been reconnected is:

$$U_f = \tfrac{1}{2}C_{eq}V_f^2 = \tfrac{1}{2}(16.0\,\mu\text{F})(4.50\,\text{V})^2 = \boxed{162\,\mu\text{J}}$$

55 •• A 1.2-μF capacitor is charged to 30 V. After charging, the capacitor is disconnected from the voltage source and is connected across the terminals of a second capacitor that had previously been discharged. The final voltage across the 1.2-μF capacitor is 10 V. (*a*) What is the capacitance of the second capacitor? (*b*) How much energy was dissipated when the connection was made?

Picture the Problem Let C_1 represent the capacitance of the 1.2-μF capacitor and C_2 the capacitance of the 2nd capacitor. Note that when they are connected as described in the problem statement they are in parallel and, hence, share a common potential difference. We can use the equation for the equivalent capacitance of two capacitors in parallel and the definition of capacitance to relate C_2 to C_1 and to the charge stored in and the potential difference across the equivalent capacitor. In Part (*b*) we can use $U = \tfrac{1}{2}CV^2$ to find the energy before and after the connection was made and, hence, the energy dissipated when the connection was made.

(*a*) Using the definition of capacitance, find the charge on capacitor C_1:

$$Q_1 = C_1V = (1.2\,\mu\text{F})(30\,\text{V}) = 36\,\mu\text{C}$$

Because the capacitors are in parallel:

$$C_{eq} = C_1 + C_2 \Rightarrow C_2 = C_{eq} - C_1 \quad (1)$$

Using the definition of capacitance, express C_{eq} in terms of the charge Q_2 on the second capacitor and the common potential difference V_2 across the two capacitors:

$$C_{eq} = \frac{Q_2}{V_2} = \frac{Q_1}{V_2}$$

Substituting for C_{eq} in equation (1) yields:

$$C_2 = \frac{Q_1}{V_2} - C_1$$

Substitute numerical values and evaluate C_2:

$$C_2 = \frac{36\,\mu\text{C}}{10\,\text{V}} - 1.2\,\mu\text{F} = \boxed{2.4\,\mu\text{F}}$$

(*b*) The energy dissipated when the connections are made in terms of the energy stored in the capacitors before and after their connection is given by:

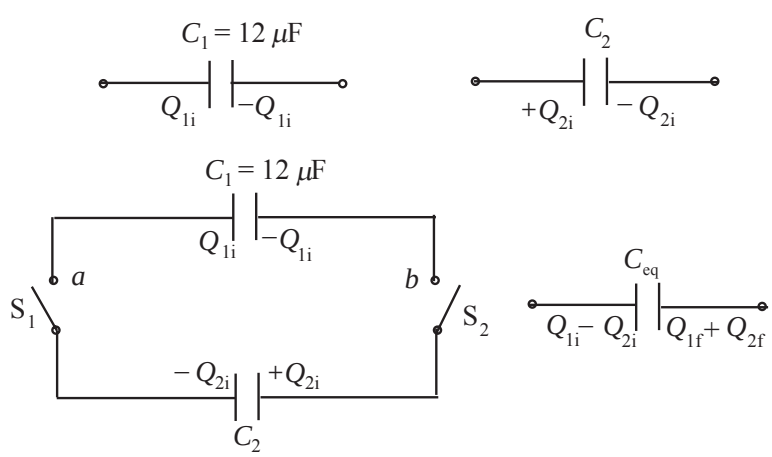
$$U_{\text{dissipated}} = U_{\text{before}} - U_{\text{after}} = \tfrac{1}{2}C_1V_1^2 - \tfrac{1}{2}C_{\text{eq}}V_{\text{f}}^2 = \tfrac{1}{2}\left(C_1V_1^2 - C_{\text{eq}}V_{\text{f}}^2\right)$$

Substitute numerical values and evaluate $U_{\text{dissipated}}$:

$$U_{\text{dissipated}} = \tfrac{1}{2}\left[(1.2\,\mu\text{F})(30\,\text{V})^2 - (3.6\,\mu\text{F})(10\,\text{V})^2\right] = \boxed{0.4\,\text{mJ}}$$

56 •• A 12-μF capacitor and a capacitor of unknown capacitance are both charged to 2.00 kV. After charging, the capacitors are disconnected from the voltage source. The capacitors are then connected to each other—positive plate to negative plate and negative plate to positive plate. The final voltage across the 12-μF capacitor is 1.00 kV. (*a*) What is the capacitance of the second capacitor? (*b*) How much energy was dissipated when the connection was made?

Picture the Problem Let C_1 represent the capacitance of the 12-μF capacitor and C_2 the capacitance of the second capacitor. (*a*) Just after the two capacitors are disconnected from the voltage source, the 12-μF capacitor carries a charge of $Q_{1\text{i}} = 24$ mC and the unknown capacitor carries a charge $Q_{2\text{i}}$. After switches S_1 and S_2 in the circuit are closed, the capacitors are in parallel between points *a* and *b*, and the equivalent capacitance of the system is $C_{\text{eq}} = C_1 + C_2$. The plates to the right in the diagram below form a single conductor with a charge of $Q_{1\text{i}} - Q_{2\text{i}}$, and the plates to the left form a conductor with charge $Q_{1\text{f}} + Q_{2\text{f}}$, where $Q_{1\text{i}}$, $Q_{2\text{i}}$, $Q_{1\text{f}}$ and $Q_{2\text{f}}$ are all positive . Because they are in parallel, the potential difference across both C_1 and C_2 when they are connected is 1.00 kV. In Part (*b*) we can find the energy dissipated when the connections are made by subtracting the energy stored in the system after they are connected from the energy stored in the system before they are connected.



(*a*) When the switches are closed and the capacitors are connected together, their initial charges redistribute. Apply conservation of charge to obtain:

$$Q_{1i} - Q_{2i} = Q_{1f} + Q_{2f}$$

or

$$C_1V_i - C_2V_i = C_1V_f + C_2V_f$$

Solving for C_2 yields:

$$C_2 = \frac{V_i - V_f}{V_i + V_f}C_1$$

Substitute numerical values and evaluate C_2:

$$C_2 = \left(\frac{2.00\text{ kV} - 1.00\text{ kV}}{2.00\text{ kV} + 1.00\text{ kV}}\right)(12\ \mu\text{F}) = \boxed{4.0\ \mu\text{F}}$$

(*b*) The energy dissipated when the connections are made is the difference between the initial and final energies stored by the system:

$$U_{\text{dissipated}} = U_i - U_f \qquad (1)$$

U_i and U_f are given by:

$$U_i = \tfrac{1}{2}C_1V_i^2 + \tfrac{1}{2}C_2V_i^2 = \tfrac{1}{2}(C_1 + C_2)V_i^2$$

and

$$U_f = \tfrac{1}{2}C_{eq}V_f^2$$

Because C_1 and C_2 are in parallel:

$$C_{eq} = C_1 + C_2$$

Substituting for C_{eq} yields:

$$U_f = \tfrac{1}{2}(C_1 + C_2)V_f^2$$

Substitute for U_i and U_f in equation (1) to obtain:

$$U_{\text{dissipated}} = \tfrac{1}{2}(C_1 + C_2)V_i^2 - \tfrac{1}{2}(C_1 + C_2)V_f^2 = \tfrac{1}{2}(C_1 + C_2)\left(V_i^2 - V_f^2\right)$$

Substitute numerical values and evaluate $U_{\text{dissipated}}$:

$$U_{\text{dissipated}} = \tfrac{1}{2}(12\ \mu\text{F} + 4.0\ \mu\text{F})\left[(2.00\text{ kV})^2 - (1.00\text{ kV})^2\right] = \boxed{24\text{ J}}$$

57 •• Two capacitors, one that has a capacitance of 4.00 μF and one that has a capacitance of 12.0 μF, are connected in parallel. The parallel combination is then connected across the terminals of a 12.0-V battery. Next they are carefully disconnected so that they are not discharged. They are then reconnected to each other—the positive plate of each capacitor connected to the negative plate of the other. (*a*) Find the potential difference across each capacitor after they are reconnected. (*b*) Find the energy stored in the capacitors before they are

disconnected from the battery, and find the energy stored after they are reconnected.

Picture the Problem When the capacitors are reconnected, each will have a charge equal to the difference between the charges they acquired while they were connected in parallel across the 12.0-V battery. We can use the definition of capacitance and their equivalent capacitance to find the common potential difference across them. In Part (*b*) we can use $U = \frac{1}{2}CV^2$ to find the initial and final energy stored in the capacitors.

(*a*) Using the definition of capacitance, express the potential difference across the capacitors when they are reconnected:

$$V_f = V_{4.00} = V_{12.0} = \frac{Q_f}{C_{eq}} = \frac{Q_f}{C_1 + C_2} \quad (1)$$

where Q_f is the common charge on the capacitors *after* they are reconnected.

Express the final charge Q_f on each capacitor:

$$Q_f = Q_2 - Q_1$$

Use the definition of capacitance to substitute for Q_2 and Q_1:

$$Q_f = C_2V - C_1V = (C_2 - C_1)V$$

Substitute in equation (1) to obtain:

$$V_f = \frac{C_2 - C_1}{C_1 + C_2}V$$

Substitute numerical values and evaluate V_f:

$$V_f = V_{4.00} = V_{12.0} = \frac{12.0\,\mu\text{F} - 4.00\,\mu\text{F}}{12.0\,\mu\text{F} + 4.00\,\mu\text{F}}(12.0\,\text{V}) = \boxed{6.0\,\text{V}}$$

(*b*) The energy stored in the capacitors initially is given by:

$$U_i = \tfrac{1}{2}C_1V^2 + \tfrac{1}{2}C_2V^2 = \tfrac{1}{2}V^2(C_1 + C_2)$$

Substitute numerical values and evaluate U_i:

$$U_i = \tfrac{1}{2}(12.0\,\text{V})^2(12.0\,\mu\text{F} + 4.00\,\mu\text{F}) = \boxed{1.15\,\text{mJ}}$$

The energy stored in the capacitors when they have been reconnected is given by:

$$U_f = \tfrac{1}{2}C_1V_f^2 + \tfrac{1}{2}C_2V_f^2 = \tfrac{1}{2}V_f^2(C_1 + C_2)$$

Substitute numerical values and evaluate U_f:

$$U_f = \tfrac{1}{2}(6.0\,\text{V})^2(12.0\,\mu\text{F} + 4.00\,\mu\text{F}) = \boxed{0.29\,\text{mJ}}$$

58 •• A 20-pF capacitor is charged to 3.0 kV and then removed from the battery and connected to an uncharged 50-pF capacitor. (*a*) What is the new charge on each capacitor? (*b*) Find the energy stored in the 20-pF capacitor before it is disconnected from the battery, and the energy stored in the two capacitors after they are connected to each other. Does the stored energy increase or decrease when the two capacitors are connected to each other?

Picture the Problem Let the numeral 1 refer to the 20-pF capacitor and the numeral 2 to the 50-pF capacitor. We can use conservation of charge and the fact that the connected capacitors will have the same potential difference across them to find the charge on each capacitor. We can decide whether stored energy increases or decreases when the two capacitors are connected by calculating the change in the electrostatic energy during this process.

(*a*) The final charges on the capacitors are given by:

$$Q_{1f} = C_1 V_f \qquad (1)$$

and

$$Q_{2f} = C_2 V_f \qquad (2)$$

Using the fact that charge is conserved when the capacitors are connected, relate the charge Q_{1i} initially on the 20-pF capacitor to the charges on the two capacitors when they have been connected:

$$Q_{1i} = Q_{1f} + Q_{2f}$$

or

$$C_1 V_{1i} = C_1 V_f + C_2 V_f$$

Solving for V_f yields:

$$V_f = \frac{C_1}{C_1 + C_2} V_{1i}$$

Substitute for V_f in equations (1) and (2) to obtain:

$$Q_{1f} = \frac{C_1^2}{C_1 + C_2} V_{1i}$$

and

$$Q_{2f} = \frac{C_1 C_2}{C_1 + C_2} V_{1i}$$

Substitute numerical values and evaluate Q_{1f}:

$$Q_{1f} = \left(\frac{(20\,\text{pF})^2}{20\,\text{pF} + 50\,\text{pF}}\right)(3.0\,\text{kV}) = 17.1\,\text{nC} = \boxed{17\,\text{nC}}$$

Substitute numerical values and evaluate Q_{2f}:

$$Q_{2f} = \left(\frac{(20\text{ pF})(50\text{ pF})}{20\text{ pF}+50\text{ pF}}\right)(3.0\text{ kV})$$
$$= 42.9\text{ nC} = \boxed{43\text{ nC}}$$

(*b*) The energy stored in the 20-pF capacitor before it is disconnected from the battery is given by:

$$U_i = U_{1i} = \tfrac{1}{2}C_1V_{1i}^2$$
$$= \tfrac{1}{2}(20\text{ pF})(3.0\text{ kV})^2 = \boxed{90\ \mu\text{J}}$$

The energy stored in the two capacitors after they are connected to each other is given by:

$$U_f = U_{1f} + U_{2f} = \frac{Q_{1f}^2}{2C_1} + \frac{Q_{2f}^2}{2C_2}$$

Substitute numerical values and evaluate U_f:

$$U_f = \frac{(17.1\text{ nC})^2}{2(20\text{ pF})} + \frac{(42.9\text{ nC})^2}{2(50\text{ pF})} = \boxed{26\ \mu\text{J}}$$

Because $U_f < U_i$, the stored energy decreases when the two capacitors are connected.

59 •• [SSM] Capacitors 1, 2 and 3, have capacitances equal to 2.00 μF, 4.00 μF, and 6.00 μF, respectively. The capacitors are connected in parallel, and the parallel combination is connected across the terminals of a 200-V source. The capacitors are then disconnected from both the voltage source and each other, and are connected to three switches as shown in Figure 24-42. (*a*) What is the potential difference across each capacitor when switches S_1 and S_2 are closed but switch S_3 remains open? (*b*) After switch S_3 is closed, what is the final charge on the leftmost plate of each capacitor? (*c*) Give the final potential difference across each capacitor after switch S_3 is closed.

Picture the Problem Let lower case *q*s refer to the charges before S_3 is closed and upper case *Q*s refer to the charges after this switch is closed. We can use conservation of charge to relate the charges on the capacitors before S_3 is closed to their charges when this switch is closed. We also know that the sum of the potential differences around the circuit when S_3 is closed must be zero and can use this to obtain a fourth equation relating the charges on the capacitors after the switch is closed to their capacitances. Solving these equations simultaneously will yield the charges Q_1, Q_2, and Q_3. Knowing these charges, we can use the definition of capacitance to find the potential difference across each of the capacitors.

(*a*) With S_1 and S_2 closed, but S_3 open, the charges on and the potential differences across the capacitors do not change. Hence:

$$V_1 = V_2 = V_3 = \boxed{200\,\text{V}}$$

(*b*) When S_3 is closed, the charges can redistribute; express the conditions on the charges that must be satisfied as a result of this redistribution:

$$q_2 - q_1 = Q_2 - Q_1,$$
$$q_3 - q_2 = Q_3 - Q_2,$$
and
$$q_1 - q_3 = Q_1 - Q_3.$$

Express the condition on the potential differences that must be satisfied when S_3 is closed:

$$V_1 + V_2 + V_3 = 0$$
where the subscripts refer to the three capacitors.

Use the definition of capacitance to eliminate the potential differences:

$$\frac{Q_1}{C_1} + \frac{Q_2}{C_2} + \frac{Q_3}{C_3} = 0 \qquad (1)$$

Use the definition of capacitance to find the initial charge on each capacitor:

$$q_1 = C_1V = (2.00\,\mu\text{F})(200\,\text{V}) = 400\,\mu\text{C},$$
$$q_2 = C_2V = (4.00\,\mu\text{F})(200\,\text{V}) = 800\,\mu\text{C},$$
and
$$q_3 = C_3V = (6.00\,\mu\text{F})(200\,\text{V}) = 1200\,\mu\text{C}$$

Let $q = q_1$. Then:

$$q_2 = 2Q \text{ and } q_3 = 3Q$$

Express Q_2 and Q_3 in terms of Q_1 and Q:

$$Q_2 = Q + Q_1 \qquad (2)$$
and
$$Q_3 = Q_1 + 2Q \qquad (3)$$

Substitute in equation (1) to obtain:

$$\frac{Q_1}{C_1} + \frac{Q + Q_1}{C_2} + \frac{Q_1 + 2Q}{C_3} = 0$$
or
$$\frac{Q_1}{2.00\,\mu\text{F}} + \frac{Q + Q_1}{4.00\,\mu\text{F}} + \frac{Q_1 + 2Q}{6.00\,\mu\text{F}} = 0$$

Solve for and evaluate Q_1 to obtain:

$$Q_1 = -\tfrac{7}{11}Q = -\tfrac{7}{11}(400\,\mu\text{C}) = \boxed{-255\,\mu\text{C}}$$

Substitute in equation (2) to obtain:

$$Q_2 = 400\,\mu\text{C} - 255\,\mu\text{C} = \boxed{145\,\mu\text{C}}$$

Substitute in equation (3) to obtain:

$$Q_3 = -255\,\mu\text{C} + 2(400\,\mu\text{C}) = \boxed{545\,\mu\text{C}}$$

(*c*) Use the definition of capacitance to find the potential difference across each capacitor with S_3 closed:

$$V_1 = \frac{Q_1}{C_1} = \frac{-255\,\mu\text{C}}{2.00\,\mu\text{F}} = \boxed{-127\,\text{V}},$$

$$V_2 = \frac{Q_2}{C_2} = \frac{145\,\mu\text{C}}{4.00\,\mu\text{F}} = \boxed{36.4\,\text{V}},$$

and

$$V_3 = \frac{Q_3}{C_3} = \frac{545\,\mu\text{C}}{6.00\,\mu\text{F}} = \boxed{90.9\,\text{V}}$$

60 •• A capacitor has a capacitance *C* and a charge *Q* on its positively charged plate. A student connects one terminal of the capacitor to a terminal of an identical capacitor whose plates are electrically neutral. When the remaining two terminals are connected, charge flows until electrostatic equilibrium is reestablished and both capacitors have charge *Q*/2 on them. Compare the total energy initially stored in the one capacitor to the total energy stored in the two when electrostatic equilibrium is reestablished. If there is less energy afterward, where do you think the missing energy went? *Hint: Wires that transport charge can heat up, which is called Joule heating and is discussed in detail in Chapter 25.*

Picture the Problem We can use the expression for the energy stored in a capacitor to express the ratio of the energy stored in the system after the discharge of the first capacitor to the energy stored in the system prior to the discharge.

Express the energy *U* initially stored in the capacitor whose capacitance is *C*:

$$U = \frac{Q^2}{2C}$$

The energy *U′* stored in the two capacitors after the first capacitor has discharged is:

$$U' = \frac{\left(\frac{Q}{2}\right)^2}{2C} + \frac{\left(\frac{Q}{2}\right)^2}{2C} = \frac{Q^2}{4C}$$

Express the ratio of *U′* to *U*:

$$\frac{U'}{U} = \frac{\frac{Q^2}{4C}}{\frac{Q^2}{2C}} = \frac{1}{2} \Rightarrow U' = \boxed{\tfrac{1}{2}U}$$

The missing energy was converted into thermal energy by the resistance of the connecting wires.

Dielectrics

61 • You are a laboratory assistant in a physics department that has budget problems. Your supervisor wants to construct cheap parallel-plate capacitors for use in introductory laboratory experiments. The design uses polyethylene, which has a dielectric constant of 2.30, between two sheets of aluminum foil. The area of each sheet of foil is 400 cm^2 and the thickness of the polyethylene is 0.300 mm. Find the capacitance of this arrangement.

Picture the Problem The capacitance of a parallel-plate capacitor filled with a dielectric of constant κ is given by $C = \frac{\kappa \epsilon_0 A}{d}$.

Relate the capacitance of the parallel-plate capacitor to the area of its plates, their separation, and the dielectric constant of the material between the plates:

$$C = \frac{\kappa \epsilon_0 A}{d}$$

Substitute numerical values and evaluate C:

$$C = \frac{(2.30)\left(8.854\times10^{-12}\,\frac{\mathrm{C}^2}{\mathrm{N\cdot m}^2}\right)(400\,\mathrm{cm}^2)}{0.300\,\mathrm{mm}} = \boxed{2.72\,\mathrm{nF}}$$

62 •• The radius and the length of the central wire in a Geiger tube are 0.200 mm and 12.0 cm, respectively. The outer surface of the tube is a conducing cylindrical shell that has an inner radius of 1.50 cm. The shell is coaxial with the wire and has the same length (12.0 cm).The tube is filled with a gas that has a dielectric constant of 1.08 and a dielectric strength of 2.00×10^6 V/m. (*a*) What is the maximum potential difference that can be maintained between the wire and shell? (*b*) What is the maximum charge per unit length on the wire?

Picture the Problem The capacitance of a cylindrical capacitor is given by $C = 2\pi\kappa\epsilon_0 L/\ln(r_2/r_1)$, where L is its length and r_1 and r_2 the radii of the inner and outer conductors. We can use this expression, in conjunction with the definition of capacitance, to express the potential difference between the wire and the cylindrical shell in the Geiger tube. Because the electric field E in the tube is related to the linear charge density λ on the wire according to $E = 2k\lambda/\kappa r$, we can use this expression to find $2k\lambda/\kappa$ for $E = E_{max}$. In Part (*b*) we'll use this relationship to find the charge per unit length λ on the wire.

(*a*) Use the definition of capacitance and the expression for the capacitance of a cylindrical capacitor to express the potential difference between the wire and the cylindrical shell in the tube:

$$\Delta V = \frac{Q}{C} = \frac{Q}{\dfrac{2\pi\kappa\,\epsilon_0\,L}{\ln(R/r)}} = \frac{2\lambda}{4\pi\,\epsilon_0\,\kappa}\ln\left(\frac{R}{r}\right) = \frac{2k\lambda}{\kappa}\ln\left(\frac{R}{r}\right)$$

where λ is the linear charge density, κ is the dielectric constant of the gas in the Geiger tube, r is the radius of the wire, and R the radius of the coaxial cylindrical shell of length L.

Express the electric field at a distance r greater than its radius from the center of the wire:

$$E = \frac{2k\lambda}{\kappa r} \Rightarrow \frac{2k\lambda}{\kappa} = Er \qquad (1)$$

Substituting for $\dfrac{2k\lambda}{\kappa}$ yields:

$$\Delta V = Er\ln\left(\frac{R}{r}\right)$$

Noting that E is a maximum at r = 0.200 mm, substitute numerical values and evaluate ΔV_{max}:

$$\Delta V_{max} = \left(2.00\times10^6\ \text{V/m}\right)\left(0.200\,\text{mm}\right)\ln\left(\frac{1.50\,\text{cm}}{0.200\,\text{mm}}\right) = \boxed{1.73\,\text{kV}}$$

(*b*) Solve equation (1) for λ:

$$\lambda = \frac{E_{max}\kappa r}{2k}$$

Substitute numerical values and evaluate λ:

$$\lambda = \frac{(1.08)\left(2.00\times10^6\ \text{V/m}\right)\left(0.200\,\text{mm}\right)}{2\left(8.988\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2\right)} = \boxed{24.0\,\text{nC/m}}$$

63 •• You are a materials science engineer and your group has fabricated a new dielectric, that has an exceptionally large dielectric constant of 24 and a dielectric strength of 4.0×10^7 V/m. Suppose you want to use this material to construct a 0.10-μF parallel plate capacitor that can withstand a potential difference of 2.0 kV. (*a*) What is the minimum plate separation required to do this? (*b*) What is the area of each plate at this separation?

Picture the Problem We can use the relationship between the electric field between the plates of a capacitor, their separation, and the potential difference between them to find the minimum plate separation. We can use the expression

for the capacitance of a dielectric-filled parallel-plate capacitor to determine the necessary area of the plates.

(*a*) Relate the electric field of the capacitor to the potential difference across its plates:

$$E = \frac{V}{d} \Rightarrow d = \frac{V}{E}$$

where d is the plate separation.

Noting that $d_{\min}$ corresponds to $E_{\max}$, evaluate $d_{\min}$:

$$d_{\min} = \frac{V}{E_{\max}} = \frac{2000\,\text{V}}{4.0\times10^{7}\ \text{V/m}} = \boxed{50\,\mu\text{m}}$$

(*b*) Relate the capacitance of a parallel-plate capacitor to the area of its plates:

$$C = \frac{\kappa\,\epsilon_0\,A}{d} \Rightarrow A = \frac{Cd}{\kappa\,\epsilon_0}$$

Substitute numerical values and evaluate A:

$$A = \frac{(0.10\,\mu\text{F})(50\,\mu\text{m})}{24\left(8.854\times10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2\right)}$$
$$= 2.35\times10^{-2}\ \text{m}^2 = \boxed{240\,\text{cm}^2}$$

64 •• A parallel-plate capacitor has plates separated by a distance d. The capacitance of this capacitor is C_0 when no dielectric is in the space between the plates. However, the space between the plates is completely filled by two different dielectrics. One dielectric has a thickness $\frac{1}{4}d$ and a dielectric constant κ_1, and the other dielectric has a thickness $\frac{3}{4}d$ and a dielectric constant κ_2. Find the capacitance of this capacitor.

Picture the Problem We can model this system as two capacitors in series, one of thickness $\frac{1}{4}d$ and the other of thickness $\frac{3}{4}d$ and use the equation for the equivalent capacitance of two capacitors connected in series. Let the capacitance of the capacitor whose dielectric constant is κ_1 be C_1 and the capacitance of the capacitor whose dielectric constant is κ_2 be C_2.

Express the equivalent capacitance of the two capacitors connected in series:

$$\frac{1}{C_{\text{eq}}} = \frac{1}{C_1} + \frac{1}{C_2} \Rightarrow C_{\text{eq}} = \frac{C_1C_2}{C_1 + C_2}$$

Relate the capacitance of C_1 to its dielectric constant and thickness:

$$C_1 = \frac{\kappa_1\,\epsilon_0\,A}{\frac{1}{4}d} = \frac{4\kappa_1\,\epsilon_0\,A}{d}$$

Relate the capacitance of C_2 to its dielectric constant and thickness:

$$C_2 = \frac{\kappa_2\,\epsilon_0\,A}{\frac{3}{4}d} = \frac{4\kappa_2\,\epsilon_0\,A}{3d}$$

Substitute for C_1 and C_2 and simplify to obtain:

$$C_{eq} = \frac{\left(\frac{4\kappa_1 \epsilon_0 A}{d}\right)\left(\frac{4\kappa_2 \epsilon_0 A}{3d}\right)}{\frac{4\kappa_1 \epsilon_0 A}{d} + \frac{4\kappa_2 \epsilon_0 A}{3d}} = \frac{\left(\frac{\kappa_1}{d}\right)\left(\frac{4\kappa_2}{3d}\right)}{\frac{3\kappa_1}{3d} + \frac{\kappa_2}{3d}} \epsilon_0 A = \frac{\frac{4\kappa_1\kappa_2}{d}}{3\kappa_1 + \kappa_2} \epsilon_0 A$$

$$= \frac{4\kappa_1\kappa_2}{3\kappa_1 + \kappa_2}\left(\frac{\epsilon_0 A}{d}\right) = \boxed{\left(\frac{4\kappa_1\kappa_2}{3\kappa_1 + \kappa_2}\right)C_0}$$

65 •• Two capacitors each have two conducting plates of surface area A and an air gap of width d. They are connected in parallel, as shown in Figure 24-43 and each has a charge Q on the positively charged plate. A slab that has a width d, an area A, and a dielectric constant κ is inserted between the plates of *one* of the capacitors. Calculate the new charge Q' on the positively charged plate of that capacitor after electrostatic equilibrium is re-established.

Picture the Problem Let the charge on the capacitor with the air gap be Q_1 and the charge on the capacitor with the dielectric gap be Q_2. If the capacitances of the capacitors were initially C, then the capacitance of the capacitor with the dielectric inserted is $C' = \kappa C$. We can use the conservation of charge and the equivalence of the potential difference across the capacitors to obtain two equations that we can solve simultaneously for Q_1 and Q_2.

Apply conservation of charge during the insertion of the dielectric to obtain:

$$Q_1 + Q_2 = 2Q \qquad (1)$$

Because the capacitors have the same potential difference across them:

$$\frac{Q_1}{C} = \frac{Q_2}{\kappa C} \qquad (2)$$

Solve equations (1) and (2) simultaneously to obtain:

$$Q_1 = \boxed{\frac{2Q}{1+\kappa}} \text{ and } Q_2 = \boxed{\frac{2Q\kappa}{1+\kappa}}$$

66 •• A parallel-plate capacitor has a plate separation d has a capacitance equal to C_0 where there is only empty space in the space between the plates. A slab of thickness t, where $t < d$, that has a dielectric constant κ is placed in the space between the plates completely covering one of the plates. What is the capacitance with the slab inserted?

Picture the Problem We can model this system as two capacitors in series, C_1 of thickness t and C_2 of thickness $d - t$ and use the equation for the equivalent capacitance of two capacitors connected in series.

Express the equivalent capacitance of the two capacitors connected in series:

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} \text{ or } C_{eq} = \frac{C_1 C_2}{C_1 + C_2}$$

Relate the capacitance of C_1 to its dielectric constant and thickness:

$$C_1 = \frac{\kappa \epsilon_0 A}{t}$$

Relate the capacitance of C_2 to its dielectric constant and thickness:

$$C_2 = \frac{\epsilon_0 A}{d - t}$$

Substitute for C_1 and C_2 and simplify to obtain:

$$C_{eq} = \frac{\left(\frac{\kappa \epsilon_0 A}{t}\right)\left(\frac{\epsilon_0 A}{d-t}\right)}{\frac{\kappa \epsilon_0 A}{t} + \frac{\epsilon_0 A}{d-t}} = \frac{\left(\frac{\kappa}{t}\right)\left(\frac{1}{d-t}\right)}{\frac{\kappa}{t} + \frac{1}{d-t}} \epsilon_0 A = \frac{\left(\frac{\kappa}{t}\right)\left(\frac{1}{d-t}\right)}{\frac{\kappa}{t} + \frac{1}{d-t}} \epsilon_0 A$$

$$= \frac{\kappa}{\kappa(d-t)+t} \epsilon_0 A = \boxed{\left(\frac{\kappa d}{\kappa(d-t)+t}\right) C_0}$$

67 •• The membrane of the axon of a nerve cell can be modeled as a thin cylindrical shell of radius 1.00×10^{-5} m, having a length of 10.0 cm and a thickness of 10.0 nm. The membrane has a positive charge on one side and a negative charge on the other, and the membrane acts as a parallel-plate capacitor of area $2\pi rL$ and separation d. Assume the membrane is filled with a material whose dielectric constant is 3.00. (*a*) Find the capacitance of the membrane. If the potential difference across the membrane is 70.0 mV, find (*b*) the charge on the positively charged side of the membrane.

Picture the Problem Because $d << r$, we can model the membrane as a parallel-plate capacitor. We can use the definition of capacitance to find the charge on each side of the membrane in Part (*b*).

(*a*) Express the capacitance of a parallel-plate capacitor:

$$C = \frac{\kappa \epsilon_0 A}{d}$$

Substitute for the area of the plates:

$$C = \frac{2\pi\kappa \epsilon_0 rL}{d} = \frac{\kappa rL}{2kd}$$

Substitute numerical values and evaluate C:

$$C = \frac{(3.00)(1.00\times10^{-5}\,\text{m})(0.100\,\text{m})}{2(8.988\times10^{9}\,\text{N}\cdot\text{m}^2/\text{C}^2)(10.0\,\text{nm})}$$
$$= 16.69\,\text{nF} = \boxed{16.7\,\text{nF}}$$

(*b*) Use the definition of capacitance to find the charge on each side of the membrane:

$$Q = CV = (16.69\,\text{nF})(70.0\,\text{mV})$$
$$= \boxed{1.17\,\text{nC}}$$

68 •• The space between the plates of the capacitor that is connected across the terminals of a battery is filled with a dielectric material. Determine the dielectric constant of the material if the induced bound–charge-per-unit-area on it is (*a*) 80 percent of the free–charge-per-unit-area on the plates, (*b*) 20 percent of the free–charge-per-unit-area on the plates, and (*c*) 98 percent of the free–charge-per-unit-area on the plates.

Picture the Problem The bound charge density is related to the dielectric constant and the free charge density by the equation $\sigma_b = \left(1-\frac{1}{\kappa}\right)\sigma_f$.

Solve the equation relating σ_b, σ_f, and κ for κ to obtain:

$$\kappa = \frac{1}{1-\sigma_b/\sigma_f}$$

(*a*) Evaluate this expression for $\sigma_b/\sigma_f = 0.80$:

$$\kappa = \frac{1}{1-0.80} = \boxed{5.0}$$

(*b*) Evaluate this expression for $\sigma_b/\sigma_f = 0.20$:

$$\kappa = \frac{1}{1-0.20} = \boxed{1.3}$$

(*c*) Evaluate this expression for $\sigma_b/\sigma_f = 0.98$:

$$\kappa = \frac{1}{1-0.98} = \boxed{50}$$

69 •• The positively charge plate of a parallel-plate capacitor has a charge equal to Q. When the space between the plates is evacuated of air, the electric field strength between the plates is 2.5×10^5 V/m. When the space is filled with a certain dielectric material, the field strength between the plates is reduced to 1.2×10^5 V/m. (*a*) What is the dielectric constant of the material? (*b*) If $Q = 10$ nC, what is the area of the plates? (*c*) What is the total induced bound charge on either face of the dielectric material?

Picture the Problem We can use the definition of the dielectric constant to find its value. In Part (*b*) we can use the expression for the electric field in the space between the charged capacitor plates to find the area of the plates and in Part (*c*) we can relate the surface charge densities to the induced charges on the plates.

(*a*) Using the definition of the dielectric constant, relate the electric field without a dielectric E_0 to the field with a dielectric E:

$$E = \frac{E_0}{\kappa} \Rightarrow \kappa = \frac{E_0}{E}$$

Solve for and evaluate κ:

$$\kappa = \frac{2.5\times10^5\ \text{V/m}}{1.2\times10^5\ \text{V/m}} = 2.08 = \boxed{2.1}$$

(*b*) Relate the electric field in the region between the plates to the surface charge density of the plates:

$$E_0 = \frac{\sigma}{\epsilon_0} = \frac{Q/A}{\epsilon_0} \Rightarrow A = \frac{Q}{E_0\,\epsilon_0}$$

Substitute numerical values and evaluate A:

$$A = \frac{10\,\text{nC}}{\left(2.5\times10^5\ \text{V/m}\right)\left(8.854\times10^{-12}\ \dfrac{\text{C}^2}{\text{N}\cdot\text{m}^2}\right)}$$

$$= 4.52\times10^{-3}\ \text{m}^2 = \boxed{45\,\text{cm}^2}$$

(*c*) Relate the surface charge densities to the induced charges on the plates:

$$\sigma_\text{b} = \left(1-\frac{1}{\kappa}\right)\sigma_\text{f}$$

or

$$\frac{\sigma_\text{b}}{\sigma_\text{f}} = \frac{Q_\text{b}}{Q_\text{f}} = 1-\frac{1}{\kappa} \Rightarrow Q_\text{b} = \left(1-\frac{1}{\kappa}\right)Q_\text{f}$$

Substitute numerical values and evaluate Q_b:

$$Q_\text{b} = \left(1-\frac{1}{2.08}\right)(10\,\text{nC}) = \boxed{5.2\,\text{nC}}$$

70 •• Find the capacitance of the parallel-plate capacitor shown in Figure 24-44.

Picture the Problem We can model this parallel-plate capacitor as a combination of two capacitors C_1 and C_2 in series with capacitor C_3 in parallel.

Express the capacitance of two series-connected capacitors in parallel with a third:

$$C = C_3 + C_s \qquad (1)$$

where

$$C_s = \frac{C_1 C_2}{C_1 + C_2} \qquad (2)$$

Express each of the capacitances C_1, C_2, and C_3 in terms of the dielectric constants, plate areas, and plate separations:

$$C_1 = \frac{\kappa_1 \epsilon_0 \left(\frac{1}{2}A\right)}{\frac{1}{2}d} = \frac{\kappa_1 \epsilon_0 A}{d},$$

$$C_2 = \frac{\kappa_2 \epsilon_0 \left(\frac{1}{2}A\right)}{\frac{1}{2}d} = \frac{\kappa_2 \epsilon_0 A}{d},$$

and

$$C_3 = \frac{\kappa_3 \epsilon_0 \left(\frac{1}{2}A\right)}{d} = \frac{\kappa_3 \epsilon_0 A}{2d}$$

Substitute in equation (2) to obtain:

$$C_s = \frac{\left(\frac{\kappa_1 \epsilon_0 A}{d}\right)\left(\frac{\kappa_2 \epsilon_0 A}{d}\right)}{\frac{\kappa_1 \epsilon_0 A}{d} + \frac{\kappa_2 \epsilon_0 A}{d}} = \frac{\kappa_1 \kappa_2}{\kappa_1 + \kappa_2}\left(\frac{\epsilon_0 A}{d}\right)$$

Substitute in equation (1) to obtain:

$$C = \frac{\kappa_3 \epsilon_0 A}{2d} + \frac{\kappa_1 \kappa_2}{\kappa_1 + \kappa_2}\left(\frac{\epsilon_0 A}{d}\right) = \boxed{\left(\kappa_3 + \frac{2\kappa_1 \kappa_2}{\kappa_1 + \kappa_2}\right)\left(\frac{\epsilon_0 A}{2d}\right)}$$

General Problems

71 • You are given four identical capacitors and a 100-V battery. When only one of the capacitors is connected to this battery the energy stored is U_0. Combine the four capacitors in such a way that the total energy stored in all four capacitors is U_0? Describe the combination and explain your answer.

Picture the Problem We can use the expression $U_0 = \frac{1}{2}C_{eq}V^2$ to express the total energy stored in the combination of four capacitors in terms of their equivalent capacitance C_{eq}.

The energy stored in one capacitor when it is connected to the 100-V battery is:

$$U_0 = \tfrac{1}{2}CV^2$$

When the four capacitors are connected to the battery in some combination, the total energy stored in them is:

$$U = \tfrac{1}{2}C_{eq}V^2$$

Equate U and U_0 and solve for C_{eq}:

$$\tfrac{1}{2}C_{eq}V^2 = \tfrac{1}{2}CV^2 \Rightarrow C_{eq} = C$$

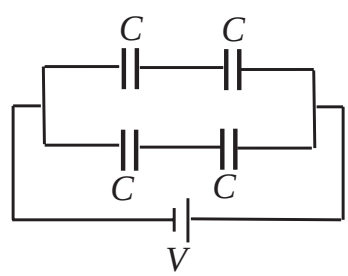
The equivalent capacitance C' of two capacitors of capacitance C connected in series is their product divided by their sum:

$$C' = \frac{C^2}{C+C} = \tfrac{1}{2}C$$

If we connect two of the capacitors in series in parallel with the other two capacitors connected in series, their equivalent capacitance will be:

$$C_{eq} = C' + C' = \tfrac{1}{2}C + \tfrac{1}{2}C = C$$

A series combination of two of the capacitors connected in parallel with a series combination of the other two capacitors will result in total energy U_0 stored in all four capacitors. The circuit diagram is shown to the right.



72 • Three capacitors have capacitances of 2.00 μF, 4.00 μF, and 8.00 μF. Find the equivalent capacitance if (*a*) the capacitors are connected in parallel and (*b*) if the capacitors are connected in series.

Picture the Problem We can use the equations for the equivalent capacitance of three capacitors connected in parallel and in series to find these equivalent capacitances.

(*a*) Express the equivalent capacitance of three capacitors connected in parallel:

$$C_{eq} = C_1 + C_2 + C_3$$

Substitute numerical values and evaluate C_{eq}:

$$C_{eq} = 2.00\,\mu\text{F} + 4.00\,\mu\text{F} + 8.00\,\mu\text{F} = \boxed{14.00\,\mu\text{F}}$$

(*b*) The equivalent capacitance of the three capacitors connected in series is given by:

$$C_{eq} = \frac{C_1C_2C_3}{C_1C_2 + C_2C_3 + C_1C_3}$$

Substitute numerical values and evaluate C_{eq}:

$$C_{eq} = \frac{(2.00\,\mu\text{F})(4.00\,\mu\text{F})(8.00\,\mu\text{F})}{(2.00\,\mu\text{F})(4.00\,\mu\text{F}) + (4.00\,\mu\text{F})(8.00\,\mu\text{F}) + (2.00\,\mu\text{F})(8.00\,\mu\text{F})} = \boxed{1.14\,\mu\text{F}}$$

73 • A 1.00-μF capacitor is connected in parallel with a 2.00-μF capacitor, and this combination is connected in series with a 6.00-μF capacitor. What is the equivalent capacitance of this combination?

Picture the Problem We can first use the equation for the equivalent capacitance of two capacitors connected in parallel and then the equation for two capacitors connected in series to find the equivalent capacitance.

Find the equivalent capacitance of a 1.00-μF capacitor connected in parallel with a 2.00-μF capacitor:

$$C_{eq,1} = C_1 + C_2 = 1.00\,\mu\text{F} + 2.00\,\mu\text{F} = 3.00\,\mu\text{F}$$

Find the equivalent capacitance of a 3.00-μF capacitor connected in series with a 6.00-μF capacitor:

$$C_{eq,2} = \frac{C_{eq,1}C_6}{C_{eq,1} + C_6} = \frac{(3.00\,\mu\text{F})(6.00\,\mu\text{F})}{3.00\,\mu\text{F} + 6.00\,\mu\text{F}} = \boxed{2.00\,\mu\text{F}}$$

74 • The voltage across a parallel-plate capacitor that has a plate separation equal to 0.500 mm is 1.20 kV. The capacitor is disconnected from the voltage source and the separation between the plates is increased until the energy stored in the capacitor has been doubled. Determine the final separation between the plates.

Picture the Problem The charge Q and the charge density σ are independent of the separation of the plates and do not change during the process described in the problem statement. Because the electric field E depends on σ, it too is constant. We can use $U = \frac{1}{2}CV^2$ and the relationship between V and E, together with the expression for the capacitance of a parallel-plate capacitor, to show that $U \propto d$.

Express the energy stored in the capacitor in terms of its capacitance C and the potential difference across its plates:

$$U = \tfrac{1}{2}CV^2 \qquad (1)$$

Express V in terms of E:

$$V = Ed$$

where d is the separation of the plates.

Express the capacitance of a parallel-plate capacitor:

$$C = \frac{\kappa \epsilon_0 A}{d}$$

Substitute for C and V in equation (1) to obtain:

$$U = \tfrac{1}{2}\frac{\kappa \epsilon_0 A}{d}(Ed)^2 = \left(\tfrac{1}{2}\kappa \epsilon_0 AE^2\right)d$$

Because $U \propto d$, to double U one must double d. Hence:

$$d_f = 2d = 2(0.500\,\text{mm}) = \boxed{1.00\,\text{mm}}$$

75 •• Determine the equivalent capacitance, in terms of C_0, of each of the combinations of capacitors shown in Figure 24-45.

Picture the Problem We can use the equations for the equivalent capacitance of capacitors connected in parallel and in series to find the single capacitor that will store the same amount of charge as each of the networks shown above.

(*a*) Find the capacitance of the two capacitors in parallel:

$$C_{\text{eq},1} = C_0 + C_0 = 2C_0$$

Find the capacitance equivalent to $2C_0$ in series with C_0:

$$C_{\text{eq},2} = \frac{C_{\text{eq},1}C_0}{C_{\text{eq},1} + C_0} = \frac{(2C_0)C_0}{2C_0 + C_0} = \boxed{\tfrac{2}{3}C_0}$$

(*b*) Find the capacitance of two capacitors of capacitance C_0 in parallel:

$$C_{\text{eq},1} = 2C_0$$

Find the capacitance equivalent to $2C_0$ in series with $2C_0$:

$$C_{\text{eq},2} = \frac{C_{\text{eq},1}C_0}{C_{\text{eq},1} + C_0} = \frac{(2C_0)(2C_0)}{2C_0 + 2C_0} = \boxed{C_0}$$

(*c*) Find the equivalent capacitance of three equal capacitors connected in parallel:

$$C_{\text{eq}} = C_0 + C_0 + C_0 = \boxed{3C_0}$$

76 •• Figure 24-46 shows four capacitors connected in the arrangement known as a capacitance bridge. The capacitors are initially uncharged. What must the relation between the four capacitances be so that the potential difference between points *c* and *d* remains zero when a voltage V is applied between points *a* and *b*?

Picture the Problem Note that with V applied between a and b, C_1 and C_3 are in series, and so are C_2 and C_4. Because in a series combination the potential differences across the two capacitors are inversely proportional to the capacitances, we can establish proportions involving the capacitances and potential differences for the left- and right-hand side of the network and then use the condition that $V_c = V_d$ to eliminate the potential differences and establish the relationship between the capacitances.

Letting Q represent the charge on capacitors 1 and 2, relate the potential differences across the capacitors to their common charge and capacitances:

$$V_1 = \frac{Q}{C_1} \text{ and } V_3 = \frac{Q}{C_3}$$

Divide the first of these equations by the second to obtain:

$$\frac{V_1}{V_3} = \frac{C_3}{C_1} \qquad (1)$$

Proceed similarly to obtain:

$$\frac{V_2}{V_4} = \frac{C_4}{C_2} \qquad (2)$$

Divide equation (1) by equation (2) to obtain:

$$\frac{V_1V_4}{V_3V_2} = \frac{C_3C_2}{C_1C_4} \qquad (3)$$

If $V_c = V_d$ then we must have:

$$V_1 = V_2 \text{ and } V_3 = V_4$$

Substitute in equation (3) and rearrange to obtain:

$$\boxed{C_2C_3 = C_1C_4}$$

77 •• The plates of a parallel-plate capacitor are separated by distance d, and each plate has area A. The capacitor is charged to a potential difference V and then disconnected from the voltage source. The plates are then pulled apart until the separation is $3d$. Find (*a*) the new capacitance, (*b*) the new potential difference, and (*c*) the new stored energy. (*d*) How much work was required to change the plate separation from d to $3d$?

Picture the Problem We can use the expression for the capacitance of a parallel-plate capacitor as a function of A and d to determine the effect on the capacitance of doubling the plate separation. We can use $V = Ed$ to determine the effect on the potential difference across the capacitor of doubling the plate separation. Finally, we can use $U = \frac{1}{2}CV^2$ to determine the effect of doubling the plate separation on the energy stored in the capacitor.

(*a*) The capacitance of a capacitor whose plates are separated by a distance 3*d* is given by:

$$C_{\text{new}} = \boxed{\frac{\epsilon_0 A}{3d}}$$

(*b*) Express the potential difference across a parallel-plate capacitor whose plates are separated by a distance *d*:

$V = Ed$
where the electric field *E* depends solely on the charge on the capacitor plates.

Express the new potential difference across the plates resulting from tripling their separation:

$$V_{\text{new}} = E(3d) = 3(Ed) = \boxed{3V}$$

(*c*) Relate the energy stored in a parallel-plate capacitor to the separation of the plates:

$$U = \frac{1}{2}CV^2 = \frac{1}{2}\frac{\epsilon_0 A}{d}V^2$$

When the plate separation is tripled we have:

$$U_{\text{new}} = \frac{1}{2}\frac{\epsilon_0 A}{3d}(3V)^2 = \boxed{\frac{3\epsilon_0 AV^2}{2d}}$$

(*d*) Relate the work required to change the plate separation from *d* to 3*d* to the change in the electrostatic potential energy of the system:

$$\begin{aligned} W &= U_{\text{new}} - U_{\text{i}} \\ &= \frac{3\epsilon_0 AV^2}{2d} - \frac{1}{2}\frac{\epsilon_0 AV^2}{d} \\ &= \boxed{\frac{\epsilon_0 AV^2}{d}} \end{aligned}$$

78 •• A parallel-plate capacitor has capacitance C_0 when there is no dielectric in the space between the plates. The space between the plates is then filled with a material that has a dielectric constant of κ. When a second capacitor of capacitance C' is connected in series with the first, the capacitance of the series combination is C_0. Find C' in terms of C_0.

Picture the Problem We can use the equation for the equivalent capacitance of two capacitors in series to relate C_0 to C' and the capacitance of the dielectric-filled parallel-plate capacitor and then solve the resulting equation for C'.

Express the equivalent capacitance of the system in terms of C' and C, where C is the dielectric-filled capacitor:

$$C_0 = \frac{C'C}{C'+C} \Rightarrow C' = \frac{C_0 C}{C - C_0}$$

Express the capacitance of the dielectric-filled capacitor:

$$C = \frac{\kappa \in_0 A}{d} = \kappa C_0$$

Substitute for C in the equation for C' and simplify to obtain:

$$C' = \frac{C_0(\kappa C_0)}{\kappa C_0 - C_0} = \boxed{\frac{\kappa}{\kappa - 1} C_0}$$

79 •• **[SSM]** A parallel combination of two identical 2.00-μF parallel-plate capacitors (no dielectric is in the space between the plates) is connected to a 100-V battery. The battery is then removed and the separation between the plates of one of the capacitors is doubled. Find the charge on the positively charged plate of each of the capacitors.

Picture the Problem When the battery is removed, after having initially charged both capacitors, and the separation of one of the capacitors is doubled, the charge is redistributed subject to the condition that the total charge remains constant; that is, $Q = Q_1 + Q_2$ where Q is the initial charge on both capacitors and Q_2 is the charge on the capacitor whose plate separation has been doubled. We can use the conservation of charge during the plate separation process and the fact that, because the capacitors are in parallel, they share a common potential difference.

Find the equivalent capacitance of the two 2.00-μF parallel-plate capacitors connected in parallel:

$$C_{eq} = 2.00\,\mu\text{F} + 2.00\,\mu\text{F} = 4.00\,\mu\text{F}$$

Use the definition of capacitance to find the charge on the equivalent capacitor:

$$Q = C_{eq}V = (4.00\,\mu\text{F})(100\,\text{V}) = 400\,\mu\text{C}$$

Relate this total charge to charges distributed on capacitors 1 and 2 when the battery is removed and the separation of the plates of capacitor 2 is doubled:

$$Q = Q_1 + Q_2 \qquad (1)$$

Because the capacitors are in parallel:

$$V_1 = V_2 \text{ and } \frac{Q_1}{C_1} = \frac{Q_2}{C_2'} = \frac{Q_2}{\frac{1}{2}C_2} = \frac{2Q_2}{C_2}$$

Solve for Q_1 to obtain:

$$Q_1 = 2\left(\frac{C_1}{C_2}\right)Q_2 \qquad (2)$$

Substitute equation (2) in equation (1) and solve for Q_2 to obtain:

$$Q_2 = \frac{Q}{2(C_1/C_2)+1}$$

Substitute numerical values and evaluate Q_2:

$$Q_2 = \frac{400\,\mu\text{C}}{2(2.00\,\mu\text{F}/2.00\,\mu\text{F})+1} = \boxed{133\,\mu\text{C}}$$

Substitute numerical values in equation (1) or equation (2) and evaluate Q_1:

$$Q_1 = \boxed{267\,\mu\text{C}}$$

80 •• An parallel-plate capacitor with no dielectric in the space between the plates has a capacitance C_0 and a plate separation d. Two dielectric slabs that have dielectric constants of κ_1 and κ_2 respectively are then inserted between the plates as shown in Figure 24-47. Each slab is has a thickness $\frac{1}{2}d$ and has area A, the same area as each capacitor plate. When the charge on the positively charged capacitor plate is Q, find (*a*) the electric field in each dielectric, and (*b*) the potential difference between the plates. (*c*) Show that the capacitance of the system after the slabs are inserted is given by $\left[2\kappa_1\kappa_2/(\kappa_1+\kappa_2)\right]C_o$. (*d*) Show that $\left[2\kappa_1\kappa_2/(\kappa_1+\kappa_2)\right]C_o$ is the equivalent capacitance of a series combination of two capacitors, each having plates of area A and a gap width equal to $d/2$. The space between the plates of one is filled with a material that has a dielectric constant equal to κ_1 and the space between the plates of the other is filled with a material that has a dielectric constant equal to κ_2.

Picture the Problem We can relate the electric field in the dielectric to the electric field between the capacitor's plates in the absence of a dielectric using $E = E_0/\kappa$. In Part (*b*) we can express the potential difference between the plates as the sum of the potential differences across the dielectrics and then express the potential differences in terms of the electric fields in the dielectrics. In Part (*c*) we can use our result from (*b*) and the definition of capacitance to express the capacitance of the dielectric-filled capacitor. In Part (*d*) we can confirm the result of Part (*c*) by using the addition formula for capacitors in series.

(*a*) Express the electric field E in a dielectric of constant κ in terms of the electric field E_0 in the absence of the dielectric:

$$E = \frac{E_0}{\kappa}$$

Express the electric field E_0 in the absence of the dielectrics:

$$E_0 = \frac{\sigma}{\epsilon_0} = \frac{Q}{\epsilon_0 A}$$

Substitute for E_0 to obtain:

$$E = \frac{Q}{\kappa \epsilon_0 A}$$

Use this relationship to express the electric fields in dielectrics whose constants are κ_1 and κ_2:

$$E_1 = \boxed{\frac{Q}{\kappa_1 \epsilon_0 A}} \text{ and } E_2 = \boxed{\frac{Q}{\kappa_2 \epsilon_0 A}}$$

(*b*) Express the potential difference between the plates as the sum of the potential differences across the dielectrics:

$$V = V_1 + V_2$$

Relate the potential differences to the electric fields and the thicknesses of the dielectrics:

$$V_1 = E_1 \frac{d}{2} = \frac{Qd}{2\kappa_1 \epsilon_0 A}$$

and

$$V_2 = E_2 \frac{d}{2} = \frac{Qd}{2\kappa_2 \epsilon_0 A}$$

Substitute for V_1 and V_2 and simplify to obtain:

$$V = \frac{Qd}{2\kappa_1 \epsilon_0 A} + \frac{Qd}{2\kappa_2 \epsilon_0 A} = \boxed{\frac{Qd}{2\epsilon_0 A}\left(\frac{1}{\kappa_1} + \frac{1}{\kappa_2}\right)}$$

(*c*) Use the definition of capacitance to obtain:

$$C = \frac{Q}{V} = \frac{Q}{\frac{Qd}{2\epsilon_0 A}\left(\frac{1}{\kappa_1} + \frac{1}{\kappa_2}\right)} = \frac{2\epsilon_0 A}{d\left(\frac{\kappa_1 + \kappa_2}{\kappa_1 \kappa_2}\right)} = \frac{2\epsilon_0 A}{d}\left(\frac{\kappa_1 \kappa_2}{\kappa_1 + \kappa_2}\right) = \boxed{2C_0\left(\frac{\kappa_1 \kappa_2}{\kappa_1 + \kappa_2}\right)}$$

where $C_0 = \epsilon_0 A/d$.

(*d*) Express the equivalent capacitance C of capacitors C_1 and C_2 in series:

$$C = \frac{C_1 C_2}{C_1 + C_2}$$

Express C_1:

$$C_1 = \frac{\kappa_1 \epsilon_0 A}{d/2} = \frac{2\kappa_1 \epsilon_0 A}{d} = 2\kappa_1 C_0$$

Express C_2:

$$C_2 = \frac{\kappa_2 \epsilon_0 A}{d/2} = \frac{2\kappa_2 \epsilon_0 A}{d} = 2\kappa_2 C_0$$

Substitute for C_1 and C_2 and simplify to obtain:

$$C = \frac{(2\kappa_1 C_0)(2\kappa_2 C_0)}{2\kappa_1 C_0 + 2\kappa_2 C_0} = \boxed{2C_0\left(\frac{\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right)},$$

a result in agreement with Part (*c*).

81 •• The plates of a parallel-plate capacitor are separated by distance d_0, and each plate has area *A*. A metal slab of thickness *d* and area *A* is inserted between the plates in such a way that the slab is parallel with the capacitor plates. (*a*) Show that the new capacitance is given by $\epsilon_0 A/(d_0 - d)$, regardless of the distance between the metal slab and the positively charged plate. (*b*) Show that this arrangement can be modeled as a capacitor that has plate separation *a* in series with a capacitor of separation *b*, where $a + b + d = d_0$.

Picture the Problem Recall that within a conductor $E = 0$. We can use the definition of capacitance to express *C* in terms of the charge on the capacitor *Q* and the potential difference across the plates *V*. We can then express *V* in terms of *E* and the thickness of the air gap between the plates. Finally, we can express the electric field between the plates in terms of the charge on them and their area. Substitution in our expression for *C* will give us *C* in terms of $d_0 - d$. In Part (*b*) we can use the expression for the equivalent capacitance of two capacitors connected in series to derive the same expression for *C*.

(*a*) Use its definition to express the capacitance of this parallel-plate capacitor:

$$C = \frac{Q}{V} \Rightarrow V = \frac{Q}{C}$$

where *Q* is the charge on the capacitor.

Relate the electric potential between the plates to the electric field between the plates:

$$V = E(d_0 - d)$$

Substituting for *V* yields:

$$\frac{Q}{C} = E(d_0 - d) \Rightarrow C = \frac{Q}{E(d_0 - d)}$$

Express the electric field *E* between the plates but outside the metal slab:

$$E = \frac{\sigma}{\epsilon_0} = \frac{Q}{\epsilon_0 A}$$

Substitute for E and simplify to obtain:

$$C = \frac{Q}{\frac{Q}{\epsilon_0 A}(d_0 - d)} = \boxed{\frac{\epsilon_0 A}{d_0 - d}}$$

(*b*) Express the equivalent capacitance C of two capacitors C_1 and C_2 connected in series:

$$C = \frac{C_1 C_2}{C_1 + C_2}$$

Express the capacitances C_1 and C_2 of the plates separated by a and b, respectively:

$$C_1 = \frac{\epsilon_0 A}{a} \text{ and } C_2 = \frac{\epsilon_0 A}{b}$$

Substitute for C_1 and C_2 and simplify to obtain:

$$C = \frac{\left(\frac{\epsilon_0 A}{a}\right)\left(\frac{\epsilon_0 A}{b}\right)}{\frac{\epsilon_0 A}{a} + \frac{\epsilon_0 A}{b}} = \frac{\epsilon_0 A}{a + b}$$

Solve the constraint that $a + b + d = d_0$ for $a + b$ to obtain:

$$a + b = d_0 - d$$

Substitute for $a + b$ to obtain:

$$C = \boxed{\frac{\epsilon_0 A}{d_0 - d}}$$

a result in agreement with Part (*a*).

82 •• A parallel-plate capacitor that has plate area A is filled with two dielectrics of equal size, as shown in Figure 24-48. (*a*) Show that this system can be modeled as two capacitors that are connected in parallel and each have an area $\frac{1}{2}A$. (*b*) Show that the capacitance is given by $\frac{1}{2}(\kappa_1 + \kappa_2)C_0$, where C_0 is the capacitance if there were no dielectric materials in the space between the plates.

Picture the Problem We can express the ratio of C_{eq} to C_0 to show that the capacitance with the dielectrics in place is $(\kappa_1 + \kappa_2)/2$ times greater than that of the capacitor in the absence of the dielectrics.

(*a*) Because the capacitor plates are conductors, the potentials are the same across the entire upper and lower plates. Hence, the system is equivalent to two capacitors, each of area $\frac{1}{2}A$, in parallel.

(*b*) Relate the capacitance C_0, in the absence of the dielectrics, to the plate area and separation:

$$C_0 = \frac{\epsilon_0 A}{d}$$

Express the equivalent capacitance of capacitors C_1 and C_2, each with plate area $\frac{1}{2}A$, connected in parallel:

$$C_{eq} = C_1 + C_2 = \frac{\kappa_1 \epsilon_0 \left(\frac{1}{2}A\right)}{d} + \frac{\kappa_2 \epsilon_0 \left(\frac{1}{2}A\right)}{d} = \frac{\epsilon_0 A}{2d}(\kappa_1 + \kappa_2)$$

Express the ratio of C_{eq} to C_0 and simplify to obtain:

$$\frac{C_{eq}}{C_0} = \frac{\frac{\epsilon_0 A}{2d}(\kappa_1 + \kappa_2)}{\frac{\epsilon_0 A}{d}} = \boxed{\tfrac{1}{2}(\kappa_1 + \kappa_2)}$$

83 •• A parallel-plate capacitor with no dielectric in the space between the plates has a plate area A and a gap width x. A charge Q is on the positively charged plate. (*a*) Find the stored electrostatic energy as a function of x. (*b*) Find the increase in energy dU due to an increase in plate separation dx from $dU = (dU/dx)\,dx$. (*c*) If F is the force exerted by one plate on the other, the work needed to move one plate a distance dx is $F\,dx = dU$. Show that $F = Q_2/(2\,\epsilon_0 A)$. (*d*) Show that the force in Part (*c*) equals $\frac{1}{2}EQ$, where Q is the charge on one plate and E is the electric field between the plates. Give a conceptual explanation for the factor $\frac{1}{2}$ in this result.

Picture the Problem We can use $U = Q^2/2C$ and the expression for the capacitance as a function of plate separation to express U as a function of x. Differentiation of this result with respect to x will yield dU. Because the work done in increasing the plate separation a distance dx equals the change in the electrostatic potential energy of the capacitor, we can evaluate F from dU/dx. Finally, we can express F in terms of Q and E by relating E to x using $E = V/x$ and using the definition of capacitance and the expression for the capacitance of a parallel-plate capacitor.

(*a*) Relate the electrostatic energy U stored in the capacitor to its capacitance C:

$$U = \frac{1}{2}\frac{Q^2}{C}$$

Express the capacitance as a function of the plate separation:

$$C = \frac{\epsilon_0 A}{x}$$

Substitute for C to obtain:

$$U = \boxed{\frac{Q^2}{2\,\epsilon_0 A}x}$$

(*b*) Use the result obtained in (*a*) to evaluate *dU*:

$$dU = \frac{dU}{dx}dx = \frac{d}{dx}\left[\frac{Q^2}{2\epsilon_0 A}x\right]dx = \boxed{\frac{Q^2}{2\epsilon_0 A}dx}$$

(*c*) Relate the work needed to move one plate a distance *dx* to the change in the electrostatic potential energy of the system:

$$W = dU = Fdx$$

Solve for and evaluate *F*:

$$F = \frac{dU}{dx} = \frac{d}{dx}\left[\frac{Q^2}{2\epsilon_0 A}x\right] = \boxed{\frac{Q^2}{2\epsilon_0 A}}$$

(*d*) Express the electric field between the plates in terms of their separation and their potential difference:

$$E = \frac{V}{x}$$

Use the definition of capacitance to eliminate *V*:

$$E = \frac{Q}{Cx}$$

Use the expression for the capacitance of a parallel-plate capacitor to eliminate *C*:

$$E = \frac{Q}{\frac{\epsilon_0 A}{x}x} = \frac{Q}{\epsilon_0 A}$$

Substitute in our result from Part (*c*) to obtain:

$$F = \frac{Q(\epsilon_0 AE)}{2\epsilon_0 A} = \boxed{\tfrac{1}{2}QE}$$

The field *E* is due to the sum of the fields from charges +*Q* and –*Q* on the opposite plates of the capacitor. Each plate produces a field $\frac{1}{2}E$. and the force is the product of charge *Q* and the field $\frac{1}{2}E$.

84 •• A rectangular parallel-plate capacitor that has a length *a* and a width *b* has a dielectric that has a width *b* partially inserted a distance *x* between the plates, as shown in Figure 24-49. (*a*) Find the capacitance as a function of *x*. Neglect edge effects. (*b*) Show that your answer gives the expected results for *x* = 0 and *x* = *a*.

Picture the Problem We can model this capacitor as the equivalent of two capacitors connected in parallel. Let the numeral 1 denote the capacitor with the dielectric material whose constant is κ and the numeral 2 the air-filled capacitor.

(*a*) Express the equivalent capacitance of the two capacitors in parallel:

$$C(x) = C_1 + C_2 \qquad (1)$$

Use the expression for the capacitance of a parallel-plate capacitor to express C_1:

$$C_1 = \frac{\kappa \epsilon_0 A_1}{d} = \frac{\kappa \epsilon_0 bx}{d}$$

Express the capacitance C_0 of the capacitor with the dielectric removed, i.e., $x = 0$:

$$C_0 = \frac{\epsilon_0 ab}{d}$$

Divide C_1 by C_0 and simplify to obtain:

$$\frac{C_1}{C_0} = \frac{\frac{\kappa \epsilon_0 bx}{d}}{\frac{\epsilon_0 ab}{d}} = \frac{\kappa x}{a} \Rightarrow C_1 = \frac{\kappa x}{a} C_0$$

Use the expression for the capacitance of a parallel-plate capacitor to express C_2:

$$C_2 = \frac{\epsilon_0 A_2}{d} = \frac{\epsilon_0 b(a-x)}{d}$$

Divide C_2 by C_0 to obtain:

$$\frac{C_2}{C_0} = \frac{\frac{\epsilon_0 b(a-x)}{d}}{\frac{\epsilon_0 ab}{d}} = \frac{a-x}{a}$$

or, solving for C_2,

$$C_2 = \frac{a-x}{a} C_0$$

Substitute for C_1 and C_2 in equation (1) and simplify to obtain:

$$\begin{aligned} C(x) &= \frac{\kappa x}{a} C_0 + \frac{a-x}{a} C_0 \\ &= \frac{C_0}{a}\left[a + (\kappa - 1)x\right] \\ &= \boxed{\frac{\epsilon_0 b}{d}\left[a + (\kappa - 1)x\right]} \end{aligned}$$

(*b*) Evaluate C for $x = 0$:

$$C(0) = \frac{\epsilon_0 b}{d}[a] = \frac{\epsilon_0 ab}{d} = \boxed{C_0}$$

as expected.

Evaluate C for $x = a$:

$$C(a) = \frac{\epsilon_0 b}{d}[a + (\kappa - 1)a]$$
$$= \boxed{\frac{\kappa \epsilon_0 ab}{d}} \textit{ as expected.}$$

85 ••• [SSM] An electrically isolated capacitor that has charge Q on its positively charged plate is partly filled with a dielectric substance as shown in Figure 24-51. The capacitor consists of two rectangular plates that have edge lengths a and b and are separated by distance d. The dielectric is inserted into the gap a distance x. (*a*) What is the energy stored in the capacitor? *Hint: the capacitor can be modeled as two capacitors connected in parallel*. (*b*) Because the energy of the capacitor decreases as x increases, the electric field must be doing work on the dielectric, meaning that there must be an electric force pulling it in. Calculate this force by examining how the stored energy varies with x. (*c*) Express the force in terms of the capacitance and potential difference V between the plates. (*d*) From where does this force originate?

Picture the Problem We can model this capacitor as the equivalent of two capacitors connected in parallel, one with an air gap and other filled with a dielectric of constant κ. Let the numeral 1 denote the capacitor with the dielectric material whose constant is κ and the numeral 2 the air-filled capacitor.

(*a*) Using the hint, express the energy stored in the capacitor as a function of the equivalent capacitance C_{eq}:

$$U = \frac{1}{2}\frac{Q^2}{C_{eq}}$$

The capacitances of the two capacitors are:

$$C_1 = \frac{\kappa \epsilon_0 ax}{d} \text{ and } C_2 = \frac{\epsilon_0 a(a-x)}{d}$$

Because the capacitors are in parallel, C_{eq} is the sum of C_1 and C_2:

$$C_{eq} = C_1 + C_2 = \frac{\kappa \epsilon_0 ax}{d} + \frac{\epsilon_0 a(a-x)}{d}$$
$$= \frac{\epsilon_0 a}{d}(\kappa x + a - x)$$
$$= \frac{\epsilon_0 a}{d}[(\kappa - 1)x + a]$$

Substitute for C_{eq} in the expression for U and simplify to obtain:

$$U = \boxed{\frac{Q^2 d}{2\epsilon_0 a[(\kappa - 1)x + a]}}$$

(*b*) The force exerted by the electric field is given by:

$$F = -\frac{dU}{dx}$$
$$= -\frac{d}{dx}\left[\frac{1}{2}\frac{Q^2 d}{\epsilon_0\, a[(\kappa-1)x+a]}\right]$$
$$= -\frac{Q^2 d}{2\,\epsilon_0\, a}\frac{d}{dx}\left\{[(\kappa-1)x+a]^{-1}\right\}$$
$$= \boxed{\frac{(\kappa-1)Q^2 d}{2a\,\epsilon_0\,[(\kappa-1)x+a]^2}}$$

(*c*) Rewrite the result in (*b*) to obtain:

$$F = \frac{(\kappa-1)Q^2\left(\frac{a\,\epsilon_0}{d}\right)}{2\left(\frac{a\,\epsilon_0}{d}\right)^2[(\kappa-1)x+a]^2}$$
$$= \frac{(\kappa-1)Q^2\left(\frac{a\,\epsilon_0}{d}\right)}{2C_{eq}^2}$$
$$= \boxed{\frac{(\kappa-1)a\,\epsilon_0\,V^2}{2d}}$$

Note that this expression is independent of *x*.

(*d*) The force originates from the fringing fields around the edges of the capacitor. The effect of the force is to pull the polarized dielectric into the space between the capacitor plates.

86 ••• A spherical capacitor consists of an solid conducting sphere that has a radius *a* and a charge +*Q* and an concentric conducting spherical shell that has an inner radius *b* and a charge of *Q*. The space between the two is filled with two different dielectric materials of dielectric constants κ_1 and κ_2. The boundary between the two dielectrics occurs a distance $\frac{1}{2}(a+b)$ from the center.

(*a*) Calculate the electric field in the regions $a < r < \frac{1}{2}(a+b)$ and $\frac{1}{2}(a+b) < r < b$. (*b*) Integrate the expression $\boldsymbol{dV} = -\vec{\boldsymbol{E}}\cdot\vec{\boldsymbol{d}}\ell$ to obtain the potential difference, *V*, between the two conductors. (*c*) Use $C = Q/V$ to obtain an expression for the capacitance of this system. (*d*) Show that your answer from Part (*c*) simplifies to the expected one if the κ_1 equals κ_2.

Picture the Problem (*a*) Gauss's law tells us that, in the absence of dielectric materials, the electric field between the conductors is $E_0 = kQ/r^2$. In the dielectric materials, the field is reduced by the appropriate dielectric constant.

(*a*) The electric fields in the regions

$$a < r < (a+b)/2$$

and

$$(a+b)/2 < r < b$$

are given by:

$$E_1 = \boxed{\frac{kQ}{\kappa_1 r^2}} \text{ and } E_2 = \boxed{\frac{kQ}{\kappa_2 r^2}}$$

(*b*) Because the electric field is directed radially away from the center, the potential of the outer spherical shell is less than the potential of the inner sphere. Hence the difference in potential between the spheres is given by:

$$\begin{aligned}|V| &= |V_1 - V_2| \\ &= \left| -\int_a^{\frac{1}{2}(a+b)} E_1 dr - \left(-\int_{\frac{1}{2}(a+b)}^{b} E_2 dr \right) \right| \\ &= \left| -\frac{kQ}{\kappa_1} \int_a^{\frac{1}{2}(a+b)} r^{-2} dr + \frac{kQ}{\kappa_2} \int_{\frac{1}{2}(a+b)}^{b} r^{-2} dr \right|\end{aligned}$$

Evaluating the integrals and simplifying yields:

$$\begin{aligned}|V| &= \left| \frac{kQ}{\kappa_1} \frac{1}{r} \Big]_a^{\frac{1}{2}(a+b)} + \frac{kQ}{\kappa_2} \frac{1}{r} \Big]_{\frac{1}{2}(a+b)}^{b} \right| = \left| \frac{kQ}{\kappa_1} \left(\frac{2}{a+b} - \frac{1}{a} \right) + \frac{kQ}{\kappa_2} \left(\frac{1}{b} - \frac{2}{a+b} \right) \right| \\ &= \left| \frac{kQ(a-b)}{a+b} \frac{\kappa_1 a + \kappa_2 b}{\kappa_1 \kappa_2 ab} \right| = \boxed{\frac{kQ(b-a)}{a+b} \frac{\kappa_1 a + \kappa_2 b}{\kappa_1 \kappa_2 ab}}\end{aligned}$$

(*c*) Use the definition of capacitance and simplify to obtain:

$$\begin{aligned}C &= \frac{Q}{|V|} = \frac{Q}{\frac{kQ(a-b)}{a+b} \frac{\kappa_1 a + \kappa_2 b}{\kappa_1 \kappa_2 ab}} \\ &= \boxed{\frac{\kappa_1 \kappa_2 ab(a+b)}{k(b-a)(\kappa_1 a + \kappa_2 b)}}\end{aligned}$$

(*d*) Letting $\kappa_1 = \kappa_2 = \kappa$, the result in Part (*c*) reduces to the expression for a spherical capacitor filled with just one dielectric material. In particular, if the capacitor is air filled ($\kappa = 1.00$), the expression for C is that for an air-filled spherical capacitor.

$$\begin{aligned}C &= \frac{\kappa\kappa ab(a+b)}{k(b-a)(\kappa a + \kappa b)} \\ &= \frac{\kappa ab}{k(b-a)} \\ &= \boxed{\frac{4\pi \epsilon_0 \kappa ab}{b-a}}\end{aligned}$$

87 ••• A capacitance balance is shown in Figure 24-50. The balance has a weight attached to one side and a capacitor that has a variable gap width on the other side. Assume the upper plate of the capacitor has negligible mass. When the capacitor potential difference between the plates is V_0, the attractive force between

the plates balances the weight of the hanging mass. (*a*) Is the balance stable? That is, if we balance it out, and then move the plates a little closer together, will they snap shut or move back to the equilibrium point? (*b*) Calculate the value of V_0 required to balance an object of mass *M*, assuming the plates are separated by distance d_0 and have area *A*. *HINT: A useful relation is that the force between the plates is equal to the derivative of the stored electrostatic energy with respect to the plate separation.*

Picture the Problem To avoid have to write *dd* (as in $F = -dE/dd$) in relating the force on the electrostatic balance plates to the electric field in the region between them, let ℓ be the variable separation of the plates. We can use the definition of the work done in charging the capacitor to relate the force on the upper plate to the energy stored in the capacitor. Solving this expression for the force and substituting for the energy stored in a parallel-plate capacitor will yield an expression that we can use to decide whether the balance is stable. We can use this same expression and a condition for equilibrium to find the voltage required to balance the object whose mass is *M*.

(*a*) Express the work done in charging the capacitor (the energy stored in it) in terms of the force between the plates:

$$dW = dE = -Fd\ell \Rightarrow F = -\frac{dE}{d\ell}$$

The energy stored in the capacitor is given by:

$$E = \frac{1}{2}CV_0^2 = \frac{1}{2}\left(\frac{\epsilon_0 A}{\ell}\right)V_0^2$$

Differentiate *E* with respect to ℓ to obtain:

$$F = -\frac{d}{d\ell}\left[\frac{1}{2}\left(\frac{\epsilon_0 A}{\ell}\right)V_0^2\right] = \left(\frac{\epsilon_0 A}{2\ell^2}\right)V_0^2$$

or, changing back to the variable *d*,

$$F = \left(\frac{\epsilon_0 A}{2d^2}\right)V_0^2$$

Because *F* increases as ℓ decreases, a decrease in plate separation will unbalance the system. Hence, the balance is unstable.

(*b*) Apply $\sum F = 0$ to the object whose mass is *M* when the plate separation is d_0 to obtain:

$$Mg - \left(\frac{\epsilon_0 A}{2d_0^2}\right)V^2 = 0 \Rightarrow V = \boxed{d_0\sqrt{\frac{2Mg}{\epsilon_0 A}}}$$

88 ••• You are an intern at an engineering company that makes capacitors used for energy storage in pulsed lasers. Your manger asks your team to construct a parallel-plate, air-gap capacitor that will store 100 kJ of energy. (*a*) What

minimum volume is required between the plates of the capacitor? (*b*) Suppose you have developed a dielectric that has a dielectric strength of 3.00×10^8 V/m and a dielectric constant of 5.00. What volume of this dielectric, between the plates of the capacitor, is required for it to be able to store 100 kJ of energy?

Picture the Problem Recall that the dielectric strength of air is 3.00 MV/m. We can express the maximum energy to be stored in terms of the capacitance of the air-gap capacitor and the maximum potential difference between its plates. This maximum potential can, in turn, be expressed in terms of the maximum electric field (dielectric strength) possible in the air gap. We can solve the resulting equation for the volume of the space between the plates. In Part (*b*) we can modify the equation we derive in Part (*a*) to accommodate a dielectric with a constant other than 1.

(*a*) Express the energy stored in the capacitor in terms of its capacitance and the potential difference across it:

$$U = \tfrac{1}{2}CV^2$$

Express the capacitance of the air-gap parallel-plate capacitor:

$$C = \frac{\epsilon_0 A}{d}$$

Relate the potential difference across the plates to the electric field between them:

$$V = Ed$$

Substitute for C and V in the expression for U to obtain:

$$U = \tfrac{1}{2}\left(\frac{\epsilon_0 A}{d}\right)(Ed)^2$$
$$= \tfrac{1}{2}\epsilon_0 (Ad)E^2 = \tfrac{1}{2}\epsilon_0 \upsilon E^2$$

where $\upsilon = Ad$ is the volume between the plates.

Solving for $\upsilon = \upsilon_{max}$ yields:

$$\upsilon_{max} = \frac{2U_{max}}{\epsilon_0 E_{max}^2} \qquad (1)$$

Substitute numerical values and evaluate υ_{max}:

$$\upsilon_{max} = \frac{2(100\,\text{kJ})}{(8.85\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2)(3\,\text{MV/m})^2}$$
$$= \boxed{2.51\times10^3\,\text{m}^3}$$

(*b*) With the dielectric in place equation (1) becomes:

$$\upsilon_{\max} = \frac{2U_{\max}}{\kappa \in_0 E_{\max}^2} \qquad (2)$$

Evaluate equation (2) with $\kappa = 5.00$ and $E_{\max} = 3.00 \times 10^8$ V/m:

$$\upsilon_{\max} = \frac{2(100\,\text{kJ})}{(5.00)\left(8.854\times10^{-12}\,\frac{\text{C}^2}{\text{N}\cdot\text{m}^2}\right)\left(3.00\times10^8\,\frac{\text{V}}{\text{m}}\right)^2} = \boxed{5.02\times10^{-2}\ \text{m}^3}$$

89 ••• Consider two parallel-plate capacitors, C_1 and C_2, that are connected in parallel. The capacitors are identical except that C_2 has a dielectric inserted between its plates. A 200 V battery is connected across the combination until electrostatic equilibrium is established, and then the battery is disconnected. (*a*) What is the charge on each capacitor? (*b*) What is the total stored energy of the capacitors? (*c*) The dielectric is removed from C_2. What is the final stored energy of the capacitors? (*d*) What is the final voltage across the two capacitors?

Picture the Problem We can use the definition of capacitance to find the charge on each capacitor in Part (*a*). In Part (*b*) we can express the total energy stored as the sum of the energy stored on the two capacitors by using our result from (*a*) for the charge on each capacitor. When the dielectric is removed in Part (*c*) each capacitor will carry half the charge carried by the capacitor system previously and we can proceed as in (*b*). Knowing the total charge stored by the capacitors, we can use the definition of capacitance to find the final voltage across the two capacitors in Part (*d*).

(*a*) Use the definition of capacitance to express the charge on each capacitor as a function of its capacitance:

$$Q_1 = C_1V = \boxed{(200\,\text{V})C_1}$$

and

$$Q_2 = C_2V = \kappa C_1V = \boxed{(200\,\text{V})\kappa C_1}$$

(*b*) Express the total stored energy of the capacitors as the sum of stored energy in each capacitor:

$$\begin{aligned} U &= U_1 + U_2 = \tfrac{1}{2}C_1V^2 + \tfrac{1}{2}C_2V^2 \\ &= \tfrac{1}{2}C_1V^2 + \tfrac{1}{2}\kappa C_1V^2 \\ &= \tfrac{1}{2}C_1V^2(1+\kappa) \\ &= \tfrac{1}{2}(200\,\text{V})^2C_1(1+\kappa) \\ &= \boxed{(2.00\times10^4\ \text{V}^2)(1+\kappa)C_1} \end{aligned}$$

(*c*) With the dielectric removed, each capacitor carries charge $Q/2$. Express the final energy stored by the capacitors under this condition:

$$U_f = \frac{1}{2}\frac{Q_1^2}{C_1} + \frac{1}{2}\frac{Q_2^2}{C_2} = \frac{1}{2}\frac{Q^2}{4C_1} + \frac{1}{2}\frac{Q^2}{4C_1} = \frac{Q^2}{4C_1}$$

Using the definition of capacitance, express the total charge carried by the capacitors with the dielectric in place in C_2:

$$Q = Q_1 + Q_2 = C_1V + C_2V = C_1V + \kappa C_1V = C_1V(1+\kappa) = (200\text{ V})C_1(1+\kappa)$$

Substitute for Q in the expression for U_f to obtain:

$$U_f = \frac{[(200\text{ V})C_1(1+\kappa)]^2}{4C_1} = \boxed{(1.00\times10^4\text{ V}^2)C_1(1+\kappa)^2}$$

(*d*) Use the definition of capacitance to express the final voltage across the capacitors:

$$V_f = \frac{Q}{C_{eq}} = \frac{(200\text{ V})C_1(1+\kappa)}{2C_1} = \boxed{100(1+\kappa)\text{V}}$$

90 ••• A capacitor is constructed of two coaxial conducting thin cylindrical shells of radii a and b ($b > a$), which have a length $L >> b$. A charge of $+Q$ is on the inner cylinder, and a charge of $-Q$ is on the outer cylinder. The region between the two cylinders is filled with a material that has a dielectric constant κ. (*a*) Find the potential difference between the cylinders. (*b*) Find the density of the free charge σ_f on the inner cylinder and the outer cylinder. (*c*) Find the bound charge density σ_b on the inner cylindrical surface of the dielectric and on the outer cylindrical surface of the dielectric. (*d*) Find the total stored energy. (*e*) If the dielectric will move without friction, how much mechanical work is required to remove the dielectric cylindrical shell?

Picture the Problem We can use the definition of capacitance and the expression for the capacitance of a cylindrical capacitor to find the potential difference between the cylinders. In Part (*b*) we can apply the definition of surface charge density to find the density of the free charge σ_f on the inner and outer cylindrical surfaces. We can use the fact that that Q and Q_b are proportional to E and E_b to express Q_b at a and b and then apply the definition of surface charge density to express $\sigma_b(a)$ and $\sigma_b(b)$. In Part (*d*) we can use $U = \frac{1}{2}QV$ to find the total stored electrostatic energy and in (*e*) find the mechanical work required from the change in energy of the system resulting from the removal of the dielectric cylindrical shell.

(*a*) Using the definition of capacitance, relate the potential difference between the cylinders to their charge and capacitance:

$$V = \frac{Q}{C}$$

Express the capacitance of a cylindrical capacitor as a function of its radii *a* and *b* and length *L*:

$$C = \frac{2\pi \in_0 \kappa L}{\ln(b/a)}$$

Substituting for *C* and simplifying yields:

$$V = \frac{Q\ln(b/a)}{2\pi \in_0 \kappa L} = \boxed{\frac{2kQ\ln(b/a)}{\kappa L}}$$

(*b*) Apply the definition of surface charge density to obtain:

$$\sigma_f(a) = \boxed{\frac{Q}{2\pi aL}}$$

and

$$\sigma_f(b) = \boxed{\frac{-Q}{2\pi bL}}$$

(*c*) Noting that *Q* and Q_b are proportional to *E* and E_b, express Q_b at *a* and *b*:

$$Q_b(a) = \frac{-Q(\kappa - 1)}{\kappa}$$

and

$$Q_b(b) = \frac{Q(\kappa - 1)}{\kappa}$$

Apply the definition of surface charge density to express $\sigma_b(a)$ and $\sigma_b(b)$:

$$\sigma_b(a) = \frac{Q_b(a)}{A} = \frac{\frac{-Q(\kappa-1)}{\kappa}}{2\pi aL} = \boxed{\frac{-Q(\kappa-1)}{2\pi aL\kappa}}$$

and

$$\sigma_b(b) = \frac{Q_b(b)}{A} = \frac{\frac{Q(\kappa-1)}{\kappa}}{2\pi bL} = \boxed{\frac{Q(\kappa-1)}{2\pi bL\kappa}}$$

(*d*) Express the total stored energy in terms of the charge stored and the potential difference between the cylinders:

$$U = \tfrac{1}{2}QV = \tfrac{1}{2}Q\left[\frac{2kQ\ln(b/a)}{\kappa L}\right] = \boxed{\frac{kQ^2\ln(b/a)}{\kappa L}}$$

(*e*) Express the work required to remove the dielectric cylindrical shell in terms of the change in the potential energy of the system:

$$W = \Delta U = U' - U$$

where $U' = \kappa U$ is the potential energy of the system with the dielectric shell in place.

Substitute for U and U' and simplify to obtain:

$$W = \kappa U - U = U(\kappa - 1) = \boxed{\frac{kQ^2(\kappa-1)\ln(b/a)}{\kappa L}}$$

91 ••• Before Switch S is closed, as shown in Figure 24-51 the voltage across the terminals of the switch is 120 V and the voltage across the capacitor labeled C_1 is 40.0 V. The capacitance of C_1 is 0.200 μF. The total energy stored in the two capacitors is 1.44 mJ. After closing the switch, the voltage across each capacitor is 80.0 V, and the energy stored by the two capacitors has dropped to 960 μJ. Determine the capacitance of C_2 and the charge on that capacitor before the switch was closed.

Picture the Problem Note that, with switch S closed, C_1 and C_2 are in parallel and we can use $U_{\text{closed}} = \frac{1}{2}C_{\text{eq}}V^2$ and $C_{\text{eq}} = C_1 + C_2$ to obtain an equation we can solve for C_2. We can use the definition of capacitance to express Q_2 in terms of V_2 and C_2 and $U_{\text{open}} = \frac{1}{2}C_1V_1^2 + \frac{1}{2}C_2V_2^2$ to obtain an equation from which we can determine V_2.

Express the energy stored in the capacitors after the switch is closed:

$$U_{\text{closed}} = \tfrac{1}{2}C_{\text{eq}}V^2$$

Express the equivalent capacitance of C_1 and C_2 in parallel:

$$C_{\text{eq}} = C_1 + C_2$$

Substitute for C_{eq} to obtain:

$$U_{\text{closed}} = \tfrac{1}{2}(C_1 + C_2)V^2$$

Solving for C_2 yields:

$$C_2 = \frac{2U_{\text{closed}}}{V^2} - C_1$$

Substitute numerical values and evaluate C_2:

$$C_2 = \frac{2(960\ \mu\text{J})}{(80\ \text{V})^2} - 0.200\ \mu\text{F} = \boxed{0.100\ \mu\text{F}}$$

Express the charge on C_2 when the switch is open:

$$Q_2 = C_2V_2 \qquad (1)$$

Express the energy stored in the capacitors with the switch open:

$$U_{\text{open}} = \tfrac{1}{2}C_1V_1^2 + \tfrac{1}{2}C_2V_2^2$$

Solving for V_2 yields:

$$V_2 = \sqrt{\frac{2U_{\text{open}} - C_1V_1^2}{C_2}}$$

Substitute for V_2 in equation (1) to obtain:

$$Q_2 = C_2\sqrt{\frac{2U_{\text{open}} - C_1V_1^2}{C_2}} = \sqrt{C_2\left(2U_{\text{open}} - C_1V_1^2\right)}$$

Substitute numerical values and evaluate Q_2:

$$Q_2 = \sqrt{(0.100\,\mu\text{F})\left[2(1.44\,\text{mJ}) - (0.200\,\mu\text{F})(40.0\,\text{V})^2\right]} = \boxed{16.0\,\mu\text{C}}$$

92 ••• An air-filled parallel-plate capacitor that has gap-width d has plates which each have an area A. The capacitor is charged to a potential difference V and is then removed from the voltage source. A dielectric slab that has a dielectric constant of 2.00, a thickness d, and an area $\frac{1}{2}A$ is then inserted, as shown in Figure24-52. Let σ_1 be the free charge density at the conductor–dielectric surface, and let σ_2 be the free charge density at the conductor–air surface. (*a*) Explain why the electric field must have the same value inside the dielectric as in the free space between the plates. (*b*) Show that $\sigma_1 = 2\sigma_2$. (*c*) Show that the final capacitance (after the slab is inserted) is 1.50 times the capacitance when the capacitor is filled with air. (*d*) Show that the final potential difference is $\frac{2}{3}V$. (*e*) Show that energy stored after the slab is inserted is only two-thirds of the energy stored before insertion.

Picture the Problem (*b*) We can express the electric fields in the dielectric and in the free space in terms of the charge densities and then use the fact that the electric field has the same value inside the dielectric as in the free space between the plates to establish that $\sigma_1 = 2\sigma_2$. In Parts (*c*) and (*d*) we can model the system as two capacitors in parallel to show that the equivalent capacitance is $3\epsilon_0A/(2d)$ and then use the definition of capacitance to show that the new potential difference is $\frac{2}{3}V$.

(*a*) The potential difference between the plates is the same for both halves (the plates are equipotential surfaces). Therefore, $E = V/d$ must be the same everywhere between the plates.

(*b*) Relate the electric field in each region to σ and κ:

$$E = \frac{\sigma}{\kappa \epsilon_0} \Rightarrow \sigma = \kappa \epsilon_0 E$$

Express σ_1 and σ_2:

$$\sigma_1 = \kappa_1 \epsilon_0 E_1 = 2 \epsilon_0 E_1$$

and

$$\sigma_2 = \kappa_2 \epsilon_0 E_2 = \epsilon_0 E_1$$

Divide the 1st of these equations by the 2nd and simplify to obtain:

$$\boxed{\sigma_1 = 2\sigma_2}$$

(*c*) Model the partially dielectric-filled capacitor as two capacitors in parallel to obtain:

$$C_{eq} = C_1 + C_2$$

where

$$C_1 = \frac{\kappa \epsilon_0 \left(\frac{1}{2} A\right)}{d} = \frac{\kappa \epsilon_0 A}{2d}$$

and

$$C_2 = \frac{\epsilon_0 \left(\frac{1}{2} A\right)}{d} = \frac{\epsilon_0 A}{2d}$$

Substitute for C_1 and C_2 and simplify to obtain:

$$C_{eq} = \frac{\kappa \epsilon_0 A}{2d} + \frac{\epsilon_0 A}{2d} = \frac{2 \epsilon_0 A}{2d} + \frac{\epsilon_0 A}{2d} = \frac{3 \epsilon_0 A}{2d} = \boxed{1.50 C_{\text{air-filled}}}$$

(*d*) Use the definition of capacitance to relate V_f, Q_f, and C_f:

$$V_f = \frac{Q_f}{C_f}$$

Because the capacitors are in parallel:

$$Q_f = Q_i = VC_i = \frac{V \epsilon_0 A}{d}$$

Substitute for Q_f and C_f and simplify to obtain:

$$V_f = \frac{V \epsilon_0 A}{C_f d} = \frac{V \epsilon_0 A}{\left(\frac{3 \epsilon_0 A}{2d}\right) d} = \boxed{\tfrac{2}{3} V}$$

(*e*) The energy stored after the slab is inserted is given by:

$$U_f = \tfrac{1}{2} C_f V_f^2$$

Substituting for C_f and V_f and simplifying yields:

$$U_f = \tfrac{1}{2}\left(\tfrac{3}{2} C_i\right)\left(\tfrac{2}{3} V\right)^2 = \tfrac{1}{3} C_i V^2 = \boxed{\tfrac{2}{3} U_i}$$

The presence of the dielectric slab reduces the potential difference between the capacitor plates and, hence, the energy stored in the capacitor.

93 ••• A capacitor has rectangular plates of length a and width b. The top plate is inclined at a small angle, as shown in Figure 24-53. The plate separation varies from y_0 at the left to $2y_0$ at the right, where y_0 is much less than a or b. Calculate the capacitance of this arrangement. *HINT*: *Break the problem up into a parallel combination. Choose strips of width dx and length b to approximate small (differential) capacitors (each having a value of dC). Each will have a plate area of bdx and separation distance* $y_0 + (y_0/a)x$. *Then argue that these differential capacitors are connected in parallel.*

Picture the Problem Choose a coordinate system in which the $+x$ direction is the right and the origin is at the left edge of the capacitor. We can express an element of capacitance dC and then integrate this expression to find C for this capacitor.

Express an element of capacitance dC of length b, width dx and separation $d = y_0 + (y_0/a)x$:

$$dC = \frac{\epsilon_0 b}{d}dx = \frac{\epsilon_0 b}{y_0(1+x/a)}dx$$

These elements are all in parallel, so the total capacitance is obtained by integration:

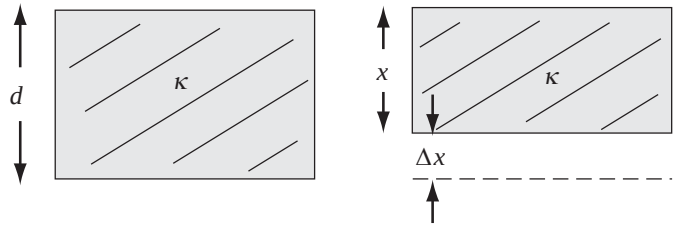
$$C = \frac{\epsilon_0 b}{y_0}\int_0^{y_0}\frac{1}{1+x/a}dx = \boxed{\frac{\epsilon_0 ab}{y_0}\ln(2)}$$

94 ••• Not all dielectrics that separate the plates of a capacitor are rigid. For example, the membrane of a nerve axon is a bi-lipid layer that has a finite compressibility. Consider a parallel-plate capacitor whose plate separation is maintained by a material that has a dielectric constant of 3.00, a dielectric strength of 40.0 kV/mm and a Young's modulus[1] for compressive stress of 5.00×10^6 N/m^2. When the potential difference between the capacitor plates is zero, the thickness of the dielectric is equal to 0.200 mm and the capacitance of the capacitor is given by C_0. (*a*) Derive an expression for the capacitance, as a function of the potential difference between the capacitor plates. (*b*) What is the maximum value of this potential difference? (Assume that the dielectric constant and the dielectric strength do *not* change under compression.)

Picture the Problem The diagram below and to the left shows the dielectric-filled parallel-plate capacitor before compression and the diagram to the right shows the capacitor when the plate separation has been reduced to x. We can use the definition of capacitance and the expression for the capacitance of a parallel-plate capacitor to derive an expression for the capacitance as a function of voltage across the capacitor. We can find the maximum voltage that can be applied from the dielectric strength of the dielectric and the separation of the plates. In Part (*c*) we can find the fraction of the total energy that is electrostatic field energy and

[1] Young's modulus is discussed in Section 12-8.

the fraction that is mechanical stress energy by expressing either of these as a fraction of their sum.



(*a*) Use its definition to express the capacitance as a function of the voltage across the capacitor:

$$C(V) = \frac{Q}{V} \qquad (1)$$

The limiting value of the capacitance is:

$$C_0 = \frac{\kappa \epsilon_0 A}{d}$$

Substitute numerical values and evaluate C_0:

$$C_0 = \frac{3\left(8.854\times10^{-12}\,\frac{\mathrm{C}^2}{\mathrm{N\cdot m}^2}\right)A}{0.200\,\mathrm{mm}} = 0.133\times10^{-6}\,A\,\frac{\mathrm{C}^2}{\mathrm{N\cdot m}^3}$$

Let x be the variable separation. Because κ is independent of x:

$$C(x) = \frac{\kappa \epsilon_0 A}{x}$$

and

$$Q(x) = C(x)V = \frac{\kappa \epsilon_0 A}{x}V$$

Substitute in equation (1) to obtain:

$$C(V) = \frac{\frac{\kappa \epsilon_0 A}{x}V}{V} = \frac{\kappa \epsilon_0 A}{x} = \frac{\kappa \epsilon_0 A}{d-\Delta x} \qquad (2)$$

The force of attraction between the plates is given in Problem 95 (*c*):

$$F = \frac{Q^2(x)}{2\kappa \epsilon_0 A}$$

Substitute to obtain:

$$F = \frac{\left(\frac{\kappa \epsilon_0 A}{x} V\right)^2}{2\kappa \epsilon_0 A} = -\frac{\kappa \epsilon_0 A V^2}{2x^2}$$

where the minus sign is used to indicate that the force acts to decrease the plate separation x.

Apply Hooke's law to relate the stress to the strain:

$$Y = \frac{F/A}{\Delta x/x} \text{ or } \frac{\Delta x}{x} = \frac{F}{YA}$$

Substitute for F to obtain:

$$\frac{\Delta x}{x} = -\frac{\kappa \epsilon_0 V^2}{2Yx^2}$$

and

$$\Delta x = -\frac{\kappa \epsilon_0 V^2}{2Yx} = -\frac{\kappa \epsilon_0 V^2}{2Yd} \quad (3)$$

provided $\Delta x << d$

The voltage across the capacitor is:

$$V = E_{max} d = (40.0\,\text{kV/mm})(0.200\,\text{mm}) = 8.00\,\text{kV}$$

Substitute numerical values in equation (3) and evaluate Δx:

$$\Delta x = -\frac{3\left(8.854\times10^{-12}\,\frac{\text{C}^2}{\text{N}\cdot\text{m}^2}\right)(8.00\,\text{kV})^2}{2(5.00\times10^6\,\text{N/m}^2)(0.200\,\text{mm})} = 8.50\times10^{-7}\,\text{m} = 8.50\times10^{-4}\,\text{mm}$$

Substitute in equation (2) to obtain:

$$C(V) = \frac{\kappa \epsilon_0 A}{d - \frac{\kappa \epsilon_0 V^2}{2Yd}} = \frac{\kappa \epsilon_0 A}{d\left(1 - \frac{\kappa \epsilon_0 V^2}{2Yd^2}\right)} = C_0\left(1 - \frac{\kappa \epsilon_0 V^2}{2Yd^2}\right)^{-1} \approx \boxed{C_0\left(1 + \frac{\kappa \epsilon_0 V^2}{2Yd^2}\right)}$$

provided $\Delta x << d$.

(*b*) Express the maximum voltage that can be applied in terms of the maximum electric field:

$$V_{max} = E_{max}(d - \Delta x) = (40.0\,\text{kV/mm})(0.200\,\text{mm} - 8.50\times10^{-4}\,\text{mm}) = \boxed{7.97\,\text{kV}}$$