Chapter 11
Gravity

Conceptual Problems

1 • **[SSM]** True or false:
(*a*) For Kepler's law of equal areas to be valid, the force of gravity must vary inversely with the square of the distance between a given planet and the Sun.
(*b*) The planet closest to the Sun has the shortest orbital period.
(*c*) Venus's orbital speed is larger than the orbital speed of Earth.
(*d*) The orbital period of a planet allows accurate determination of that planet's mass.

(*a*) False. Kepler's law of equal areas is a consequence of the fact that the gravitational force acts along the line joining two bodies but is independent of the manner in which the force varies with distance.

(*b*) True. The periods of the planets vary with the three-halves power of their distances from the sun. So the shorter the distance from the sun, the shorter the period of the planet's motion.

(*c*) True. Setting up a proportion involving the orbital speeds of the two planets in terms of their orbital periods and mean distances from the Sun (see Table 11-1) shows that $\boldsymbol{v}_{\text{Venus}} = 1.17\boldsymbol{v}_{\text{Earth}}$.

(*d*) False. The orbital period of a planet is independent of the planet's mass.

2 • If the mass of a small Earth-orbiting satellite is doubled, the radius of its orbit can remain constant if the speed of the satellite (*a*) increases by a factor of 8, (*b*) increases by a factor of 2, (*c*) does not change, (*d*) is reduced by a factor of 8, (*e*) is reduced by a factor of 2.

Determine the Concept We can apply Newton's 2nd law and the law of gravity to the satellite to obtain an expression for its speed as a function of the radius of its orbit.

Apply Newton's 2nd law to the satellite to obtain:

$$\sum F_{\text{radial}} = \frac{GMm}{r^2} = m\frac{v^2}{r}$$

where M is the mass of the object the satellite is orbiting and m is the mass of the satellite.

Solving for v yields:

$$v = \sqrt{\frac{GM}{r}}$$

Thus the speed of the satellite is independent of its mass and $\boxed{(c)}$ is correct.

3 • **[SSM]** During what season in the northern hemisphere does Earth attain its maximum orbital speed about the Sun? What season is related to its minimum orbital speed? HINT: The major factor determining the seasons on Earth is *not* the variation in distance from the Sun.

Determine the Concept Earth is closest to the Sun during winter in the northern hemisphere. This is the time of fastest orbital speed. Summer would be the time for minimum orbital speed.

4 • Haley's comet is in a highly elliptical orbit about the Sun with a period of about 76 y. Its last closest approach to the Sun occurred in 1987. In what years of the twentieth century was it traveling at its fastest or slowest orbital speed about the Sun?

Determine the Concept Haley's comet was traveling at its fastest orbital speed in 1987, and at its slowest orbital speed 38 years previously in 1949.

5 • Venus has no natural satellites. However artificial satellites have been placed in orbit around it. To use one of their orbits to determine the mass of Venus, what orbital parameters would you have to measure? How would you then use them to do the mass calculation?

Determine the Concept To obtain the mass M of Venus you need to measure the period T and semi-major axis a of the orbit of one of the satellites, substitute the measured values into $T^2/a^2 = 4\pi^2/(GM)$ (Kepler's 3rd law), and solve for M.

6 • A majority of the asteroids are in approximately circular orbits in a ″belt″ between Mars and Jupiter. Do they all have the same orbital period about the Sun? Explain.

Determine the Concept No. As described by Kepler's 3rd law, the asteroids closer to the Sun have a shorter ″year″ and are orbiting faster.

7 • **[SSM]** At the surface of the moon, the acceleration due to the gravity of the moon is a. At a distance from the center of the moon equal to four times the radius of the moon, the acceleration due to the gravity of the moon is (*a*) 16a, (*b*) a/4, (*c*) a/3, (*d*) a/16, (*e*) None of the above.

Picture the Problem The acceleration due to gravity varies inversely with the square of the distance from the center of the moon.

Express the dependence of the acceleration due to the gravity of the moon on the distance from its center:

$$a' \propto \frac{1}{r^2}$$

Express the dependence of the acceleration due to the gravity of the moon at its surface on its radius:

$$a \propto \frac{1}{R_M^2}$$

Divide the first of these expressions by the second to obtain:

$$\frac{a'}{a} = \frac{R_M^2}{r^2}$$

Solving for a' and simplifying yields:

$$a' = \frac{R_M^2}{r^2} a = \frac{R_M^2}{(4R_M)^2} a = \tfrac{1}{16} a$$

and $\boxed{(\boldsymbol{d})}$ is correct.

8 • At a depth equal to half the radius of Earth, the acceleration due to gravity is about (*a*) *g* (*b*) 2*g* (*c*) *g*/2, (*d*) *g*/4, (*e*) *g*/8, (*f*) You cannot determine the answer based on the data given.

Picture the Problem We can use Newton's law of gravity and the assumption of uniform density to express the ratio of the acceleration due to gravity at a depth equal to half the radius of Earth to the acceleration due to gravity at the surface of Earth.

The acceleration due to gravity at a depth equal to half the radius of Earth is given by:

$$\boldsymbol{g}_{\frac{1}{2}r} = \frac{\boldsymbol{GM'}}{\left(\frac{1}{2}\boldsymbol{r}\right)^2} = \frac{4\boldsymbol{GM'}}{\boldsymbol{r}^2}$$

where M' is the mass of Earth between the location of interest and the center of Earth.

The acceleration due to gravity at the surface of Earth is given by:

$$\boldsymbol{g} = \frac{\boldsymbol{GM}}{\boldsymbol{r}^2}$$

Dividing the first of these equations by the second and simplifying yields:

$$\frac{g_{\frac{1}{2}r}}{g} = \frac{\frac{4GM'}{r^2}}{\frac{GM}{r^2}} = \frac{4M'}{M} \qquad (1)$$

Express M' in terms of the average density of Earth ρ and the volume V' of Earth between the location of interest and the center of Earth:

$$M' = \rho V' = \rho\left[\tfrac{4}{3}\pi\left(\tfrac{1}{2}r\right)^3\right] = \tfrac{1}{6}\pi\rho r^3$$

Express M in terms of the average density of Earth ρ and the volume V of Earth:

$$M = \rho V = \rho\left(\tfrac{4}{3}\pi r^3\right) = \tfrac{4}{3}\pi\rho r^3$$

Substitute for M and M' in equation (1) and simplify to obtain:

$$\frac{g_{\frac{1}{2}r}}{g} = \frac{4\left(\frac{1}{6}\pi\rho r^3\right)}{\frac{4}{3}\pi\rho r^3} = \frac{1}{2}$$

and $\boxed{(c)}$ is correct.

9 •• Two stars orbit their common center of mass as a *binary* star system. If each of their masses were doubled, what would have to happen to the distance between them in order to maintain the same gravitational force? The distance would have to (*a*) remain the same (*b*) double (*c*) quadruple (*d*) be reduced by a factor of 2 (*e*) You cannot determine the answer based on the data given.

Picture the Problem We can use Newton's law of gravity to express the ratio of the forces and then solve this proportion for the separation of the stars that would maintain the same gravitational force.

The gravitational force acting on the stars before their masses are doubled is given by:

$$F_{\rm g} = \frac{GmM}{r^2}$$

The gravitational force acting on the stars after their masses are doubled is given by:

$$F_{\rm g}' = \frac{G(2m)(2M)}{r'^2} = \frac{4GmM}{r'^2}$$

Dividing the second of these equations by the first yields:

$$\frac{F_{\rm g}'}{F_{\rm g}} = 1 = \frac{\dfrac{4GmM}{r'^2}}{\dfrac{GmM}{r^2}} = \frac{4r^2}{r'^2}$$

Solving for r' yields:

$r' = 2r$ and $\boxed{(b)}$ is correct.

10 •• If you had been working for NASA in the 1960's and planning the trip to the moon, you would have determined that there exists a unique location somewhere between Earth and the moon, where a spaceship is, for an instant, truly weightless. [Consider only the moon, Earth and the Apollo spaceship, and neglect other gravitational forces.] Explain this phenomenon and explain whether this location is closer to the moon, midway on the trip, or closer to Earth.

Determine the Concept Between Earth and the moon, the gravitational pulls on the spaceship are oppositely directed. Because of the moon's relatively small mass compared to the mass of Earth, the location where the gravitational forces cancel (thus producing no net gravitational force, a weightless condition) is considerably closer to the moon.

11 •• [SSM] Suppose the escape speed from a planet was only slightly larger than the escape speed from Earth, yet it was considerably larger than Earth. How would the planet's (average) density compare to Earth's (average) density? (*a*) It must be more dense. (*b*) It must be less dense. (*c*) It must be the same density. (*d*) You cannot determine the answer based on the data given.

Picture the Problem The densities of the planets are related to the escape speeds from their surfaces through $v_e = \sqrt{2GM/R}$.

The escape speed from the planet is given by:

$$v_{\text{planet}} = \sqrt{\frac{2GM_{\text{planet}}}{R_{\text{planet}}}}$$

The escape speed from Earth is given by:

$$v_{\text{Earth}} = \sqrt{\frac{2GM_{\text{Earth}}}{R_{\text{Earth}}}}$$

Expressing the ratio of the escape speed from the planet to the escape speed from Earth and simplifying yields:

$$\frac{v_{\text{planet}}}{v_{\text{Earth}}} = \frac{\sqrt{\dfrac{2GM_{\text{planet}}}{R_{\text{planet}}}}}{\sqrt{\dfrac{2GM_{\text{Earth}}}{R_{\text{Earth}}}}} = \sqrt{\frac{R_{\text{Earth}}}{R_{\text{planet}}}\frac{M_{\text{planet}}}{M_{\text{Earth}}}}$$

Because $v_{\text{planet}} \approx v_{\text{Earth}}$:

$$1 \approx \sqrt{\frac{R_{\text{Earth}}}{R_{\text{planet}}}\frac{M_{\text{planet}}}{M_{\text{Earth}}}}$$

Squaring both sides of the equation yields:

$$1 \approx \frac{R_{\text{Earth}}}{R_{\text{planet}}}\frac{M_{\text{planet}}}{M_{\text{Earth}}}$$

Express M_{planet} and M_{Earth} in terms of their densities and simplify to obtain:

$$1 \approx \frac{R_{\text{Earth}}}{R_{\text{planet}}}\frac{\rho_{\text{planet}}V_{\text{planet}}}{\rho_{\text{Earth}}V_{\text{Earth}}} = \frac{R_{\text{Earth}}}{R_{\text{planet}}}\frac{\rho_{\text{planet}}V_{\text{planet}}}{\rho_{\text{Earth}}V_{\text{Earth}}} = \frac{R_{\text{Earth}}}{R_{\text{planet}}}\frac{\rho_{\text{planet}}\frac{4}{3}\pi R_{\text{planet}}^3}{\rho_{\text{Earth}}\frac{4}{3}\pi R_{\text{Earth}}^3} = \frac{\rho_{\text{planet}}R_{\text{planet}}^2}{\rho_{\text{Earth}}R_{\text{Earth}}^2}$$

Solving for the ratio of the densities yields:

$$\frac{\rho_{\text{planet}}}{\rho_{\text{Earth}}} \approx \frac{R_{\text{Earth}}^2}{R_{\text{planet}}^2}$$

Because the planet is considerably larger than Earth:

$$\frac{\rho_{\text{planet}}}{\rho_{\text{Earth}}} << 1$$

and $\boxed{(b)}$ is correct.

12 •• Suppose that, using telescope in your backyard, you discovered a distant object approaching the Sun, and were able to determine both its distance from the Sun and its speed. How would you be able to predict whether the object will remain ″bound″ to the Solar System, or if it is an interstellar interloper and would come in, turn around and escape, never to return?

Determine the Concept You could take careful measurements of its position as a function of time in order to determine whether its trajectory is an ellipse, a hyperbola, or a parabola. If the path is an ellipse, it will return; if its path is hyperbolic or parabolic, it will not return. Alternatively, by measuring its distance from the Sun, you can estimate the gravitational potential energy (per kg of its mass, and neglecting the planets) of the object, and by determining its position on several successive nights, the speed of the object can be determined. From this, its kinetic energy (per kg) can be determined. The sum of these two gives the comet's total energy (per kg) and if it is positive, it will likely swing once around the Sun and then leave the Solar System forever.

13 •• **[SSM]** Near the end of their useful lives, several large Earth-orbiting satellites have been maneuvered so as to burn up as they enter Earth's atmosphere. These maneuvers have to be done carefully so large fragments do not impact populated land areas. You are in charge of such a project. Assuming the satellite of interest has on-board propulsion, in what direction would you fire the rockets for a short burn time to start this downward spiral? What would happen to the kinetic energy, gravitational potential energy and total mechanical energy following the burn as the satellite came closer and closer to Earth?

Determine the Concept You should fire the rocket in a direction to oppose the orbital motion of the satellite. As the satellite gets closer to Earth after the burn, the potential energy will decrease as the satellite gets closer to Earth. However, the total mechanical energy will decrease due to the frictional drag forces transforming mechanical energy into thermal energy. The kinetic energy will increase until the satellite enters the atmosphere where the drag forces slow its motion.

14 •• During a trip back from the moon, the Apollo spacecraft fires its rockets to leave its lunar orbit. Then it coasts back to Earth where it enters the atmosphere at high speed, survives a blazing re-entry and parachutes safely into the ocean. In what direction do you fire the rockets to initiate this return trip? Explain the changes in kinetic energy, gravitational potential and total mechanical energy that occur to the spacecraft from the beginning to the end of this journey.

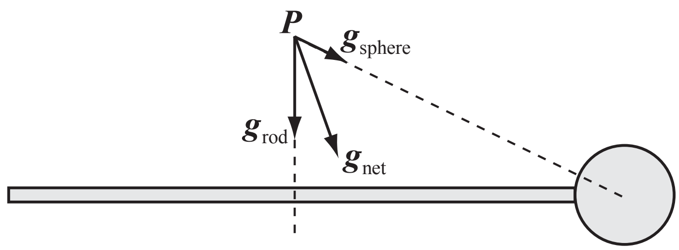
Determine the Concept Near the moon you would fire the rockets to accelerate the spacecraft with the thrust acting in the direction of your ship's velocity at the time. When the rockets have shut down, as you leave the lunar orbit, your kinetic energy will initially decrease (the moon's gravitational pull exceeds that of Earth), and your potential energy will increase. When you reach a certain point, Earth's gravitational attraction will begin accelerating the ship and its kinetic energy will increase at the expense of the gravitational potential energy of the spacecraft-Earth-moon system. The spacecraft will enter Earth's atmosphere with its maximum kinetic energy. Eventually, landing in the ocean, the kinetic energy will be zero, the gravitational potential energy a minimum, and the total mechanical energy of the ship will have been dramatically reduced due to air drag forces producing heat and light during re-entry.

15 •• Explain why the gravitational field inside a solid sphere of uniform mass is directly proportional to r rather than inversely proportional to r.

Determine the Concept At a point inside the sphere a distance r from its center, the gravitational field strength is directly proportional to the amount of mass within a distance r from the center, and inversely proportional to the square of the distance r from the center. The mass within a distance r from the center is proportional to the cube of r. Thus, the gravitational field strength is directly proportional to r.

16 •• In the movie *2001 A Space Odyssey*, a spaceship containing two astronauts is on a long-term mission to Jupiter. A model of their ship could be a uniform pencil-like rod (containing the propulsion systems) with a uniform sphere (the crew habitat and flight deck) attached to one end. The design is such that the radius of the sphere is much smaller than the length of the rod. At a location a few meters away from the ship, on the perpendicular bisector of the rod-like section, what would be the direction of the gravitational field due to the ship alone (that is, assuming all other gravitational fields are negligible)? Explain your answer. At a large distance from the ship, what would be the dependence of its gravitational field on the distance from the ship?

Determine the Concept The pictorial representation shows the point of interest P and the gravitational fields $\vec{g}_{\text{rod}}$ and $\vec{g}_{\text{sphere}}$ due to the rod and the sphere as well as the resultant field $\vec{g}_{\text{net}}$. Note that the net field (the sum of $\vec{g}_{\text{rod}}$ and $\vec{g}_{\text{sphere}}$) points slightly toward the habitat end of the ship. At very large distances, the rod-sphere mass distribution looks like a point mass and so the field's distance dependence is an inverse square dependence.



Estimation and Approximation

17 • **[SSM]** Estimate the mass of our galaxy (the Milky Way) if the Sun orbits the center of the galaxy with a period of 250 million years at a mean distance of 30,000 $c\cdot$y. Express the mass in terms of multiples of the solar mass M_S. (Neglect the mass farther from the center than the Sun, and assume that the mass closer to the center than the Sun exerts the same force on the Sun as would a point particle of the same mass located at the center of the galaxy.)

Picture the Problem To approximate the mass of the galaxy we'll assume the galactic center to be a point mass with the sun in orbit about it and apply Kepler's 3rd law.

Using Kepler's 3rd law, relate the period of the sun T to its mean distance r from the center of the galaxy:

$$T^2 = \frac{4\pi^2}{GM_{\text{galaxy}}}r^3 = \frac{\dfrac{4\pi^2}{M_s}}{G\dfrac{M_{\text{galaxy}}}{M_s}}r^3$$

Solve for $\dfrac{r^3}{T^2}$ and simplify to obtain:

$$\frac{r^3}{T^2} = \frac{G\dfrac{M_{\text{galaxy}}}{M_s}}{\dfrac{4\pi^2}{M_s}} = \frac{\dfrac{M_{\text{galaxy}}}{M_s}}{\dfrac{4\pi^2}{GM_s}}$$

If we measure distances in AU and times in years:

$$\frac{4\pi^2}{GM_s} = 1\frac{\text{y}^2}{(\text{AU})^3}$$

and

$$\frac{r^3}{T^2} = \frac{M_{\text{galaxy}}}{M_s}\frac{(\text{AU})^3}{\text{y}^2}$$

Substitute numerical values and evaluate $M_{\text{galaxy}}/M_{\text{s}}$:

$$\frac{M_{\text{galaxy}}}{M_{\text{s}}}=\frac{\left(3.00\times10^{4}\,c\cdot\text{y}\times\dfrac{9.461\times10^{15}\text{ m}}{c\cdot\text{y}}\times\dfrac{1\text{ AU}}{1.50\times10^{11}\text{ m}}\right)^{3}\dfrac{\text{y}^{2}}{(\text{AU})^{3}}}{\left(250\times10^{6}\text{ y}\right)^{2}}=1.08\times10^{11}$$

or

$$M_{\text{galaxy}}=\boxed{1.08\times10^{11}M_{\text{s}}}$$

18 •• Besides studying samples of the lunar surface, the Apollo astronauts had several ways of determining that the moon is *not* made of green cheese. Among these ways are measurements of the gravitational acceleration at the lunar surface. Estimate the gravitational acceleration at the lunar surface if the moon were, in fact, a solid block of green cheese and compare it to the known value of the gravitational acceleration at the lunar surface.

Picture the Problem The density of a planet or other object determines the strength of the gravitational force it exerts on other objects. We can use Newton's law of gravity to express the acceleration due to gravity at the surface of the moon as a function of the density of the moon. Estimating the density of cheese will then allow us to calculate what the acceleration due to gravity at the surface of the moon would be if the moon were made of cheese. Finally, we can compare this value to the measured value of 1.62 m/s^2.

Apply Newton's law of gravity to an object of mass m at the surface of the moon to obtain:

$$F_{\text{g}}=ma_{\text{g}}=\frac{GmM}{r_{\text{moon}}^{2}}\Rightarrow a_{\text{g}}=\frac{GM}{r_{\text{moon}}^{2}}$$

Assuming the moon to be made of cheese, substitute for its mass to obtain:

$$a_{\text{g,cheese}}=\frac{G\rho_{\text{cheese}}V_{\text{moon}}}{r_{\text{moon}}^{2}}$$

Substituting for the volume of the moon and simplifying yields:

$$a_{\text{g,cheese}}=\frac{G\rho_{\text{cheese}}\frac{4}{3}\pi r_{\text{moon}}^{3}}{r_{\text{moon}}^{2}}=\frac{4\pi}{3}G\rho_{\text{cheese}}r_{\text{moon}}$$

Substitute numerical values and evaluate a_{g}:

$$\boldsymbol{a}_{\text{g,cheese}}=\frac{4\pi}{3}\left(6.673\times10^{-11}\text{ N}\cdot\text{m}^{2}/\text{kg}^{2}\right)\left(0.80\frac{\text{g}}{\text{cm}^{3}}\times\frac{1\text{ kg}}{10^{3}\text{ g}}\times\frac{10^{6}\text{ cm}^{3}}{\text{m}^{3}}\right)\left(1.738\times10^{6}\text{ m}\right)=\boxed{0.39\text{ m/s}^{2}}$$

Express the ratio of $a_{g,cheese}$ to the measured value of $a_{g,moon}$:

$$\frac{a_{g,cheese}}{a_{g,moon}} = \frac{0.388\ \text{m/s}^2}{1.62\ \text{m/s}^2} = 0.24$$

or

$$a_{g,cheese} \approx \boxed{0.24 a_{g,moon}}$$

19 •• You are in charge of the first manned exploration of an asteroid. You are concerned that, due to the weak gravitational field and resulting low escape speed, tethers might be required to bind the explorers to the surface of the asteroid. Therefore, if you do not wish to use tethers, you have to be careful about which asteroids to choose to explore. Estimate the largest radius the asteroid can have that would still allow you to escape its surface by jumping. Assume spherical geometry and reasonable rock density.

Picture the Problem The density of an asteroid determines the strength of the gravitational force it exerts on other objects. We can use the equation for the escape speed from an asteroid of mass $M_{asteroid}$ and radius $R_{asteroid}$ to derive an expression for the radius of an asteroid as a function of its escape speed and density. We can approximate the escape speed from the asteroid by determining one's push-off speed for a jump at the surface of Earth.

The escape speed from an asteroid is given by:

$$v_{e,asteroid} = \sqrt{\frac{2GM_{asteroid}}{R_{asteroid}}}$$

In terms of the density of the asteroid, $v_{e,asteroid}$ becomes:

$$v_{e,asteroid} = \sqrt{\frac{2G\rho_{asteroid}\,\frac{4}{3}\pi R_{asteroid}^3}{R_{asteroid}}} = \sqrt{\tfrac{8}{3}\pi G\rho_{asteroid}}\,R_{asteroid}$$

Solving for $R_{asteroid}$ yields:

$$R_{asteroid} = \frac{v_{e,asteroid}}{\sqrt{\frac{8}{3}\pi G\rho_{asteroid}}} \qquad (1)$$

Using a constant-acceleration equation, relate the height h to which you can jump on the surface of Earth to your push-off speed:

$$v^2 = v_0^2 - 2gh$$

or, because $v = 0$,

$$0 = v_0^2 - 2gh \Rightarrow v_0 = \sqrt{2gh}$$

Letting $v_0 = v_{e,asteroid}$, substitute in equation (1) and simplify to obtain:

$$R_{asteroid} = \frac{\sqrt{2gh}}{\sqrt{\frac{8}{3}\pi G\rho_{asteroid}}} = \sqrt{\frac{3gh}{4\pi G\rho_{asteroid}}}$$

Assuming that you can jump 0.75 m and that the average density of an asteroid is 3.0 g/cm^3, substitute numerical values and evaluate R_{asteroid} :

$$R_{\text{asteroid}} = \sqrt{\frac{3\left(9.81\,\text{m/s}^2\right)\left(0.75\,\text{m}\right)}{4\pi\left(6.673\times10^{-11}\,\text{N}\cdot\text{m}^2/\text{kg}^2\right)\left(3.0\,\frac{\text{g}}{\text{cm}^3}\times\frac{1\,\text{kg}}{10^3\,\text{g}}\times\frac{10^6\,\text{cm}^3}{\text{m}^3}\right)}} \approx \boxed{3.0\,\text{km}}$$

20 ••• One of the great discoveries in astronomy in the past decade is the detection of planets outside the Solar System. Since 1996, 100 planets have been detected orbiting stars other than the Sun. While the planets themselves cannot be seen directly, telescopes can detect the small periodic motion of the star as the star and planet orbit around their common center of mass. (This is measured using the *Doppler effect*, which is discussed in Chapter 15.) Both the period of this motion and the variation in the speed of the star over the course of time can be determined observationally. The mass of the star is found from its observed luminance and from the theory of stellar structure. *Iota Draconis* is the 8th brightest star in the constellation Draco. Observations show that a planet, with an orbital period of 1.50 y, is orbiting this star. The mass of Iota Draconis is $1.05M_{\text{Sun}}$. (*a*) Estimate the size (in AU) of the semimajor axis of this planet's orbit. (*b*) The radial speed of the star is observed to vary by 592 m/s. Use conservation of momentum to find the mass of the planet. Assume the orbit is circular, we are observing the orbit edge-on, and no other planets orbit Iota Draconis. Express the mass as a multiple of the mass of Jupiter.

Picture the Problem We can use Kepler's 3rd law to find the size of the semi-major axis of the planet's orbit and the conservation of momentum to find its mass.

(*a*) Using Kepler's 3rd law, relate the period of this planet T to the length r of its semi-major axis and simplify to obtain:

$$T^2 = \frac{4\pi^2}{GM_{\text{Iota Draconis}}}r^3 = \frac{\frac{4\pi^2}{M_{\text{s}}}}{G\frac{M_{\text{Iota Draconis}}}{M_{\text{s}}}}r^3 = \frac{\frac{4\pi^2}{GM_{\text{s}}}}{\frac{M_{\text{Iota Draconis}}}{M_{\text{s}}}}r^3$$

If we measure time in years, distances in AU, and masses in terms of the mass of the sun:

$$\frac{4\pi^2}{MG_{\text{s}}} = 1 \text{ and } T^2 = \frac{1}{\frac{M_{\text{Iota Draconis}}}{M_{\text{s}}}}r^3$$

Solving for r yields:

$$r = \sqrt[3]{\frac{M_{\text{Iota Draconis}}}{M_{\text{s}}}T^2}$$

Substitute numerical values and evaluate r:

$$r=\sqrt[3]{\left(\frac{1.05M_s}{M_s}\right)(1.50\,\text{y})^2}=\boxed{1.33\,\text{AU}}$$

(*b*) Apply conservation of momentum to the planet (mass m and speed v) and the star (mass $M_{\text{Iota Draconis}}$ and speed V) to obtain:

$$mv=M_{\text{Iota Draconis}}V$$

Solve for m to obtain:

$$m=M_{\text{Iota Draconis}}\frac{V}{v} \qquad (1)$$

The speed v of the orbiting planet is given by:

$$v=\frac{\Delta d}{\Delta t}=\frac{2\pi r}{T}$$

Substitute numerical values and evaluate v:

$$\boldsymbol{v}=\frac{2\boldsymbol{\pi}\left(1.33\,\text{AU}\times\dfrac{1.50\times10^{11}\,\text{m}}{\text{AU}}\right)}{1.50\,\text{y}\times\dfrac{365.24\,\text{d}}{\text{y}}\times\dfrac{24\,\text{h}}{\text{d}}\times\dfrac{3600\,\text{s}}{\text{h}}}$$
$$=2.648\times10^4\,\text{m/s}$$

Substitute numerical values in equation (1) and evaluate m:

$$\boldsymbol{m}=(1.05\boldsymbol{M}_{\text{sun}})\left(\frac{296\,\text{m/s}}{2.648\times10^4\,\text{m/s}}\right)$$
$$=(1.05)(1.99\times10^{30}\,\text{kg})(0.0112)$$
$$=2.336\times10^{28}\,\text{kg}$$

Express m as a fraction of the mass M_J of Jupiter:

$$\frac{\boldsymbol{m}}{\boldsymbol{M}_J}=\frac{2.336\times10^{28}\,\text{kg}}{1.90\times10^{27}\,\text{kg}}=12.3$$

or

$$m=\boxed{12.3M_J}$$

Remarks: A more sophisticated analysis, using the eccentricity of the orbit, leads to a lower bound of 8.7 Jovian masses. (Only a lower bound can be established, as the plane of the orbit is not known.)

21 ••• One of the biggest unresolved problems in the theory of the formation of the solar system is that, while the mass of the Sun is 99.9 percent of the total mass of the Solar System, it carries only about 2 percent of the total angular momentum. The most widely accepted theory of solar system formation has as its central hypothesis the collapse of a cloud of dust and gas under the force of gravity, with most of the mass forming the Sun. However, because the net angular momentum of this cloud is conserved, a simple theory would indicate that the Sun should be rotating much more rapidly than it currently is. In this problem, you will show why it is important that most of the angular momentum was somehow transferred to the planets. (*a*) The Sun is a cloud of gas held together by the force of gravity. If the Sun were rotating too rapidly, gravity couldn't hold it together. Using the known mass of the Sun (1.99×10^{30} kg) and its radius (6.96×10^{8} m), estimate the maximum angular speed that the Sun can have if it is to stay intact. What is the period of rotation corresponding to this rotation rate? (*b*) Calculate the orbital angular momentum of Jupiter and of Saturn from their masses (318 and 95.1 Earth masses, respectively), mean distances from the Sun (778 and 1430 million km, respectively), and orbital periods (11.9 and 29.5 y, respectively). Compare them to the experimentally measured value of the Sun's angular momentum of 1.91×10^{41} kg·m^2/s. (*c*) If we were to somehow transfer all of Jupiter's and Saturn's angular momentum to the Sun, what would be the Sun's new rotational period? The Sun is not a uniform sphere of gas, and its moment of inertia is given by the formula $I = 0.059MR^2$. Compare this to the maximum rotational period of Part (*a*).

Picture the Problem We can apply Newton's law of gravity to estimate the maximum angular speed which the sun can have if it is to stay together and use the definition of angular momentum to find the orbital angular momenta of Jupiter and Saturn. In Part (*c*) we can relate the final angular speed of the sun to its initial angular speed, its moment of inertia, and the orbital angular momenta of Jupiter and Saturn.

(*a*) Gravity must supply the centripetal force which keeps an element of the sun's mass *m* rotating around it. Letting the radius of the sun be *R*, apply Newton's law of gravity to an object of mass *m* on the surface of the Sun to obtain:

$$m\omega^2 R < \frac{GMm}{R^2}$$

or

$$\omega^2 R < \frac{GM}{R^2} \Rightarrow \omega < \sqrt{\frac{GM}{R^3}}$$

where we've used the inequality because we're estimating the *maximum* angular speed which the sun can have if it is to stay together.

Substitute numerical values and evaluate ω:

$$\omega < \sqrt{\frac{\left(6.673\times10^{-11}\ \mathrm{N\cdot m^2/kg^2}\right)\left(1.99\times10^{30}\ \mathrm{kg}\right)}{\left(6.96\times10^{8}\ \mathrm{m}\right)^3}} = \boxed{6.28\times10^{-4}\ \mathrm{rad/s}}$$

Calculate the maximum period of this motion from its angular speed:

$$T_{\max} = \frac{2\pi}{\omega} = \frac{2\pi}{6.28\times10^{-4}\ \mathrm{rad/s}}$$

$$= 1.00\times10^{4}\ \mathrm{s}\times\frac{1\,\mathrm{h}}{3600\,\mathrm{s}} = \boxed{2.78\,\mathrm{h}}$$

(*b*) Express the orbital angular momenta of Jupiter and Saturn:

$$L_J = m_J r_J v_J \text{ and } L_S = m_S r_S v_S$$

Express the orbital speeds of Jupiter and Saturn in terms of their periods and distances from the sun:

$$v_J = \frac{2\pi r_J}{T_J} \text{ and } v_S = \frac{2\pi r_S}{T_S}$$

Substitute to obtain:

$$L_J = \frac{2\pi m_J r_J^2}{T_J} \text{ and } L_S = \frac{2\pi m_S r_S^2}{T_S}$$

Substitute numerical values and evaluate L_J and L_S:

$$\boldsymbol{L}_J = \frac{2\boldsymbol{\pi}(318\boldsymbol{M}_E)\boldsymbol{r}_J^2}{\boldsymbol{T}_J} = \frac{2\boldsymbol{\pi}(318)(5.98\times10^{24}\text{ kg})(778\times10^9\text{ m})^2}{11.9\text{ y}\times\frac{365.24\text{ d}}{\text{y}}\times\frac{24\text{ h}}{\text{d}}\times\frac{3600\text{ s}}{\text{h}}}$$

$$= \boxed{1.93\times10^{43}\text{ kg}\cdot\text{m}^2/\text{s}}$$

and

$$\boldsymbol{L}_S = \frac{2\boldsymbol{\pi}(95.1\boldsymbol{M}_E)\boldsymbol{r}_S^2}{\boldsymbol{T}_S} = \frac{2\boldsymbol{\pi}(95.1)(5.98\times10^{24}\text{ kg})(1430\times10^9\text{ m})^2}{29.5\text{ y}\times\frac{365.24\text{ d}}{\text{y}}\times\frac{24\text{ h}}{\text{d}}\times\frac{3600\text{ s}}{\text{h}}}$$

$$= \boxed{7.85\times10^{42}\text{ kg}\cdot\text{m}^2/\text{s}}$$

Express the angular momentum of the sun as a fraction of the sum of the angular momenta of Jupiter and Saturn:

$$\frac{\boldsymbol{L}_{\text{sun}}}{\boldsymbol{L}_J + \boldsymbol{L}_S} = \frac{1.91\times10^{41}\text{ kg}\cdot\text{m}^2/\text{s}}{(19.3+7.85)\times10^{42}\text{ kg}\cdot\text{m}^2/\text{s}} = \boxed{0.70\%}$$

(*c*) The Sun's rotational period depends on its rotational speed:

$$T_{\text{Sun}} = \frac{2\pi}{\omega_f} \qquad (1)$$

Relate the final angular momentum of the sun to its initial angular momentum and the angular momenta of Jupiter and Saturn:

$$L_f = L_i + L_J + L_S$$

or

$$I_{\text{Sun}}\omega_f = I_{\text{Sun}}\omega_i + L_J + L_S$$

Solve for ω_f to obtain:

$$\omega_f = \omega_i + \frac{L_J + L_S}{I_{\text{sun}}}$$

Substitute for ω_i and I_{Sun}:

$$\omega_f = \frac{2\pi}{T_{\text{Sun}}} + \frac{L_J + L_S}{0.059 M_{\text{sun}} R_{\text{sun}}^2}$$

Substitute numerical values and evaluate ω_f:

$$\omega_f = \frac{2\pi}{30\,\text{d}\times\frac{24\,\text{h}}{\text{d}}\times\frac{3600\,\text{s}}{\text{h}}} + \frac{(19.3+7.85)\times10^{42}\,\text{kg}\cdot\text{m}^2/\text{s}}{0.059(1.99\times10^{30}\,\text{kg})(6.96\times10^{8}\,\text{m})^2}$$
$$= 4.798\times10^{-4}\,\text{rad/s}$$

Substitute numerical values in equation (1) and evaluate T_{Sun}:

$$T_{\text{Sun}} = \frac{2\pi}{4.798\times10^{-4}\,\frac{\text{rad}}{\text{s}}\times\frac{3600\,\text{s}}{\text{h}}} = \boxed{3.64\,\text{h}}$$

Compare this to the maximum rotational period of Part (*a*).

$$\frac{T_{\text{Sun}}}{T_{\text{max}}} = \frac{3.64\,\text{h}}{2.78\,\text{h}} = 1.30$$

or

$$T_{\text{Sun}} = \boxed{1.30T_{\text{max}}}$$

Kepler's Laws

22 • The new comet Alex-Casey has a very elliptical orbit with a period of 127.4 y. If the closest approach of Alex-Casey to the Sun is 0.1 AU, what is its greatest distance from the Sun?

Picture the Problem We can use the relationship between the semi-major axis and the distances of closest approach and greatest separation, together with Kepler's 3rd law, to find the greatest separation of Alex-Casey from the sun.

Letting x represent the greatest distance from the sun, express the relationship between x, the distance of closest approach, and its semi-major axis R:

$$R = \frac{x+0.1\,\text{AU}}{2} \Rightarrow x = 2R - 0.1\,\text{AU} \quad (1)$$

Apply Kepler's 3rd law, with the period T measured in years and R in AU to obtain:

$$T^2 = R^3 \Rightarrow R = \sqrt[3]{T^2}$$

Substituting for R in equation (1) yields:

$$\boldsymbol{x} = 2\sqrt[3]{\boldsymbol{T}^2} - 0.1\,\text{AU}$$

Substitute numerical values and evaluate x:

$$x = 2\sqrt[3]{(127.4\,\text{y})^2} - 0.1\,\text{AU} = \boxed{50.5\,\text{AU}}$$

23 • The radius of Earth's orbit is 1.496×10^{11} m and that of Uranus is 2.87 $\times 10^{12}$ m. What is the orbital period of Uranus?

Picture the Problem We can use Kepler's 3rd law to relate the orbital period of Uranus to the orbital period of Earth.

Using Kepler's 3rd law, relate the orbital period of Uranus to its mean distance from the sun:

$$\boldsymbol{T}_{\text{Uranus}}^2 = \boldsymbol{C}\boldsymbol{r}_{\text{Uranus}}^3$$

.

Using Kepler's 3rd law, relate the orbital period of Earth to its mean distance from the sun:

$$\boldsymbol{T}_{\text{Earth}}^2 = \boldsymbol{C}\boldsymbol{r}_{\text{Earth}}^3$$

Dividing the first of these equations by the second and simplifying yields:

$$\frac{\boldsymbol{T}_{\text{Uranus}}^2}{\boldsymbol{T}_{\text{Earth}}^2} = \frac{\boldsymbol{C}\boldsymbol{r}_{\text{Uranus}}^3}{\boldsymbol{C}\boldsymbol{r}_{\text{Earth}}^3} = \frac{\boldsymbol{r}_{\text{Uranus}}^3}{\boldsymbol{r}_{\text{Earth}}^3}$$

Solve for T_{Uranus} to obtain:

$$\boldsymbol{T}_{\text{Uranus}} = \boldsymbol{T}_{\text{Earth}}\sqrt{\frac{\boldsymbol{r}_{\text{Uranus}}^3}{\boldsymbol{r}_{\text{Earth}}^3}}$$

Substitute numerical values and evaluate T_{Uranus}:

$$\boldsymbol{T}_{\text{Uranus}} = (1.00\text{ y})\sqrt{\left(\frac{2.87\times10^{12}\text{ m}}{1.496\times10^{11}\text{ m}}\right)^3} = \boxed{84.0\text{ y}}$$

24 • The asteroid Hektor, discovered in 1907, is in a nearly circular orbit of radius 5.16 *AU* about the Sun. Determine the period of this asteroid.

Picture the Problem We can use Kepler's 3rd law to relate the orbital period of Hektor to its mean distance from the sun.

Using Kepler's 3rd law, relate the orbital period of Hektor to its mean distance from the sun:

$$\boldsymbol{T}_{\text{Hector}}^2 = \boldsymbol{C}\boldsymbol{r}_{\text{Hector}}^3 \Rightarrow \boldsymbol{T}_{\text{Hector}} = \sqrt{\boldsymbol{C}\boldsymbol{r}_{\text{Hector}}^3}$$

where $C = \dfrac{4\pi^2}{GM_{\text{s}}} = 2.973\times10^{-19}\text{ s}^2/\text{m}^3$.

Substitute numerical values and evaluate T_{Hector}:

$$\boldsymbol{T}_{\text{Hector}} = \sqrt{\left(2.973\times10^{-19}\ \text{s}^2/\text{m}^3\right)\left(5.16\,\text{AU}\times\frac{1.50\times10^{11}\ \text{m}}{\text{AU}}\right)^3}$$
$$= 3.713\times10^{8}\,\text{s}\times\frac{1\,\text{h}}{3600\,\text{s}}\times\frac{1\,\text{d}}{24\,\text{h}}\times\frac{1\,\text{y}}{365.24\,\text{d}} = \boxed{11.8\,\text{y}}$$

25 •• **[SSM]** One of the so-called ″Kirkwood gaps″ in the asteroid belt occurs at an orbital radius at which the period of the orbit is half that of Jupiter's. The reason there is a gap for orbits of this radius is because of the periodic pulling (by Jupiter) that an asteroid experiences at the same place in its orbit every *other* orbit around the sun. Repeated tugs from Jupiter of this kind would eventually change the orbit of such an asteroid – therefore all asteroids that would otherwise have orbited at this radius have presumably been cleared away from the area due to this resonance phenomenon. How far from the sun is this particular 2:1 ″Kirkwood″ gap?

Picture the Problem The period of an orbit is related to its semi-major axis (for circular orbits this distance is the orbital radius). Because we know the orbital periods of Jupiter and a hypothetical asteroid in the Kirkwood gap, we can use Kepler's 3rd law to set up a proportion relating the orbital periods and average distances of Jupiter and the asteroid from the Sun from which we can obtain an expression for the orbital radius of an asteroid in the Kirkwood gap.

Use Kepler's 3rd law to relate Jupiter's orbital period to its mean distance from the Sun:

$$\boldsymbol{T}_{\text{Jupiter}}^2 = \boldsymbol{C}\boldsymbol{r}_{\text{Jupiter}}^3$$

Use Kepler's 3rd law to relate the orbital period of an asteroid in the Kirkwood gap to its mean distance from the Sun:

$$\boldsymbol{T}_{\text{Kirkwood}}^2 = \boldsymbol{C}\boldsymbol{r}_{\text{Kirkwood}}^3$$

Dividing the second of these equations by the first yields:

$$\frac{\boldsymbol{T}_{\text{Kirkwood}}^2}{\boldsymbol{T}_{\text{Jupiter}}^2} = \frac{\boldsymbol{C}\boldsymbol{r}_{\text{Kirkwood}}^3}{\boldsymbol{C}\boldsymbol{r}_{\text{Jupiter}}^3} = \frac{\boldsymbol{r}_{\text{Kirkwood}}^3}{\boldsymbol{r}_{\text{Jupiter}}^3}$$

Solving for r_{Kirkwood} yields:

$$\boldsymbol{r}_{\text{Kirkwood}} = \sqrt[3]{\left(\frac{\boldsymbol{T}_{\text{Kirkwood}}}{\boldsymbol{T}_{\text{Jupiter}}}\right)^2}\,\boldsymbol{r}_{\text{Jupiter}}$$

Because the period of the orbit of an asteroid in the Kirkwood gap is half that of Jupiter's:

$$\boldsymbol{r}_{\text{Kirkwood}} = \sqrt[3]{\left(\frac{\frac{1}{2}\boldsymbol{T}_{\text{Jupiter}}}{\boldsymbol{T}_{\text{Jupiter}}}\right)^2}\left(77.8\times10^{10}\text{ m}\right)$$

$$= \boxed{4.90\times10^{11}\text{ m}}$$

26 •• The tiny Saturnian moon, Atlas, is locked into what is known as an orbital resonance with another moon, Mimas, whose orbit lies outside of Atlas's. The ratio between periods of these orbits is 3:2 – that means, for every 3 orbits of Atlas, Mimas completes 2 orbits. Thus, Atlas, Mimas and Saturn are aligned at intervals equal to two orbital periods of Atlas. If Mimas orbits Saturn at a radius of 186,000 km, what is the radius of Atlas's orbit?

Picture the Problem The period of an orbit is related to its semi-major axis (for circular orbits this distance is the orbital radius). Because we know the orbital periods of Atlas and Mimas, we can use Kepler's 3rd law to set up a proportion relating the orbital periods and average distances from Saturn of Atlas and Mimas from which we can obtain an expression for the radius of Atlas's orbit.

Use Kepler's 3rd law to relate Atlas's orbital period to its mean distance from Saturn:

$$T_{\text{Atlas}}^2 = Cr_{\text{Atlas}}^3$$

Use Kepler's 3rd law to relate the orbital period of Mimas to its mean distance from Saturn:

$$T_{\text{Mimas}}^2 = Cr_{\text{Mimas}}^3$$

Dividing the second of these equations by the first yields:

$$\frac{T_{\text{Mimas}}^2}{T_{\text{Atlas}}^2} = \frac{Cr_{\text{Mimas}}^3}{Cr_{\text{Atlas}}^3} = \frac{r_{\text{Mimas}}^3}{r_{\text{Atlas}}^3}$$

Solving for r_{Atlas} yields:

$$r_{\text{Atlas}} = \sqrt[3]{\left(\frac{T_{\text{Atlas}}}{T_{\text{Mimas}}}\right)^2}\,r_{\text{Mimas}}$$

Because for every 3 orbits of Atlas, Mimas has completed 2:

$$r_{\text{Atlas}} = \sqrt[3]{\left(\frac{2}{3}\right)^2}\left(1.86\times10^5\text{ km}\right)$$

$$= \boxed{1.42\times10^5\text{ km}}$$

27 •• The asteroid Icarus, discovered in 1949, was so named because its highly eccentric elliptical orbit brings it close to the Sun at perihelion. The eccentricity e of an ellipse is defined by the relation $r_p = a(1 - e)$, where r_p is the perihelion distance and a is the semimajor axis. Icarus has an eccentricity of 0.83 and a period of 1.1 y. (*a*) Determine the semimajor axis of the orbit of Icarus.

(*b*) Find the perihelion and aphelion distances of the orbit of Icarus.

Picture the Problem Kepler's 3rd law relates the period of Icarus to the length of its semimajor axis. The aphelion distance r_a is related to the perihelion distance r_p and the semimajor axis by $r_a + r_p = 2a$.

(*a*) Using Kepler's 3rd law, relate the period T of Icarus to the length a of its semimajor axis:

$$T^2 = Ca^3 \Rightarrow a = \sqrt[3]{\frac{T^2}{C}}$$

where $C = \dfrac{4\pi^2}{GM_s} = 2.973\times10^{-19}\ \mathrm{s^2/m^3}$.

Substitute numerical values and evaluate a:

$$\boldsymbol{a} = \sqrt[3]{\frac{\left(1.1\,\mathrm{y}\times\dfrac{365.24\,\mathrm{d}}{\mathrm{y}}\times\dfrac{24\,\mathrm{h}}{\mathrm{d}}\times\dfrac{3600\,\mathrm{s}}{\mathrm{h}}\right)^2}{2.973\times10^{-19}\ \mathrm{s^2/m^3}}} = \boxed{1.6\times10^{11}\ \mathrm{m}}$$

(*b*) Use the definition of the eccentricity of an ellipse to determine the perihelion distance of Icarus:

$$r_p = a(1-e) = \left(1.59\times10^{11}\ \mathrm{m}\right)(1-0.83) = 2.71\times10^{10}\ \mathrm{m} = \boxed{2.7\times10^{10}\ \mathrm{m}}$$

Express the relationship between r_p and r_a for an ellipse:

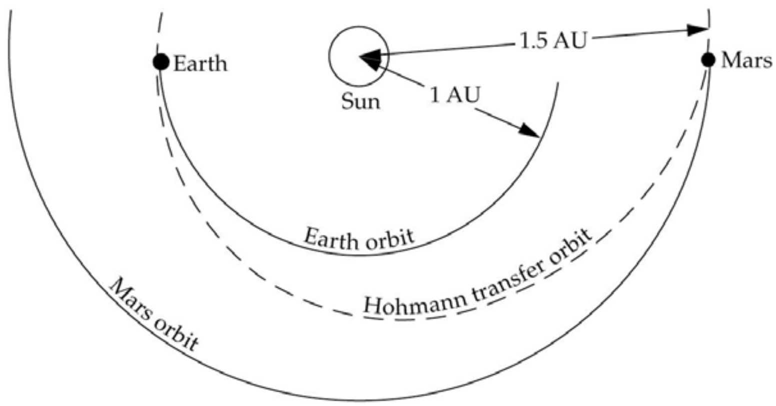
$$r_a + r_p = 2a \Rightarrow r_a = 2a - r_p$$

Substitute numerical values and evaluate r_a:

$$\boldsymbol{r}_a = 2\left(1.59\times10^{11}\ \mathrm{m}\right) - 2.71\times10^{10}\ \mathrm{m} = \boxed{2.9\times10^{11}\ \mathrm{m}}$$

28 •• A manned mission to Mars and its attendant problems due to the extremely long time the astronauts would spend weightless and without supplies have been extensively discussed. To examine this issue in a simple way, consider one possible trajectory for the spacecraft: the "Hohmann transfer orbit." This orbit consists of an elliptical orbit tangent to the orbit of Earth at its perihelion and tangent to the orbit of Mars at its aphelion. Given that Mars has a mean distance from the Sun of 1.52 times the mean Sun–Earth distance, calculate the time spent by the astronauts during the out-bound part of the trip to Mars. Many adverse biological effects (such as muscle atrophy, decreased bone density, etc.) have been observed in astronauts returning from near-Earth orbit after only a few months in space. As the flight doctor, are there any health issues you should be aware of?

Picture the Problem The Hohmann transfer orbit is shown in the diagram. We can apply Kepler's 3rd law to relate the time-in-orbit to the period of the spacecraft in its Hohmann Earth-to-Mars orbit. The period of this orbit is, in turn, a function of its semi-major axis which we can find from the average of the lengths of the semi-major axes of Earth and Mars orbits.



Using Kepler's 3rd law, relate the period T of the spacecraft to the semi-major axis of its orbit:

$$T^2 = R^3 \Rightarrow T = \sqrt{R^3}$$
where T is in years and R is in AU.

Relate the out-bound transit time to the period of this orbit:

$$t_{\text{out-bound}} = \tfrac{1}{2}T = \tfrac{1}{2}\sqrt{R^3}$$

Express the semi-major axis of the Hohmann transfer orbit in terms of the mean sun-Mars and sun-Earth distances:

$$R = \frac{1.52\,\text{AU} + 1.00\,\text{AU}}{2} = 1.26\,\text{AU}$$

Substitute numerical values and evaluate $t_{\text{out-bound}}$:

$$t_{\text{out-bound}} = \tfrac{1}{2}\sqrt{(1.26\,\text{AU})^3} = 0.707\,\text{y} \times \frac{365.24\,\text{d}}{1\,\text{y}} = \boxed{258\ \text{d}}$$

In order for bones and muscles to maintain their health, they need to be under compression as they are on Earth. Due to the long duration (well over a year) of the round trip, you would want to design an exercise program that would maintain the strength of their bones and muscles.

29 •• **[SSM]** Kepler determined distances in the Solar System from his data. For example, he found the relative distance from the Sun to Venus (as compared to the distance from the Sun to Earth) as follows. Because Venus's orbit is closer to the Sun than is Earth's, Venus is a morning or evening star—its position in the sky is never very far from the Sun (see Figure 11-24). If we consider the orbit of Venus as a perfect circle, then consider the relative orientation of Venus, Earth, and the Sun at maximum extension—when Venus is farthest from the Sun in the sky. (*a*) Under this condition, show that angle *b* in Figure 11-24 is 90º. (*b*) If the maximum elongation angle *a* between Venus and the Sun is 47º, what is the distance between Venus and the Sun in AU? (*c*) Use this result to estimate the length of a Venusian ″year.″

Picture the Problem We can use a property of lines tangent to a circle and radii drawn to the point of contact to show that $b = 90°$. Once we've established that b is a right angle we can use the definition of the sine function to relate the distance from the Sun to Venus to the distance from the Sun to Earth.

(*a*) The line from Earth to Venus′ orbit is tangent to the orbit of Venus at the point of maximum extension. Venus will appear closer to the sun in earth's sky when it passes the line drawn from Earth and tangent to its orbit. Hence $b = \boxed{90°}$

(*b*) Using trigonometry, relate the distance from the sun to Venus $d_{\rm SV}$ to the angle *a*:

$$\sin a = \frac{d_{\rm SV}}{d_{\rm SE}} \Rightarrow d_{\rm SV} = d_{\rm SE}\sin a$$

Substitute numerical values and evaluate $d_{\rm SV}$:

$$d_{\rm SV} = (1.00\,{\rm AU})\sin 47° = 0.731\,{\rm AU}$$
$$= \boxed{0.73\,{\rm AU}}$$

(*c*) Use Kepler's 3rd law to relate Venus's orbital period to its mean distance from the Sun:

$$\boldsymbol{T}^2_{\rm Venus} = \boldsymbol{Cr}^3_{\rm Venus}$$

Use Kepler's 3rd law to relate Earth's orbital period to its mean distance from the Sun:

$$\boldsymbol{T}^2_{\rm Earth} = \boldsymbol{Cr}^3_{\rm Earth}$$

Dividing the first of these equations by the second yields:

$$\frac{\boldsymbol{T}^2_{\rm Venus}}{\boldsymbol{T}^2_{\rm Earth}} = \frac{\boldsymbol{Cr}^3_{\rm Venus}}{\boldsymbol{Cr}^3_{\rm Earth}} = \frac{\boldsymbol{r}^3_{\rm Venus}}{\boldsymbol{r}^3_{\rm Earth}}$$

Solving for $T_{\rm Venus}$ yields:

$$T_{\rm Venus} = \sqrt{\left(\frac{r_{\rm Venus}}{r_{\rm Earth}}\right)^3}\,T_{\rm Earth}$$

Using the result from part (*b*) yields:

$$\boldsymbol{T}_{\rm Venus} = \sqrt{\left(\frac{0.731\,{\rm AU}}{1.00\,{\rm AU}}\right)^3}\,(1.00\ {\rm y})$$
$$= \boxed{0.63\ {\rm y}}$$

Remarks: The correct distance from the sun to Venus is closer to 0.723 AU.

30 •• At apogee the moon is 406,395 km from Earth and at perigee it is 357,643 km. What is the orbital speed of the moon at perigee and at apogee? Its orbital period is 27.3 d.

Picture the Problem Because the gravitational force Earth exerts on the moon is along the line joining their centers, the net torque acting on the moon is zero and its angular momentum is conserved in its orbit about Earth. Because energy is also conserved, we can combine these two expressions to solve for either v_p or v_a initially and then use conservation of angular momentum to find the other.

Letting m be the mass of the moon, apply conservation of angular momentum to the moon at apogee and perigee to obtain:

$$mv_pr_p = mv_ar_a \Rightarrow v_a = \frac{r_p}{r_a}v_p$$

Apply conservation of energy to the moon-Earth system to obtain:

$$\tfrac{1}{2}mv_p^2 - \frac{GMm}{r_p} = \tfrac{1}{2}mv_a^2 - \frac{GMm}{r_a}$$

or

$$\tfrac{1}{2}v_p^2 - \frac{GM}{r_p} = \tfrac{1}{2}v_a^2 - \frac{GM}{r_a}$$

Substitute for v_a to obtain:

$$\tfrac{1}{2}v_p^2 - \frac{GM}{r_p} = \tfrac{1}{2}\left(\frac{r_p}{r_a}v_p\right)^2 - \frac{GM}{r_a}$$
$$= \tfrac{1}{2}\left(\frac{r_p}{r_a}\right)^2 v_p^2 - \frac{GM}{r_a}$$

Solving for v_p yields:

$$v_p = \sqrt{\frac{2GM}{r_p}\left(\frac{1}{1 + r_p/r_a}\right)}$$

Substitute numerical values and evaluate v_p:

$$v_p = \sqrt{\frac{2\left(6.673\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2\right)\left(5.98\times10^{24}\ \text{kg}\right)}{3.576\times10^{8}\ \text{m}}\left(\frac{1}{1+\dfrac{3.576\times10^{8}\ \text{m}}{4.064\times10^{8}\ \text{m}}}\right)} = \boxed{1.09\ \text{km/s}}$$

Substitute numerical values in equation (1) and evaluate v_a:

$$\boldsymbol{v_a} = \frac{3.576\times10^8\ \text{m}}{4.064\times10^8\ \text{m}}(1.09\,\text{km/s}) = \boxed{959\ \text{m/s}}$$

Newton's Law of Gravity

31 • **[SSM]** Jupiter's satellite Europa orbits Jupiter with a period of 3.55 d at an average orbital radius of 6.71×10^8 m. (*a*) Assuming that the orbit is circular, determine the mass of Jupiter from the data given. (*b*) Another satellite of Jupiter, Callisto, orbits at an average radius of 18.8×10^8 m with an orbital period of 16.7 d. Show that this data is consistent with an inverse square force law for gravity (*Note*: DO NOT use the value of *G* anywhere in Part (*b*)).

Picture the Problem While we could apply Newton's Law of Gravitation and 2nd Law of Motion to solve this problem from first principles, we'll use Kepler's 3rd law (derived from these laws) to find the mass of Jupiter in Part (*a*). In Part (*b*) we can compare the ratio of the centripetal accelerations of Europa and Callisto to show that they are consistent with an inverse square law for gravity.

(*a*) Assuming a circular orbit, apply Kepler's 3rd law to the motion of Europa to obtain:

$$T_{\text{E}}^2 = \frac{4\pi^2}{GM_{\text{J}}}R_{\text{E}}^3 \Rightarrow M_{\text{J}} = \frac{4\pi^2}{GT_{\text{E}}^2}R_{\text{E}}^3$$

Substitute numerical values and evaluate M_{J}:

$$M_{\text{J}} = \frac{4\pi^2(6.71\times10^8\ \text{m})^3}{(6.673\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)\left(3.55\,\text{d}\times\frac{24\,\text{h}}{\text{d}}\times\frac{3600\,\text{s}}{\text{h}}\right)^2} = \boxed{1.90\times10^{27}\ \text{kg}}$$

Note that this result is in excellent agreement with the accepted value of 1.902×10^{27} kg.

(*b*) Express the centripetal acceleration of both of the moons to obtain:

$$a_{\text{centripetal}} = \frac{v^2}{R} = \frac{\left(\frac{2\pi R}{T}\right)^2}{R} = \frac{4\pi^2 R}{T^2}$$

where *R* and *T* are the radii and periods of their motion.

Using this result, express the centripetal accelerations of Europa and Callisto:

$$a_{\text{E}} = \frac{4\pi^2 R_{\text{E}}}{T_{\text{E}}^2} \text{ and } a_{\text{C}} = \frac{4\pi^2 R_{\text{C}}}{T_{\text{C}}^2}$$

Divide the first of these equations by the second and simplify to obtain:

$$\frac{a_{\mathrm{E}}}{a_{\mathrm{C}}}=\frac{\dfrac{4\pi^2 R_{\mathrm{E}}}{T_{\mathrm{E}}^2}}{\dfrac{4\pi^2 R_{\mathrm{C}}}{T_{\mathrm{C}}^2}}=\frac{T_{\mathrm{C}}^2}{T_{\mathrm{E}}^2}\frac{R_{\mathrm{E}}}{R_{\mathrm{C}}}$$

Substitute for the periods of Callisto and Europa using Kepler's 3rd law to obtain:

$$\frac{a_{\mathrm{E}}}{a_{\mathrm{C}}}=\frac{CR_{\mathrm{C}}^3}{CR_{\mathrm{E}}^3}\frac{R_{\mathrm{E}}}{R_{\mathrm{C}}}=\frac{R_{\mathrm{C}}^2}{R_{\mathrm{E}}^2}$$

This result, together with the fact that the gravitational force is directly proportional to the acceleration of the moons, demonstrates that the gravitational force varies inversely with the square of the distance.

32 • Some people think that shuttle astronauts are ″weightless″ because they are ″beyond the pull of Earth's gravity. ″ In fact, this is completely untrue. (*a*) What is the magnitude of the gravitational field in the vicinity of a shuttle orbit? A shuttle orbit is about 400 km above the ground. (*b*) Given the answer in Part (*a*), explain why shuttle astronauts do suffer from adverse biological affects such as muscle atrophy even though they are actually not ″weightless″?

Determine the Concept The weight of anything, including astronauts, is the reading of a scale from which the object is suspended or on which it rests. That is, it is the magnitude of the normal force acting on the object. If the scale reads zero, then we say the object is ″weightless.″ The pull of Earth's gravity, on the other hand, depends on the local value of the acceleration of gravity and we can use Newton's law of gravity to find this acceleration at the elevation of the shuttle.

(*a*) Apply Newton's law of gravitation to an astronaut of mass *m* in a shuttle at a distance *h* above the surface of Earth:

$$mg_{\text{shuttle}}=\frac{GmM_{\mathrm{E}}}{(h+R_{\mathrm{E}})^2}$$

Solving for g_{shuttle} yields:

$$g_{\text{shuttle}}=\frac{GM_{\mathrm{E}}}{(h+R_{\mathrm{E}})^2}$$

Substitute numerical values and evaluate g_{shuttle}:

$$g_{\text{shuttle}}=\frac{\left(6.673\times10^{-11}\ \mathrm{N\cdot m^2/kg^2}\right)\left(5.98\times10^{24}\ \mathrm{kg}\right)}{(400\ \mathrm{km}+6370\ \mathrm{km})^2}=\boxed{8.71\,\mathrm{m/s^2}}$$

(*b*) In orbit, the astronauts experience only one (the gravitational force) of the two forces (the second being the normal force – a compressive force – exerted by Earth) that normally acts on them. Lacking this compressive force, their bones and muscles, the absence of an exercise program, will weaken. In orbit the astronauts are not weightless, they are normal-forceless.

33 • **[SSM]** The mass of Saturn is 5.69×10^{26} kg. (*a*) Find the period of its moon Mimas, whose mean orbital radius is 1.86×10^{8} m. (*b*) Find the mean orbital radius of its moon Titan, whose period is 1.38×10^{6} s.

Picture the Problem While we could apply Newton's Law of Gravitation and 2nd Law of Motion to solve this problem from first principles, we'll use Kepler's 3rd law (derived from these laws) to find the period of Mimas and to relate the periods of the moons of Saturn to their mean distances from its center.

(*a*) Using Kepler's 3rd law, relate the period of Mimas to its mean distance from the center of Saturn:

$$T_M^2 = \frac{4\pi^2}{GM_S} r_M^3 \Rightarrow T_M = \sqrt{\frac{4\pi^2}{GM_S} r_M^3}$$

Substitute numerical values and evaluate T_M:

$$T_M = \sqrt{\frac{4\pi^2 (1.86\times10^8\ \text{m})^3}{(6.6726\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(5.69\times10^{26}\ \text{kg})}} = 8.18\times10^4\ \text{s} \approx \boxed{22.7\ \text{h}}$$

(*b*) Using Kepler's 3rd law, relate the period of Titan to its mean distance from the center of Saturn:

$$T_T^2 = \frac{4\pi^2}{GM_S} r_T^3 \Rightarrow r_T = \sqrt[3]{\frac{T_T^2 GM_S}{4\pi^2}}$$

Substitute numerical values and evaluate r_T:

$$r_T = \sqrt[3]{\frac{(1.38\times10^6\ \text{s})^2 (6.6726\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(5.69\times10^{26}\ \text{kg})}{4\pi^2}} = \boxed{1.22\times10^9\ \text{m}}$$

34 • Calculate the mass of Earth from the period of the moon, $T = 27.3$ d; its mean orbital radius, $r_m = 3.84 \times 10^8$ m; and the known value of G.

Picture the Problem While we could apply Newton's law of gravitation and 2nd law of motion to solve this problem from first principles, we'll use Kepler's 3rd law (derived from these laws) to relate the period of the moon to the mass of Earth and the mean Earth-moon distance.

Using Kepler's 3rd law, relate the period of the moon to its mean orbital radius:

$$T_{\rm m}^2 = \frac{4\pi^2}{GM_{\rm E}} r_{\rm m}^3 \Rightarrow M_{\rm E} = \frac{4\pi^2}{GT_{\rm m}^2} r_{\rm m}^3$$

Substitute numerical values and evaluate $M_{\rm E}$:

$$M_{\rm E} = \frac{4\pi^2\left(3.84\times10^8\ \rm m\right)^3}{\left(6.6726\times10^{-11}\ \rm N\cdot m^2/kg^2\right)\left(27.3\,d\times\frac{24\,h}{d}\times\frac{3600\ s}{h}\right)^2} = \boxed{6.02\times10^{24}\ \rm kg}$$

Remarks: This analysis neglects the mass of the moon; consequently the mass calculated here is slightly too large.

35 • Suppose you leave the Solar System and arrive at a planet that has the same mass-to-volume ratio as Earth but has 10 times Earth's radius. What would you weigh on this planet compared with what you weigh on Earth?

Picture the Problem Your weight is the local gravitational force exerted on you. We can use the definition of density to relate the mass of the planet to the mass of Earth and the law of gravity to relate your weight on the planet to your weight on Earth.

Using the definition of density, relate the mass of Earth to its radius:

$$M_{\rm E} = \rho V_{\rm E} = \tfrac{4}{3}\rho\pi R_{\rm E}^3$$

Relate the mass of the planet to its radius:

$$M_{\rm P} = \rho V_{\rm P} = \tfrac{4}{3}\rho\pi R_{\rm P}^3 = \tfrac{4}{3}\rho\pi\left(10R_{\rm E}\right)^3$$

Divide the second of these equations by the first to express $M_{\rm P}$ in terms of $M_{\rm E}$:

$$\frac{M_{\rm P}}{M_{\rm E}} = \rho\frac{\tfrac{4}{3}\rho\pi\left(10R_{\rm E}\right)^3}{\tfrac{4}{3}\rho\pi R_{\rm E}^3} \Rightarrow M_{\rm P} = 10^3 M_{\rm E}$$

Letting w' represent your weight on the planet, use the law of gravity to relate w' to your weight on Earth:

$$w' = \frac{GmM_{\rm P}}{R_{\rm P}^2} = \frac{Gm\left(10^3 M_{\rm E}\right)}{\left(10R_{\rm E}\right)^2} = 10\frac{GmM_{\rm E}}{R_{\rm E}^2} = 10w$$

Your weight would be ten times your weight on Earth.

36 • Suppose that Earth retained its present mass but was somehow compressed to half its present radius. What would be the value of g at the surface of this new, compact planet?

Picture the Problem We can relate the acceleration due to gravity of a test object at the surface of the new planet to the acceleration due to gravity at the surface of Earth through use of the law of gravity and Newton's 2nd law of motion.

Letting a represent the acceleration due to gravity at the surface of this new planet and m the mass of a test object, apply Newton's 2nd law and the law of gravity to obtain:

$$\sum F_{\text{radial}} = \frac{GmM_{\text{E}}}{\left(\frac{1}{2}R_{\text{E}}\right)^2} = ma \Rightarrow a = \frac{GM_{\text{E}}}{\left(\frac{1}{2}R_{\text{E}}\right)^2}$$

Simplify this expression to obtain:

$$a = 4\left(\frac{GM_{\text{E}}}{R_{\text{E}}^2}\right) = 4g = \boxed{39.2\,\text{m/s}^2}$$

37 • A planet orbits a massive star. When the planet is at perihelion, it has a speed of 5.0×10^4 m/s and is 1.0×10^{15} m from the star. The orbital radius increases to 2.2×10^{15} m at aphelion. What is the planet's speed at aphelion?

Picture the Problem We can use conservation of angular momentum to relate the planet's speeds at aphelion and perihelion.

Using conservation of angular momentum, relate the angular momenta of the planet at aphelion and perihelion:

$$L_{\text{a}} = L_{\text{p}}$$

or

$$mv_{\text{p}}r_{\text{p}} = mv_{\text{a}}r_{\text{a}} \Rightarrow v_{\text{a}} = \frac{v_{\text{p}}r_{\text{p}}}{r_{\text{a}}}$$

Substitute numerical values and evaluate v_{a}:

$$\boldsymbol{v}_{\text{a}} = \frac{\left(5.0\times10^4\ \text{m/s}\right)\left(1.0\times10^{15}\ \text{m}\right)}{2.2\times10^{15}\ \text{m}} = \boxed{2.3\times10^4\ \text{m/s}}$$

38 • What is the magnitude of the gravitational field at the surface of a neutron star whose mass is 1.60 times the mass of the Sun and whose radius is 10.5 km?

Picture the Problem We can use Newton's law of gravity to express the gravitational force acting on an object at the surface of the neutron star in terms of the weight of the object. We can then simplify this expression be dividing out the mass of the object … leaving an expression for the magnitude of the gravitational field at the surface of the neutron star.

Apply Newton's law of gravity to an object of mass *m* at the surface of the neutron star to obtain:

$$\frac{GM_{\text{Neutron Star}}m}{R^2_{\text{Neutron Star}}} = mg$$

where *g* represents the magnitude of the gravitational field at the surface of the neutron star.

Solve for *g* and substitute for the mass of the neutron star:

$$g = \frac{GM_{\text{Neutron Star}}}{R^2_{\text{Neutron Star}}} = \frac{G(1.60M_{\text{sun}})}{R^2_{\text{Neutron Star}}}$$

Substitute numerical values and evaluate *g*:

$$g = \frac{1.60(6.673\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(1.99\times10^{30}\ \text{kg})}{(10.5\ \text{km})^2} = \boxed{1.93\times10^{12}\ \text{m/s}^2}$$

39 •• The speed of an asteroid is 20 km/s at perihelion and 14 km/s at aphelion. (*a*) Determine the ratio of the aphelion to perihelion distances. (*b*) Is this asteroid farther from the Sun or closer to the Sun than Earth, on average? Explain.

Picture the Problem We can use conservation of angular momentum to relate the asteroid's aphelion and perihelion distances.

(*a*) Using conservation of angular momentum, relate the angular momenta of the asteroid at aphelion and perihelion:

$$\boldsymbol{L}_{\text{a}} - \boldsymbol{L}_{\text{p}} = 0$$

or

$$mv_{\text{a}}r_{\text{a}} - mv_{\text{p}}r_{\text{p}} = 0 \Rightarrow \frac{\boldsymbol{r}_{\text{a}}}{\boldsymbol{r}_{\text{p}}} = \frac{\boldsymbol{v}_{\text{p}}}{\boldsymbol{v}_{\text{a}}}$$

Substitute numerical values and evaluate the ratio of the asteroid's aphelion and perihelion distances:

$$\frac{\boldsymbol{r}_{\text{a}}}{\boldsymbol{r}_{\text{p}}} = \frac{20\,\text{km/s}}{14\,\text{km/s}} = \boxed{1.4}$$

(*b*) It is farther from the Sun than Earth. Kepler's third law ($T^2 = Cr^3_{\text{av}}$) tells us that longer orbital periods together with larger orbital radii means slower orbital speeds, so the speed of objects orbiting the Sun decreases with distance from the Sun. The average orbital speed of Earth, given by $v = 2\pi r_{\text{ES}}/T_{\text{ES}}$, is approximately 30 km/s. Because the given maximum speed of the asteroid is only 20 km/s, the asteroid is further from the Sun.

40 •• A satellite with a mass of 300 kg moves in a circular orbit 5.00×10^7 m above Earth's surface. (*a*) What is the gravitational force on the satellite? (*b*) What is the speed of the satellite? (*c*) What is the period of the satellite?

Picture the Problem We'll use the law of gravity to find the gravitational force acting on the satellite. The application of Newton's 2nd law will lead us to the speed of the satellite and its period can be found from its definition.

(*a*) Letting *m* represent the mass of the satellite and *h* its elevation, use the law of gravity to express the gravitational force acting on it:

$$F_g = \frac{GmM_E}{(R_E + h)^2}$$

Because $GM_E = gR_E^2$:

$$F_g = \frac{mR_E^2 g}{(R_E + h)^2}$$

Divide the numerator and denominator of this expression by R_E^2 to obtain:

$$F_g = \frac{mg}{\left(1 + \frac{h}{R_E}\right)^2}$$

Substitute numerical values and evaluate F_g:

$$F_g = \frac{(300\,\text{kg})(9.81\,\text{N/kg})}{\left(1 + \frac{5.00\times10^7\,\text{m}}{6.37\times10^6\,\text{m}}\right)^2} = 37.58\,\text{N}$$

$$= \boxed{37.6\,\text{N}}$$

(*b*) Using Newton's 2nd law, relate the gravitational force acting on the satellite to its centripetal acceleration:

$$F_g = m\frac{v^2}{r} \Rightarrow v = \sqrt{\frac{F_g r}{m}}$$

Substitute numerical values and evaluate *v*:

$$\boldsymbol{v} = \sqrt{\frac{(37.58\,\text{N})(6.37\times10^6\,\text{m} + 5.00\times10^7\,\text{m})}{300\,\text{kg}}} = 2.657\ \text{km/s} = \boxed{2.66\ \text{km/s}}$$

(*c*) The period of the satellite is given by:

$$T = \frac{2\pi r}{v}$$

Substitute numerical values and evaluate *T*:

$$\boldsymbol{T} = \frac{2\boldsymbol{\pi}(6.37\times10^6\,\text{m} + 5.00\times10^7\,\text{m})}{2.657\times10^3\,\text{m/s}}$$

$$= 1.333\times10^5\,\text{s}\times\frac{1\,\text{h}}{3600\,\text{s}} = \boxed{37.0\,\text{h}}$$

41 •• **[SSM]** A superconducting gravity meter can measure changes in gravity of the order $\Delta g/g = 1.00 \times 10^{-11}$. (*a*) You are hiding behind a tree holding the meter, and your 80-kg friend approaches the tree from the other side. How close to you can your friend get before the meter detects a change in *g* due to his presence? (*b*) You are in a hot air balloon and are using the meter to determine the rate of ascent (assume the balloon has constant acceleration). What is the smallest change in altitude that results in a detectable change in the gravitational field of Earth?

Picture the Problem We can determine the maximum range at which an object with a given mass can be detected by substituting the equation for the gravitational field in the expression for the resolution of the meter and solving for the distance. Differentiating $g(r)$ with respect to r, separating variables to obtain dg/g, and approximating Δr with dr will allow us to determine the vertical change in the position of the gravity meter in Earth's gravitational field is detectable.

(*a*) Express the gravitational field of Earth:

$$g_E = \frac{GM_E}{R_E^2}$$

Express the gravitational field due to the mass *m* (assumed to be a point mass) of your friend and relate it to the resolution of the meter:

$$g(r) = \frac{Gm}{r^2} = 1.00\times10^{-11}\, g_E = 1.00\times10^{-11}\,\frac{GM_E}{R_E^2}$$

Solving for *r* yields:

$$r = R_E\sqrt{\frac{1.00\times10^{11}\,m}{M_E}}$$

Substitute numerical values and evaluate *r*:

$$\boldsymbol{r} = \left(6.37\times10^6\text{ m}\right)\sqrt{\frac{1.00\times10^{11}(80\text{ kg})}{5.98\times10^{24}\text{ kg}}} = \boxed{7.37\text{ m}}$$

(*b*) Differentiate $g(r)$ and simplify to obtain:

$$\frac{dg}{dr} = \frac{-2Gm}{r^3} = -\frac{2}{r}\left(\frac{Gm}{r^2}\right) = -\frac{2}{r}g$$

Separate variables to obtain:

$$\frac{dg}{g} = -2\frac{dr}{r} = 10^{-11}$$

Approximating dr with Δr, evaluate Δr with $r = R_E$:

$$\Delta \boldsymbol{r} = \left|-\tfrac{1}{2}\left(1.00\times10^{-11}\right)\left(6.37\times10^6\text{ m}\right)\right| = \boxed{31.9\,\mu\text{m}}$$

42 •• Suppose that the attractive interaction between a star of mass M and a planet of mass $m \ll M$ is of the form $F = KMm/r$, where K is the gravitational constant. What would be the relation between the radius of the planet's circular orbit and its period?

Picture the Problem We can use the law of gravity and Newton's 2nd law to relate the force exerted on the planet by the star to its orbital speed and the definition of the period to relate it to the radius of the orbit.

The period of the planet is related to its orbital speed:

$$T = \frac{2\pi r}{v} \quad (1)$$

Using the law of gravity and Newton's 2nd law, relate the force exerted on the planet by the star to its centripetal acceleration:

$$F_{\text{net}} = \frac{KMm}{r} = m\frac{v^2}{r} \Rightarrow v = \sqrt{KM}$$

Substitute for v in equation (1) to obtain:

$$T = \boxed{\frac{2\pi}{\sqrt{KM}}r}$$

43 •• **[SSM]** Earth's radius is 6370 km and the moon's radius is 1738 km. The acceleration of gravity at the surface of the moon is 1.62 m/s^2. What is the ratio of the average density of the moon to that of Earth?

Picture the Problem We can use the definitions of the gravitational fields at the surfaces of Earth and the moon to express the accelerations due to gravity at these locations in terms of the average densities of Earth and the moon. Expressing the ratio of these accelerations will lead us to the ratio of the densities.

Express the acceleration due to gravity at the surface of Earth in terms of Earth's average density:

$$g_{\text{E}} = \frac{GM_{\text{E}}}{R_{\text{E}}^2} = \frac{G\rho_{\text{E}}V_{\text{E}}}{R_{\text{E}}^2} = \frac{G\rho_{\text{E}}\,\frac{4}{3}\pi R_{\text{E}}^3}{R_{\text{E}}^2} = \tfrac{4}{3}G\rho_{\text{E}}\pi R_{\text{E}}$$

The acceleration due to gravity at the surface of the moon in terms of the moon's average density is:

$$g_{\text{M}} = \tfrac{4}{3}G\rho_{\text{M}}\pi R_{\text{M}}$$

Divide the second of these equations by the first to obtain:

$$\frac{g_{\text{M}}}{g_{\text{E}}} = \frac{\rho_{\text{M}}R_{\text{M}}}{\rho_{\text{E}}R_{\text{E}}} \Rightarrow \frac{\rho_{\text{M}}}{\rho_{\text{E}}} = \frac{g_{\text{M}}R_{\text{E}}}{g_{\text{E}}R_{\text{M}}}$$

Substitute numerical values and evaluate $\frac{\rho_M}{\rho_E}$:

$$\frac{\rho_M}{\rho_E}=\frac{(1.62\,\text{m/s}^2)(6.37\times10^6\,\text{m})}{(9.81\,\text{m/s}^2)(1.738\times10^6\,\text{m})}=\boxed{0.605}$$

Gravitational and Inertial Mass

44 • The weight of a standard object defined as having a mass of exactly 1.00… kg is measured to be 9.81 N. In the same laboratory, a second object weighs 56.6 N. (*a*) What is the mass of the second object? (*b*) Is the mass you determined in Part (*a*) gravitational or inertial mass?

Picture the Problem Newton's 2nd law of motion relates the weights of these two objects to their masses and the acceleration due to gravity.

(*a*) Apply Newton's 2nd law to the standard object:

$$F_{\text{net}}=w_1=m_1g$$

Apply Newton's 2nd law to the object of unknown mass:

$$F_{\text{net}}=w_2=m_2g$$

Eliminate g between these two equations and solve for m_2:

$$m_2=\frac{w_2}{w_1}m_1$$

Substitute numerical values and evaluate m_2:

$$\boldsymbol{m}_2=\frac{56.6\,\text{N}}{9.81\,\text{N}}(1.00\,\text{kg})=\boxed{5.77\,\text{kg}}$$

(*b*) Because this result is determined by the effect on m_2 of Earth's gravitational field, it is the *gravitational* mass of the second object.

45 • The *Principle of Equivalence* states that the free-fall acceleration of any object in a gravitational field is independent of the mass of the object. This can be deduced from the law of universal gravitation, but how well does it hold up experimentally? The Roll-Krotkov-Dicke experiment performed in the 1960s indicates that the free-fall acceleration is independent of mass to at least 1 part in 10^{12}. Suppose two objects are simultaneously released from rest in a uniform gravitational field. Also, suppose one of the objects falls with a constant acceleration of exactly 9.81 m/s^2 while the other falls with a constant acceleration that is greater than 9.81 m/s^2 by one part in 10^{12}. How far will the first object have fallen when the second object has fallen 1.00 mm farther than it has? Note that this estimate provides only an upper bound on the difference in the accelerations; most physicists believe that there is no difference in the accelerations.

Picture the Problem Noting that $g_1 \sim g_2 \sim g$, let the acceleration of gravity on the first object be g_1, and on the second be g_2. We can use a constant-acceleration equation to express the difference in the distances fallen by each object and then relate the average distance fallen by the two objects to obtain an expression from which we can approximate the distance they would have to fall before we might measure a difference in their fall distances greater than 1 mm.

Express the difference Δd in the distances fallen by the two objects in time t:

$$\Delta d = d_1 - d_2$$

Express the distances fallen by each of the objects in time t:

$$d_1 = \tfrac{1}{2} g_1 t^2 \text{ and } d_2 = \tfrac{1}{2} g_2 t^2$$

Substitute for d_1 and d_2 to obtain:

$$\Delta d = \tfrac{1}{2} g_1 t^2 - \tfrac{1}{2} g_2 t^2 = \tfrac{1}{2}\left(g_1 - g_2\right) t^2$$

Relate the average distance d fallen by the two objects to their time of fall:

$$d = \tfrac{1}{2} g t^2 \Rightarrow t^2 = \frac{2d}{g}$$

Substitute for t^2 to obtain:

$$\Delta d \approx \tfrac{1}{2} \Delta g \frac{2d}{g} = d \frac{\Delta g}{g} \Rightarrow d = \Delta d \frac{g}{\Delta g}$$

Substitute numerical values and evaluate d:

$$d = \left(10^{-3}\text{ m}\right)\left(10^{12}\right) = \boxed{10^9\text{ m}}$$

Gravitational Potential Energy

46 • (*a*) If we take the potential energy of a 100-kg and Earth to be zero when the two are separated by an infinite distance, what is the potential energy when the object is at the surface of Earth? (*b*) Find the potential energy of the same object at a height above Earth's surface equal to Earth's radius. (*c*) Find the escape speed for a body projected from this height.

Picture the Problem Choosing the zero of gravitational potential energy to be at infinite separation yields, as the potential energy of a two-body system in which the objects are separated by a distance r, $U(r) = -GMm/r$, where M and m are the masses of the two bodies. In order for an object to just escape a gravitational field from a particular location, it must have enough kinetic energy so that its total energy is zero.

(*a*) Letting $U(\infty) = 0$, express the gravitational potential energy of Earth-object system:

$$U(r) = -\frac{GM_{\text{E}}m}{r} \qquad (1)$$

Substitute for GM_E and simplify to obtain:

$$U(R_E) = -\frac{GM_E m}{R_E} = -\frac{gR_E^2 m}{R_E} = -mgR_E$$

Substitute numerical values and evaluate $U(R_E)$:

$$U(R_E) = -(100\,\text{kg})(9.81\,\text{N/kg})(6.37\times10^6\,\text{m}) = \boxed{-6.25\times10^9\,\text{J}}$$

(*b*) Evaluate equation (1) with $r = 2R_E$:

$$U(2R_E) = -\frac{GM_E m}{2R_E} = -\frac{gR_E^2 m}{2R_E} = -\tfrac{1}{2}mgR_E$$

Substitute numerical values and evaluate $U(2R_E)$:

$$\boldsymbol{U}(2\boldsymbol{R}_E) = -\tfrac{1}{2}(100\,\text{kg})(9.81\,\text{N/kg})(6.37\times10^6\,\text{m}) = -3.124\times10^9\,\text{J} = \boxed{-3.12\times10^9\,\text{J}}$$

(*c*) Express the condition that an object must satisfy in order to escape from Earth's gravitational field from a height R_E above its surface:

$$K_e(2R_E) + U(2R_E) = 0$$

or

$$\tfrac{1}{2}mv_e^2 + U(2R_E) = 0$$

Solving for v_e yields:

$$v_e = \sqrt{\frac{-2U(2R_E)}{m}}$$

Substitute numerical values and evaluate v_e:

$$v_e = \sqrt{\frac{-2(-3.124\times10^9\,\text{J})}{100\,\text{kg}}} = \boxed{7.90\,\text{km/s}}$$

47 • **[SSM]** Find the escape speed for a projectile leaving the surface of the moon. The acceleration of gravity on the moon is 0.166 times that on Earth and the moon's radius is 0.273 R_E.

Picture the Problem The escape speed from the moon or Earth is given by $v_e = \sqrt{2GM/R}$, where M and R represent the masses and radii of the moon or Earth.

Express the escape speed from the moon:

$$v_{e.m} = \sqrt{\frac{2GM_m}{R_m}} = \sqrt{2g_m R_m} \qquad (1)$$

Express the escape speed from Earth:

$$v_{\mathrm{e.E}}=\sqrt{\frac{2GM_{\mathrm{E}}}{R_{\mathrm{E}}}}=\sqrt{2g_{\mathrm{E}}R_{\mathrm{E}}} \qquad (2)$$

Divide equation (1) by equation (2) to obtain:

$$\frac{v_{\mathrm{e.m}}}{v_{\mathrm{e.E}}}=\frac{\sqrt{g_{\mathrm{m}}R_{\mathrm{m}}}}{\sqrt{g_{\mathrm{E}}R_{\mathrm{E}}}}=\sqrt{\frac{g_{\mathrm{m}}R_{\mathrm{m}}}{g_{\mathrm{E}}R_{\mathrm{E}}}}$$

Solving for $v_{\mathrm{e,m}}$ yields:

$$v_{\mathrm{e.m}}=\sqrt{\frac{g_{\mathrm{m}}R_{\mathrm{m}}}{g_{\mathrm{E}}R_{\mathrm{E}}}}v_{\mathrm{e.E}}$$

Substitute numerical values and evaluate $v_{\mathrm{e,m}}$:

$$v_{\mathrm{e.m}}=\sqrt{(0.166)(0.273)}(11.2\,\mathrm{km/s})=\boxed{2.38\,\mathrm{km/s}}$$

48 •• What initial speed would a particle have to be given at the surface of Earth if it is to have a final speed that is equal to its escape speed when it is very far from Earth? Neglect any effects due to air resistance.

Picture the Problem Let the zero of gravitational potential energy be at infinity, *m* represent the mass of the particle, and the subscript E refer to Earth. When the particle is very far from Earth, the gravitational potential energy of the Earth-particle system is zero. We'll use conservation of energy to relate the initial potential and kinetic energies of the particle-Earth system to the final kinetic energy of the particle.

Use conservation of energy to relate the initial energy of the system to its energy when the particle is very far away:

$$K_{\mathrm{f}}-K_{\mathrm{i}}+U_{\mathrm{f}}-U_{\mathrm{i}}=0$$

or, because $U_{\mathrm{f}}=0$,

$$K(\infty)-K(R_{\mathrm{E}})-U(R_{\mathrm{E}})=0 \qquad (1)$$

Substitute in equation (1) to obtain:

$$\tfrac{1}{2}mv_\infty^2-\tfrac{1}{2}mv_{\mathrm{i}}^2+\frac{GM_{\mathrm{E}}m}{R_{\mathrm{E}}}=0$$

or, because $GM_{\mathrm{E}}=gR_{\mathrm{E}}^2$,

$$\tfrac{1}{2}mv_\infty^2-\tfrac{1}{2}mv_{\mathrm{i}}^2+mgR_{\mathrm{E}}=0$$

Solving for v_{i} yields:

$$v_{\mathrm{i}}=\sqrt{v_\infty^2+2gR_{\mathrm{E}}}$$

Substitute numerical values and evaluate v_i:

$$v_i = \sqrt{(11.2\times10^3\text{ m/s})^2 + 2(9.81\text{ m/s}^2)(6.37\times10^6\text{ m})} = \boxed{15.8\text{ km/s}}$$

49 •• While trying to work out its budget for the next fiscal years, NASA would like to report to the nation a rough estimate of the cost (per kilogram) to launch a modern satellite into near-Earth orbit. You are chosen for this task, because you know physics and accounting. (*a*) Determine the energy, in kW·h, necessary to place 1.0-kg object in low-Earth orbit. In low-Earth orbit, the height of the object above the surface of Earth is much smaller than Earth's radius. Take the orbital height to be 300 km. (*b*) If this energy can be obtained at a typical electrical energy rate of \$0.15/kW·h, what is the minimum cost of launching a 400-kg satellite into low-Earth orbit? Neglect any effects due to air resistance.

Picture the Problem We can use the expression for the total energy of a satellite to find the energy required to place in a low-Earth orbit.

(*a*) The total energy of a satellite in a low-Earth orbit is given by:

$$E = K + U_g = \tfrac{1}{2}U_g$$

Substituting for U_g yields:

$$E = -\frac{GM_{\text{Earth}}m_{\text{satellite}}}{2r}$$

where r is the orbital radius and the minus sign indicates the satellite is bound to Earth.

For a near-Earth orbit, $r \approx R_{\text{Earth}}$ and the amount of energy required to place the satellite in orbit becomes:

$$E = \frac{GM_{\text{Earth}}m_{\text{satellite}}}{2R_{\text{Earth}}}$$

Substitute numerical values and evaluate E:

$$\boldsymbol{E} = -\frac{(6.673\times10^{-11}\text{ N}\cdot\text{m}^2/\text{kg}^2)(5.98\times10^{24}\text{ kg})(1.0\text{ kg})}{2(6.37\times10^6\text{ m})} = 31.31\text{ MJ}\times\frac{1\text{ kW}\cdot\text{h}}{3.6\text{ MJ}}$$

$$= \boxed{8.7\text{ kW}\cdot\text{h}}$$

(*b*) Express the cost of this project in terms of the mass of the satellite:

$$\text{Cost} = \text{rate}\times\frac{\text{required energy}}{\text{kg}}\times \boldsymbol{m}_{\text{satellite}}$$

Substitute numerical values and find the cost:

$$\text{Cost} = \frac{\$0.15}{\text{kW}\cdot\text{h}} \times \frac{8.7\,\text{kW}\cdot\text{h}}{\text{kg}}(400\,\text{kg})$$
$$\approx \boxed{\$500}$$

50 •• The science fiction writer Robert Heinlein once said, "If you can get into orbit, then you're halfway to anywhere." Justify this statement by comparing the minimum energy needed to place a satellite into low Earth orbit (h = 400 km) to that needed to set it completely free from the bonds of Earth's gravity. Neglect any effects of air resistance.

Picture the Problem We'll consider a rocket of mass m which is initially on the surface of Earth (mass M and radius R) and compare the kinetic energy needed to get the rocket to its escape speed with its kinetic energy in a low circular orbit around Earth. We can use conservation of energy to find the escape kinetic energy and Newton's law of gravity to derive an expression for the low-Earth orbit kinetic energy.

Apply conservation of energy to relate the initial energy of the rocket to its escape kinetic energy:

$$K_\text{f} - K_\text{i} + U_\text{f} - U_\text{i} = 0$$

Letting the zero of gravitational potential energy be at infinity we have $U_\text{f} = K_\text{f} = 0$ and:

$$-K_\text{i} - U_\text{i} = 0$$
or
$$K_\text{e} = -U_\text{i} = \frac{GMm}{R}$$

Apply Newton's law of gravity to the rocket in orbit at the surface of Earth to obtain:

$$\frac{GMm}{R^2} = m\frac{v^2}{R}$$

Rewrite this equation to express the low-Earth orbit kinetic energy K_o of the rocket:

$$K_\text{o} = \tfrac{1}{2}mv^2 = \frac{GMm}{2R}$$

Express the ratio of K_o to K_e and simplify to obtain:

$$\frac{K_\text{o}}{K_\text{e}} = \frac{\dfrac{GMm}{2R}}{\dfrac{GMm}{R}} = \frac{1}{2}$$

Solving for K_e yields:

$K_\text{e} = \boxed{2K_\text{o}}$ as asserted by Heinlein.

51 •• **[SSM]** An object is dropped from rest from a height of 4.0×10^6 m above the surface of Earth. If there is no air resistance, what is its speed when it strikes Earth?

Picture the Problem Let the zero of gravitational potential energy be at infinity and let m represent the mass of the object. We'll use conservation of energy to relate the initial potential energy of the object-Earth system to the final potential and kinetic energies.

Use conservation of energy to relate the initial potential energy of the system to its energy as the object is about to strike Earth:

$$K_f - K_i + U_f - U_i = 0$$

or, because $K_i = 0$,

$$K(R_E) + U(R_E) - U(R_E + h) = 0 \qquad (1)$$

where h is the initial height above Earth's surface.

Express the potential energy of the object-Earth system when the object is at a distance r from the surface of Earth:

$$U(r) = -\frac{GM_E m}{r}$$

Substitute in equation (1) to obtain:

$$\tfrac{1}{2}mv^2 - \frac{GM_E m}{R_E} + \frac{GM_E m}{R_E + h} = 0$$

Solving for v yields:

$$v = \sqrt{2\left(\frac{GM_E}{R_E} - \frac{GM_E}{R_E + h}\right)} = \sqrt{2gR_E\left(\frac{h}{R_E + h}\right)}$$

Substitute numerical values and evaluate v:

$$\boldsymbol{v} = \sqrt{\frac{2(9.81\,\text{m/s}^2)(6.37\times10^6\,\text{m})(4.0\times10^6\,\text{m})}{6.37\times10^6\,\text{m} + 4.0\times10^6\,\text{m}}} = \boxed{6.9\,\text{km/s}}$$

52 •• An object is projected upward from the surface of Earth with an initial speed of 4.0 km/s. Find the maximum height it reaches.

Picture the Problem Let the zero of gravitational potential energy be at infinity, m represent the mass of the object, and h the maximum height reached by the object. We'll use conservation of energy to relate the initial potential and kinetic energies of the object-Earth system to the final potential energy.

Use conservation of energy to relate the initial potential energy of the system to its energy as the object is at its maximum height:

$$K_f - K_i + U_f - U_i = 0$$

or, because $K_f = 0$,

$$K(R_E) + U(R_E) - U(R_E + h) = 0 \quad (1)$$

where h is the maximum height above Earth's surface.

Express the potential energy of the object-Earth system when the object is at a distance r from the surface of Earth:

$$U(r) = -\frac{GM_E m}{r}$$

Substitute in equation (1) to obtain:

$$\tfrac{1}{2}mv^2 - \frac{GM_E m}{R_E} + \frac{GM_E m}{R_E + h} = 0$$

Solving for h yields:

$$h = \frac{R_E}{\frac{2gR_E}{v^2} - 1}$$

Substitute numerical values and evaluate h:

$$\boldsymbol{h} = \frac{6.37 \times 10^6\ \text{m}}{\frac{2(9.81\,\text{m/s}^2)(6.37 \times 10^6\ \text{m})}{(4.0 \times 10^3\ \text{m/s})^2} - 1} = \boxed{9.4 \times 10^5\ \text{m}}$$

53 •• A particle is projected from the surface of Earth with a speed twice the escape speed. When it is very far from Earth, what is its speed?

Picture the Problem Let the zero of gravitational potential energy be at infinity, m represent the mass of the particle, and the subscript E refer to Earth. When the particle is very far from Earth, the gravitational potential energy of Earth-particle system will be zero. We'll use conservation of energy to relate the initial potential and kinetic energies of the particle-Earth system to the final kinetic energy of the particle.

Use conservation of energy to relate the initial energy of the system to its energy when the particle is very far from Earth:

$$K_f - K_i + U_f - U_i = 0$$

or, because $U_f = 0$,

$$K(\infty) - K(R_E) - U(R_E) = 0 \quad (1)$$

Substitute in equation (1) to obtain:

$$\tfrac{1}{2}mv_\infty^2 - \tfrac{1}{2}m(2v_e)^2 + \frac{GM_Em}{R_E} = 0$$

or, because $GM_E = gR_E^2$,

$$\tfrac{1}{2}mv_\infty^2 - 2mv_e^2 + mgR_E = 0$$

Solving for v_∞ yields:

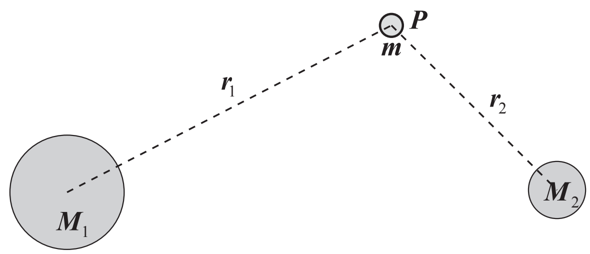
$$v_\infty = \sqrt{2\left(2v_e^2 - gR_E\right)}$$

Substitute numerical values and evaluate v_∞:

$$v_\infty = \sqrt{2\left[2\left(11.2\times10^3\ \text{m/s}\right)^2 - \left(9.81\ \text{m/s}^2\right)\left(6.37\times10^6\ \text{m}\right)\right]} = \boxed{19.4\ \text{km/s}}$$

54 ••• When we calculate escape speeds, we usually do so with the assumption that the body from which we are calculating escape speed is isolated. This is, of course, generally not true in the Solar system. Show that the escape speed at a point near a system that consists of two massive spherical bodies is equal to the square root of the sum of the squares of the escape speeds from each of the two bodies considered individually.

Picture the Problem The pictorial representation shows the two massive objects from which the object (whose mass is *m*), located at point *P*, is to escape. This object will have escaped the gravitational fields of the two massive objects provided, when its gravitational potential energy has become zero, its kinetic energy will also be zero.



Express the total energy of the system consisting of the two massive objects and the object whose mass is *m*:

$$E = \tfrac{1}{2}mv^2 - \frac{GM_1m}{r_1} - \frac{GM_2m}{r_2}$$

When the object whose mass is *m* has escaped, $E = 0$ and:

$$0 = \tfrac{1}{2}mv_e^2 - \frac{GM_1m}{r_1} - \frac{GM_2m}{r_2}$$

Solving for v_e yields:

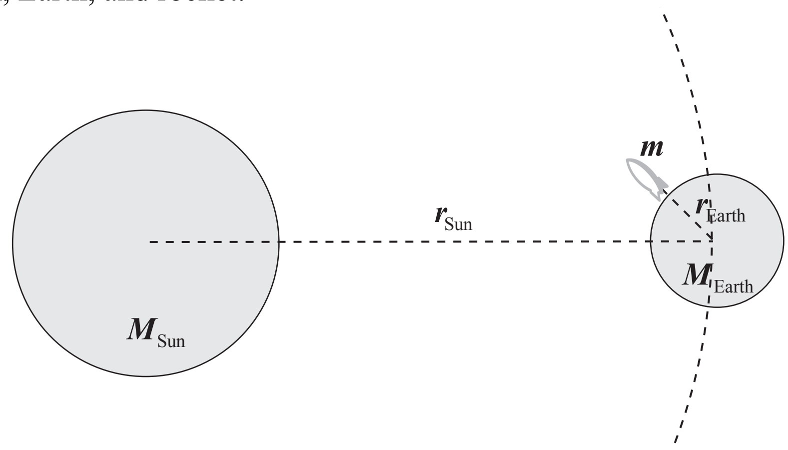
$$v_e^2 = \frac{2GM_1}{r_1} + \frac{2GM_2}{r_2}$$

The terms on the right-hand side of the equation are the squares of the escape speeds from the objects whose masses are M_1 and M_2. Hence;

$$v_e^2 = \boxed{v_{e,1}^2 + v_{e,2}^2}$$

55 ••• Calculate the minimum necessary speed, relative to Earth, for a projectile launched from the surface of Earth to escape the solar system. The answer will depend on the direction of the launch. Explain the choice of direction you'd make for the direction of the launch in order to minimize the necessary launch speed relative to Earth. Neglect Earth's rotational motion and any effects due to air resistance.

Picture the Problem The pictorial representation summarizes the initial positions of the Sun, Earth, and rocket.



From Problem 54, the escape speed from the Earth-Sun system is given by:

$$v_e^2 = \frac{2GM_{Earth}}{r_{Earth}} + \frac{2GM_{Sun}}{r_{Sun}}$$

Solving for v_e yields:

$$v_e = \sqrt{2G\left(\frac{M_{Earth}}{r_{Earth}} + \frac{M_{Sun}}{r_{Sun}}\right)}$$

Substitute numerical values and evaluate v_e:

$$\boldsymbol{v}_e = \sqrt{2\left(6.673\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2\right)\left(\frac{5.98\times10^{24}\ \text{kg}}{6.37\times10^{6}\ \text{m}} + \frac{1.99\times10^{30}\ \text{kg}}{1.496\times10^{11}\ \text{m}}\right)} = 43.6\ \text{km/s}$$

What we have just calculated is the escape speed from Earth's surface, at Earth's orbit. Because the launch will take place from a moving Earth, we need to consider Earth's motion and use it to our advantage. If we launch at sunrise from the Equator, zenith will be pointed directly along the direction of motion of Earth and the required launch speed will be the *minimum* launch speed.

Express the minimum launch speed in terms of the escape speed calculated above and Earth's orbital speed:

$$v_{min} = v_e - v_{orbital}$$

The orbital speed of Earth is given by:

$$v_{orbital} = \frac{2\pi r_{orbital}}{T_{orbital}}$$

Substituting for $v_{orbital}$ yields:

$$v_{min} = v_e - \frac{2\pi r_{orbital}}{T_{orbital}}$$

Substitute numerical values and evaluate v_{min}:

$$v_{min} = 43.6\ \text{km/s} - \frac{2\pi\left(1.496\times10^{11}\ \text{m}\right)}{1\ \text{y}\times\frac{365.24\ \text{d}}{\text{y}}\times\frac{8.64\times10^{4}\ \text{s}}{\text{d}}} = \boxed{13.8\ \text{km/s}}$$

56 ••• An object is projected vertically from the surface of Earth at less than the escape speed. Show that the maximum height reached by the object is $H = R_E H'/(R_E - H')$, where H' is the height that it would reach if the gravitational field were constant. Neglect any effects due to air resistance.

Picture the Problem Let *m* represent the mass of the body that is projected vertically from the surface of Earth. We'll begin by using conservation of energy under the assumption that the gravitational field is constant to determine H'. We'll apply conservation of energy a second time, with the zero of gravitational potential energy at infinity, to express *H*. Finally, we'll solve these two equations simultaneously to express *H* in terms of H'.

Assuming the gravitational field to be constant and letting the zero of potential energy be at the surface of Earth, apply conservation of mechanical energy to relate the initial kinetic energy and the final potential energy of the object-Earth system:

$$K_f - K_i + U_f - U_i = 0$$

or, because $K_f = U_i = 0$,

$$-K_i + U_f = 0$$

Substitute for K_i and U_f to obtain:

$$-\tfrac{1}{2}mv^2 + mgH' = 0 \Rightarrow H' = \frac{v^2}{2g} \quad (1)$$

Letting the zero of gravitational potential energy be at infinity, use conservation of mechanical energy to relate the initial kinetic energy and the final potential energy of the object-earth system:

$$K_f - K_i + U_f - U_i = 0$$

or, because $K_f = 0$,

$$-K_i + U_f - U_i = 0$$

Substitute for K_i, U_f, and U_i and simplify to obtain:

$$-\tfrac{1}{2}mv^2 - \frac{GMm}{R_E + H} + \frac{GMm}{R_E} = 0$$

or

$$-\tfrac{1}{2}v^2 - \frac{gR_E^2}{R_E + H} + \frac{gR_E^2}{R_E} = 0$$

Solving for v^2 yields:

$$v^2 = 2gR_E^2\left(\frac{1}{R_E} - \frac{1}{R_E + H}\right) = 2gR_E\left(\frac{H}{R_E + H}\right)$$

Substitute in equation (1) to obtain:

$$H' = R_E\left(\frac{H}{R_E + H}\right) \Rightarrow H = \boxed{\frac{H'R_E}{R_E - H'}}$$

Gravitational Orbits

57 •• A 100-kg spacecraft is in a circular orbit about Earth at a height $h = 2R_E$. (*a*) What is the orbital period of the spacecraft? (*b*) What is the spacecraft's kinetic energy? (*c*) Express the angular momentum L of the spacecraft about the center of Earth in terms of its kinetic energy K and find the numerical value of L.

Picture the Problem We can use its definition to express the period of the spacecraft's motion and apply Newton's 2nd law to the spacecraft to determine its orbital speed. We can then use this orbital speed to calculate the kinetic energy of the spacecraft. We can relate the spacecraft's angular momentum to its kinetic energy and moment of inertia.

(*a*) Express the period of the spacecraft's orbit about Earth:

$$T = \frac{2\pi R}{v} = \frac{2\pi(3R_E)}{v} = \frac{6\pi R_E}{v}$$

where v is the orbital speed of the spacecraft.

Use Newton's 2nd law to relate the gravitational force acting on the spacecraft to its orbital speed:

$$F_{\text{radial}} = \frac{GM_E m}{(3R_E)^2} = m\frac{v^2}{3R_E} \Rightarrow v = \sqrt{\frac{gR_E}{3}}$$

Substitute for v in our expression for T to obtain:

$$T = 6\sqrt{3}\pi\sqrt{\frac{R_E}{g}}$$

Substitute numerical values and evaluate T:

$$\boldsymbol{T} = 6\sqrt{3}\pi\sqrt{\frac{6.37\times10^6\ \text{m}}{9.81\ \text{m/s}^2}}$$

$$= 2.631\times10^4\ \text{s}\times\frac{1\ \text{h}}{3600\ \text{s}} = \boxed{7.31\ \text{h}}$$

(*b*) Using its definition, express the spacecraft's kinetic energy:

$$K = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m\left(\tfrac{1}{3}gR_E\right)$$

Substitute numerical values and evaluate K:

$$\boldsymbol{K} = \tfrac{1}{6}(100\ \text{kg})(9.81\ \text{m/s}^2)(6.37\times10^6\ \text{m})$$

$$= 1.041\ \text{GJ} = \boxed{1.04\ \text{GJ}}$$

(*c*) Express the kinetic energy of the spacecraft in terms of its angular momentum:

$$K = \frac{L^2}{2I} \Rightarrow L = \sqrt{2IK}$$

Express the moment of inertia of the spacecraft with respect to an axis through the center of Earth:

$$I = m(3R_E)^2 = 9mR_E^2$$

Substitute for I in the expression for L and simplify to obtain:

$$L = \sqrt{2(9mR_E^2)K} = 3R_E\sqrt{2mK}$$

Substitute numerical values and evaluate L:

$$\boldsymbol{L} = 3(6.37\times10^6\ \text{m})\sqrt{2(100\ \text{kg})(1.041\times10^9\ \text{J})} = \boxed{8.72\times10^{12}\ \text{J}\cdot\text{s}}$$

58 •• The orbital period of the moon is 27.3 d, the average center-to-center distance between the moon and Earth is 3.82×10^8 m, the length of a year is

365.24 d, and the average center-to-center distance between Earth and the Sun is 1.50×10^{11} m. Use this data to estimate the ratio of the mass of the Sun to the mass of Earth. Compare this to the measured ratio of 3.33×10^5. List some neglected factors that might account for any discrepancy.

Picture the Problem We can use Kepler's 3rd law to relate the periods of the moon and Earth, in their orbits about Earth and the Sun, to their mean center-to-center distances from the objects about which they orbit. We can solve these equations for the masses of the Sun and Earth and then divide one by the other to establish a value for the ratio of the mass of the Sun to the mass of Earth.

Using Kepler's 3rd law, relate the period of the moon to its mean distance from Earth:

$$T_m^2 = \frac{4\pi^2}{GM_E} r_m^3 \qquad (1)$$

where r_m is the distance between the centers of Earth and the moon.

Using Kepler's 3rd law, relate the period of Earth to its mean distance from the Sun:

$$T_E^2 = \frac{4\pi^2}{GM_s} r_E^3 \qquad (2)$$

where r_E is the distance between the centers of Earth and the Sun.

Solve equation (1) for M_E:

$$M_E = \frac{4\pi^2}{GT_m^2} r_m^3 \qquad (3)$$

Solve equation (2) for M_s:

$$M_s = \frac{4\pi^2}{GT_E^2} r_E^3 \qquad (4)$$

Divide equation (4) by equation (3) and simplify to obtain:

$$\frac{M_s}{M_E} = \left(\frac{r_E}{r_m}\right)^3 \left(\frac{T_m}{T_E}\right)^2$$

Substitute numerical values and evaluate M_s/M_E:

$$\frac{\boldsymbol{M}_s}{\boldsymbol{M}_E} = \left(\frac{1.50\times10^{11}\ \text{m}}{3.82\times10^{8}\ \text{m}}\right)^3 \left(\frac{27.3\ \text{d}}{365.24\ \text{d}}\right)^2 = \boxed{3.38\times10^5}$$

Express the difference between this value and the measured value of 3.33×10^5:

$$\%\,\text{diff} = \frac{3.38\times10^5 - 3.33\times10^5}{3.33\times10^5} = \boxed{1.50\,\%}$$

In this analysis we've neglected gravitational forces exerted by other planets and the Sun.

59 •• **[SSM]** Many satellites orbit Earth with maximum altitudes of 1000 km or less. *Geosynchronous* satellites, however, orbit at an altitude of 3579 km above Earth's surface. How much more energy is required to launch a 500-kg satellite into a geosynchronous orbit than into an orbit 1000 km above the surface of Earth?

Picture the Problem We can express the energy difference between these two orbits in terms of the total energy of a satellite at each elevation. The application of Newton's 2nd law to the force acting on a satellite will allow us to express the total energy of each satellite as function of its mass, the radius of Earth, and its orbital radius.

Express the energy difference:

$$\Delta E = E_{\text{geo}} - E_{1000} \qquad (1)$$

Express the total energy of an orbiting satellite:

$$E_{\text{tot}} = K + U = \tfrac{1}{2}mv^2 - \frac{GM_{\text{E}}m}{R} \qquad (2)$$

where R is the orbital radius.

Apply Newton's 2nd law to a satellite to relate the gravitational force to the orbital speed:

$$F_{\text{radial}} = \frac{GM_{\text{E}}m}{R^2} = m\frac{v^2}{R}$$

Solving for v^2 yields:

$$v^2 = \frac{gR_{\text{E}}^2}{R}$$

Substitute in equation (2) to obtain:

$$E_{\text{tot}} = \tfrac{1}{2}m\frac{gR_{\text{E}}^2}{R} - \frac{gR_{\text{E}}^2 m}{R} = -\frac{mgR_{\text{E}}^2}{2R}$$

Substituting in equation (1) and simplifying yields:

$$\begin{aligned}\Delta E &= -\frac{mgR_{\text{E}}^2}{2R_{\text{geo}}} + \frac{mgR_{\text{E}}^2}{2R_{1000}} = \frac{mgR_{\text{E}}^2}{2}\left(\frac{1}{R_{1000}} - \frac{1}{R_{\text{geo}}}\right)\\ &= \frac{mgR_{\text{E}}^2}{2}\left(\frac{1}{R_{\text{E}} + 1000\ \text{km}} - \frac{1}{R_{\text{E}} + 3579\ \text{km}}\right)\end{aligned}$$

Substitute numerical values and evaluate ΔE:

$$\Delta E = \tfrac{1}{2}(500\ \text{kg})(9.81\ \text{N/kg})(6.37\times10^6\ \text{m})^2\left(\frac{1}{7.37\times10^6\ \text{m}} - \frac{1}{9.95\times10^6\ \text{m}}\right) = \boxed{3.50\ \text{GJ}}$$

60 ••• The idea of a spaceport orbiting Earth is an attractive proposition for launching probes and/or manned missions to the outer planets of the Solar System. Suppose such a ″platform″ has been constructed, and orbits Earth at a distance of 450 km above Earth's surface. Your research team is launching a lunar probe in an orbit that has its perigee at the spaceport's orbital radius, and its apogee at the moon's orbital radius. (*a*) To launch the probe successfully, first determine the orbital speed for the platform. (*b*) Next you determine the necessary speed relative to the platform, for the probe's launch, to attain the desired orbit. Assume that any effects due to the gravitational pull of the moon on the probe are negligible. In addition, assume that the launch takes place in a negligible amount of time. (*c*) You have the probe designed to radio back when it has reached apogee. How long after launch should you expect to receive this signal from the probe (neglect the second or so delay for the transit time of the signal back to the platform)?

Picture the Problem We can use the fact that the kinetic energy of the orbiting platform equals half its gravitational potential energy to find the orbital speed of the platform. In Part (*b*), the required launch speed is the difference between the speed of the probe at perigee and the orbital speed of the platform. To find the speed of the probe at perigee, we can use conservation of mechanical energy and Kepler's law of equal areas. Finally, in Part (*c*), we can use Kepler's 3rd law to find the time after launch that you would expect to receive a signal from the probe announcing that it had reached apogee.

(*a*) The kinetic energy of the orbiting platform equals half its gravitational potential energy:

$$K = \frac{1}{2}U_{\mathrm{g}}$$

Substitute for K and U to obtain:

$$\frac{1}{2}mv^2 = \frac{1}{2}\frac{GM_{\mathrm{Earth}}m}{r}$$

where m is the mass of the platform and r is the distance from the center of Earth to the platform.

Solving for v yields:

$$v = \sqrt{\frac{GM_{\mathrm{Earth}}}{r}}$$

Substitute numerical values and evaluate v:

$$\boldsymbol{v} = \sqrt{\frac{\left(6.67\times10^{-11}\ \mathrm{N\cdot m^2/kg^2}\right)\left(5.98\times10^{24}\ \mathrm{kg}\right)}{6.37\times10^{6}\ \mathrm{m} + 0.450\times10^{6}\ \mathrm{m}}} = \boxed{7.65\ \mathrm{km/s}}$$

(*b*) The required launch speed is the difference between the speed of the probe at perigee and the orbital speed of the platform:

$$\boldsymbol{v}_{\text{rel to platform}} = \boldsymbol{v}_{\mathrm{p}} - \boldsymbol{v} \qquad (1)$$

Here the platform's orbital radius will be the probe's orbit's perigee; the speed of launch will point in the direction the platform is moving at launch time. Neglecting everything in the universe but Earth and probe, apply conservation of energy from perigee to apogee to obtain:

$$\tfrac{1}{2}mv_p^2 - \tfrac{1}{2}mv_a^2 - \frac{GmM_{\text{Earth}}}{r_p} - \left(-\frac{GmM_{\text{Earth}}}{r_a}\right) = 0$$

Simplify these expression to obtain:

$$\boldsymbol{v}_p^2\left[1 - \left(\frac{\boldsymbol{v}_a}{\boldsymbol{v}_p}\right)\right] = 2\boldsymbol{GM}_{\text{Earth}}\left(\frac{1}{\boldsymbol{r}_p} - \frac{1}{\boldsymbol{r}_a}\right) \quad (2)$$

Applying Kepler's law of equal areas between perigee and apogee yields:

$$\boldsymbol{v}_p\boldsymbol{r}_p = \boldsymbol{v}_a\boldsymbol{r}_a \Rightarrow \boldsymbol{v}_a = \frac{\boldsymbol{r}_p}{\boldsymbol{r}_a}\boldsymbol{v}_p$$

Substituting for v_a in equation (2) yields:

$$\boldsymbol{v}_p^2\left[1 - \left(\frac{\boldsymbol{r}_p}{\boldsymbol{r}_a}\right)\right] = 2\boldsymbol{GM}_{\text{Earth}}\left(\frac{1}{\boldsymbol{r}_p} - \frac{1}{\boldsymbol{r}_a}\right) \quad (3)$$

where

$$\boldsymbol{r}_p = \boldsymbol{R}_{\text{Earth}} + \boldsymbol{h} = 6370\text{ km} + 450\text{ km}$$
$$= 6820\text{ km}$$

and

$$\boldsymbol{r}_a = 3.84\times10^5\text{ km}$$

Because $r_p << r_a$, equation (3) becomes:

$$v_p^2 \approx 2GM_{\text{Earth}}\left(\frac{1}{r_p}\right) = \frac{2GM_{\text{Earth}}}{r_p}$$

Solving for v_p yields:

$$v_p \approx \sqrt{\frac{2GM_{\text{Earth}}}{r_p}}$$

Substitute numerical values and evaluate v_p:

$$\boldsymbol{v}_p \approx \sqrt{\frac{2(6.673\times10^{-11}\text{ N}\cdot\text{m}^2/\text{kg}^2)(5.98\times10^{24}\text{ kg})}{6820\text{ km}}} = 10.8\text{ km/s}$$

Substitute numerical values in equation (1) and evaluate $v_{\text{rel to platform}}$:

$$\boldsymbol{v}_{\text{rel to platform}} = 10.8\text{ km/s} - 7.65\text{ km/s}$$
$$= \boxed{3.2\text{ km/s}}$$

(*c*) The time after launch that you should expect to receive this signal from the probe is half the period of the probe's motion:

$$\Delta t = \tfrac{1}{2}T \qquad (4)$$

Apply Kepler's 3rd law to the probe-Earth system to obtain:

$$T^2 = \frac{4\pi^2}{GM_{\text{Earth}}}a^3 \Rightarrow T = \sqrt{\frac{4\pi^2 a^3}{GM_{\text{Earth}}}}$$

where a is the semi-major axis.

Substitute for T in equation (4) to obtain:

$$\Delta t = \sqrt{\frac{\pi^2 a^3}{GM_{\text{Earth}}}}$$

Substitute numerical values and evaluate Δt:

$$\Delta t = \sqrt{\frac{\pi^2\left[\frac{1}{2}\left(3.844\times10^8\ \text{m} + (6.37+0.45)\times10^6\ \text{m}\right)\right]^3}{\left(6.673\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2\right)\left(5.98\times10^{24}\ \text{kg}\right)}} = 4.304\times10^5\ \text{s} \approx \boxed{5.0\ \text{d}}$$

The Gravitational Field ($\vec{g}$)

61 • A 3.0-kg space probe experiences a gravitational force of $12\ \text{N}\,\hat{i}$ as it passes through point *P*. What is the gravitational field at point *P*?

Picture the Problem The gravitational field at any point is defined by $\vec{g} = \vec{F}/m$.

Using its definition, express the gravitational field at a point in space:

$$\vec{g} = \frac{\vec{F}}{m} = \frac{(12\ \text{N})\hat{i}}{3.0\ \text{kg}} = \boxed{(4.0\ \text{N/kg})\hat{i}}$$

62 • The gravitational field at some point is given by $\vec{g} = 2.5\times10^{-6}\ \text{N/kg}\ \hat{j}$. What is the gravitational force on a 0.0040-kg object located at that point?

Picture the Problem The gravitational force acting on an object of mass *m* where the gravitational field is $\vec{g}$ is given by $\vec{F} = m\vec{g}$.

The gravitational force acting on the object is the product of the mass of the object and the gravitational field:

$$\vec{F} = m\vec{g}$$

Substitute numerical values and evaluate $\vec{g}$:

$$\vec{F} = (0.0040\,\text{kg})(2.5\times10^{-6}\ \text{N/kg})\hat{j} = \boxed{(1.0\times10^{-8}\ N)\hat{j}}$$

63 •• **[SSM]** A point particle of mass m is on the x axis at $x = L$ and an identical point particle is on the y axis at $y = L$. (a) What is the gravitational field at the origin? (b) What is the magnitude of this field?

Picture the Problem We can use the definition of the gravitational field due to a point mass to find the x and y components of the field at the origin and then add these components to find the resultant field. We can find the magnitude of the field from its components using the Pythagorean theorem.

(a) The gravitational field at the origin is the sum of its x and y components:

$$\vec{g} = \vec{g}_x + \vec{g}_y \qquad (1)$$

Express the gravitational field due to the point mass at $x = L$:

$$\vec{g}_x = \frac{Gm}{L^2}\hat{i}$$

Express the gravitational field due to the point mass at $y = L$:

$$\vec{g}_y = \frac{Gm}{L^2}\hat{j}$$

Substitute in equation (1) to obtain:

$$\vec{g} = \vec{g}_x + \vec{g}_y = \boxed{\frac{Gm}{L^2}\hat{i} + \frac{Gm}{L^2}\hat{j}}$$

(b) The magnitude of $\vec{g}$ is given by:

$$|\vec{g}| = \sqrt{g_x^2 + g_y^2}$$

Substitute for g_x and g_y and simplify to obtain:

$$|\vec{g}| = \sqrt{\left(\frac{Gm}{L^2}\right)^2 + \left(\frac{Gm}{L^2}\right)^2} = \boxed{\sqrt{2}\,\frac{Gm}{L^2}}$$

64 •• Five objects, each of mass M, are equally spaced on the arc of a semicircle of radius R as in Figure 11-25. An object of mass m is located at the center of curvature of the arc. (a) If M is 3.0 kg, m is 2.0 kg, and R is 10 cm, what is the gravitational force on the particle of mass m due to the five objects? (b) If the object whose mass is m is removed, what is the gravitational field at the center of curvature of the arc?

Picture the Problem We can find the net force acting on m by superposition of the forces due to each of the objects arrayed on the circular arc. Once we have expressed the net force, we can find the gravitational field at the center of curvature from its definition. Choose a coordinate system in which the $+x$ direction is to the right and the $+y$ direction is upward.

(*a*) Express the net force acting on the object whose mass is m:

$$\vec{F} = F_x\hat{\boldsymbol{i}} + F_y\hat{\boldsymbol{j}} \qquad (1)$$

F_x is given by:

$$F_x = \frac{GMm}{R^2} - \frac{GMm}{R^2} + \frac{GMm}{R^2}\cos 45° - \frac{GMm}{R^2}\cos 45° = 0$$

F_y is given by:

$$F_y = \frac{GMm}{R^2} + \frac{GMm}{R^2}\sin 45° + \frac{GMm}{R^2}\sin 45° = \frac{GMm}{R^2}(2\sin 45° + 1)$$

Substitute numerical values and evaluate F_y:

$$\boldsymbol{F}_y = \frac{(6.673\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(3.0\,\text{kg})}{(0.10\,\text{m})^2}(2.0\,\text{kg})(2\sin 45° + 1) = 9.67\times10^{-8}\ \text{N}$$

Substitute in equation (1) to obtain:

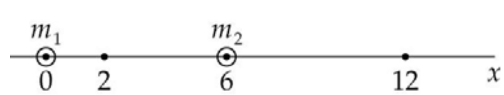
$$\vec{F} = \boxed{0\hat{\boldsymbol{i}} + (9.7\times10^{-8}\ \text{N})\hat{\boldsymbol{j}}}$$

(*b*) Using its definition, express $\vec{g}$ at the center of curvature of the arc:

$$\vec{g} = \frac{\vec{F}}{\boldsymbol{m}} = \frac{0\hat{\boldsymbol{i}} + (9.67\times10^{-8}\ \text{N})\hat{\boldsymbol{j}}}{2.0\,\text{kg}} = \boxed{(4.8\times10^{-8}\ \text{N/kg})\hat{\boldsymbol{j}}}$$

65 •• A point particle of mass $m_1 = 2.0$ kg is at the origin and a second point particle of mass $m_2 = 4.0$ kg is on the x axis at $x = 6.0$ m. Find the gravitational field g at (*a*) $x = 2.0$ m and (*b*) $x = 12$ m. (*c*) Find the point on the x axis for which $g = 0$.

Picture the Problem The configuration of point masses is shown to the right. The gravitational field at any point can be found by superimposing the fields due to each of the point masses.



(*a*) Express the gravitational field at $x = 2.0$ m as the sum of the fields due to the point masses m_1 and m_2:

$$\vec{g} = \vec{g}_1 + \vec{g}_2 \qquad (1)$$

Express $\vec{g}_1$ and $\vec{g}_2$:

$$\vec{g}_1 = -\frac{Gm_1}{x_1^2}\hat{\boldsymbol{i}} \text{ and } \vec{g}_2 = \frac{Gm_2}{x_2^2}\hat{\boldsymbol{i}}$$

Substitute in equation (1) to obtain:

$$\vec{g} = -\frac{\boldsymbol{Gm}_1}{\boldsymbol{x}_1^2}\hat{\boldsymbol{i}} + \frac{\boldsymbol{Gm}_2}{\boldsymbol{x}_2^2}\hat{\boldsymbol{i}} = -\frac{\boldsymbol{Gm}_1}{\boldsymbol{x}_1^2}\hat{\boldsymbol{i}} + \frac{\boldsymbol{Gm}_2}{(2\boldsymbol{x}_1)^2}\hat{\boldsymbol{i}} = -\frac{\boldsymbol{G}}{\boldsymbol{x}_1^2}\left(\boldsymbol{m}_1 - \tfrac{1}{4}\boldsymbol{m}_2\right)\hat{\boldsymbol{i}}$$

Substitute numerical values and evaluate $\vec{g}$:

$$\vec{g} = -\frac{6.673\times10^{-11}\ \mathrm{N\cdot m^2/kg^2}}{(2.0\,\mathrm{m})^2}\left[2.0\,\mathrm{kg} - \tfrac{1}{4}(4.0\,\mathrm{kg})\right]\hat{\boldsymbol{i}} = \boxed{\left(-1.7\times10^{-11}\ \mathrm{N/kg}\right)\hat{\boldsymbol{i}}}$$

(*b*) Express $\vec{g}_1$ and $\vec{g}_2$:

$$\vec{g}_1 = -\frac{Gm_1}{x_1^2}\hat{\boldsymbol{i}} \text{ and } \vec{g}_2 = -\frac{Gm_2}{x_2^2}\hat{\boldsymbol{i}}$$

Substitute in equation (1) and simplify to obtain:

$$\vec{g} = -\frac{Gm_1}{x_1^2}\hat{\boldsymbol{i}} - \frac{Gm_2}{x_2^2}\hat{\boldsymbol{i}} = -\frac{Gm_1}{(2x_2)^2}\hat{\boldsymbol{i}} - \frac{Gm_2}{x_2^2}\hat{\boldsymbol{i}} = -\frac{G}{x_2^2}\left(\tfrac{1}{4}m_1 + m_2\right)\hat{\boldsymbol{i}}$$

Substitute numerical values and evaluate $\vec{g}$:

$$\vec{g} = -\frac{6.673\times10^{-11}\ \mathrm{N\cdot m^2/kg^2}}{(6.0\,\mathrm{m})^2}\left[\tfrac{1}{4}(2.0\,\mathrm{kg}) + 4.0\,\mathrm{kg}\right]\hat{\boldsymbol{i}} = \boxed{\left(-8.3\times10^{-12}\ \mathrm{N/kg}\right)\hat{\boldsymbol{i}}}$$

(*c*) Express the condition that $\vec{g} = 0$:

$$\frac{\boldsymbol{Gm}_1}{\boldsymbol{x}^2} - \frac{\boldsymbol{Gm}_2}{(6.0-\boldsymbol{x})^2} = 0$$

or

$$\frac{2.0}{\boldsymbol{x}^2} - \frac{4.0}{(6.0-\boldsymbol{x})^2} = 0$$

Solve this quadratic equation to obtain:

$$x = 2.48\,\mathrm{m} \text{ and } x = -14.5\,\mathrm{m}$$

From the diagram it is clear that the physically meaningful root is the positive one at:

$$x = \boxed{2.5\,\text{m}}$$

66 •• Show that on the x axis the maximum value of g for the field of Example 11-7 occurs at points $x = \pm a/\sqrt{2}$.

Picture the Problem To show that the maximum value of g for the field of Example 11-7 occurs at the points $x = \pm a/\sqrt{2}$, we can differentiate g_x with respect to x and set the derivative equal to zero.

From Example 11-7:

$$g_x = -\frac{2GMx}{\left(x^2+a^2\right)^{3/2}}$$

Differentiate g_x with respect to x and set the derivative equal to zero to find extreme values:

$$\frac{dg_x}{dx} = -2GM\left[\left(x^2+a^2\right)^{-3/2} - 3x^2\left(x^2+a^2\right)^{-5/2}\right] = 0 \text{ for extrema.}$$

Solve for x to obtain:

$$x = \boxed{\pm\frac{a}{\sqrt{2}}}$$

Remarks: To establish that this value for x corresponds to a relative maximum, we need to either evaluate the second derivative of g_x at $x = \pm a/\sqrt{2}$ or examine the graph of $|g_x|$ at $x = \pm a/\sqrt{2}$ for concavity downward.

67 ••• **[SSM]** A nonuniform thin rod of length L lies on the x axis. One end of the rod is at the origin, and the other end is at $x = L$. The rod's mass per unit length λ varies as $\lambda = Cx$, where C is a constant. (Thus, an element of the rod has mass $dm = \lambda\,dx$.) (*a*) What is the total mass of the rod? (*b*) Find the gravitational field due to the rod on the x axis at $x = x_0$, where $x_0 > L$.

Picture the Problem We can find the mass of the rod by integrating dm over its length. The gravitational field at $x_0 > L$ can be found by integrating $d\vec{g}$ at x_0 over the length of the rod.

(*a*) The total mass of the stick is given by:

$$M = \int_0^L \lambda\,dx$$

Substitute for λ and evaluate the integral to obtain”

$$M = C\int_0^L x\,dx = \boxed{\tfrac{1}{2}CL^2}$$

(*b*) Express the gravitational field due to an element of the stick of mass *dm*:

$$d\vec{g} = -\frac{G\,dm}{(x_0 - x)^2}\hat{i} = -\frac{G\lambda\,dx}{(x_0 - x)^2}\hat{i} = -\frac{GCx\,dx}{(x_0 - x)^2}\hat{i}$$

Integrate this expression over the length of the stick to obtain:

$$\vec{g} = -GC\int_0^L \frac{x\,dx}{(x_0 - x)^2}\hat{i} = \boxed{\frac{2GM}{L^2}\left[\ln\left(\frac{x_0}{x_0 - L}\right) - \left(\frac{L}{x_0 - L}\right)\right]\hat{i}}$$

68 •• A uniform thin rod of mass *M* and length *L* lies on the positive *x* axis with one end at the origin. Consider an element of the rod of length *dx,* and mass *dm,* at point *x*, where $0 < x < L$. (*a*) Show that this element produces a gravitational field at a point x_0 on the *x* axis in the region $x_0 > L$ given by $dg_x = -\frac{GM}{L(x_0 - x)^2}dx$. (*b*) Integrate this result over the length of the rod to find the total gravitational field at the point x_0 due to the rod. (*c*) Find the gravitational force on a point particle of mass m_0 at x_0. (*d*) Show that for $x_0 >> L$, the field of the rod approximates the field of a point particle of mass *M* at $x = 0$.

Picture the Problem The elements of the rod of mass *dm* and length *dx* produce a gravitational field at any point *P* located a distance $x_0 > L$ from the origin. We can calculate the total field by integrating the magnitude of the field due to *dm* from $x = 0$ to $x = L$.

(*a*) Express the gravitational field at *P* due to the element *dm*:

$$d\vec{g}_x = -\frac{G\,dm}{r^2}\hat{i}$$

Relate *dm* to *dx*:

$$dm = \frac{M}{L}dx$$

Express the distance *r* between *dm* and point *P* in terms of *x* and x_0:

$$r = x_0 - x$$

Substitute these results to express $d\vec{g}_x$ in terms of *x* and x_0:

$$d\vec{g}_x = \boxed{\left\{-\frac{GM}{L(x_0 - x)^2}dx\right\}\hat{i}}$$

(*b*) Integrate from $x = 0$ to $x = L$ to find the total gravitational field at point *P*:

$$\vec{g}_x = -\frac{GM}{L}\int_0^L \frac{dx}{(x_0 - x)^2}\hat{i} = \left\{-\frac{GM}{L}\left[\frac{1}{x_0 - x}\right]_0^L\right\}\hat{i} = \boxed{-\frac{GM}{x_0(x_0 - L)}\hat{i}}$$

(*c*) Use the definition of gravitational field and the result from Part (*b*) to express $\vec{F}_g$ at $x = x_0$:

$$\vec{F}_g = m_0\vec{g} = \boxed{-\frac{GMm_0}{x_0(x_0 - L)}\hat{i}}$$

(*d*) Factor x_0 from the denominator of the expression for $\vec{g}_x$ to obtain:

$$\vec{g}_x = -\frac{GM}{x_0^2\left(1 - \frac{L}{x_0}\right)}\hat{i}$$

For $x_0 >> L$ the second term in parentheses is very small and:

$$\vec{g}_x \approx \boxed{-\frac{GM}{x_0^2}\hat{i}}$$

which is the gravitational field of a point mass *M* located at the origin.

The Gravitational Field ($\vec{g}$) due to Spherical Objects

69 • A uniform thin spherical shell has a radius of 2.0 m and a mass of 300 kg. What is the gravitational field at the following distances from the center of the shell: (*a*) 0.50 m, (*b*) 1.9 m, (*c*) 2.5 m?

Picture the Problem The gravitational field inside a spherical shell is zero and the field at the surface of and outside the shell is given by $g = GM/r^2$.

(*a*) Because 0.50 m < *R*:

$$g(0.50\text{ m}) = \boxed{0}$$

(*b*) Because 1.9 m < *R*:

$$g(1.9\text{ m}) = \boxed{0}$$

(*c*) Because 2.5 m > *R*:

$$g(2.5\text{ m}) = \frac{GM}{r^2} = \frac{(6.673\times10^{-11}\text{ N}\cdot\text{m}^2/\text{kg}^2)(300\text{ kg})}{(2.5\text{ m})^2} = \boxed{3.2\times10^{-9}\text{ N/kg}}$$

70 • A uniform thin spherical shell has a radius of 2.00 m and a mass of 300 kg, and its center is located at the origin of a coordinate system. Another uniform thin spherical shell with a radius of 1.00 m and a mass of 150 kg is inside the larger shell with its center at 0.600 m on the x axis. What is the gravitational force of attraction between the two shells?

Determine the Concept The gravitational force is zero. The gravitational field inside the 2.00 m shell due to that shell is zero; therefore, it exerts no force on the 1.00 m-shell, and, by Newton's 3rd law, that shell exerts no force on the larger shell.

71 •• **[SSM]** Two widely separated solid spheres, S_1 and S_2, each have radius R and mass M. Sphere S_1 is uniform, whereas the density of sphere S_2 is given by $\rho(r) = C/r$, where r is the distance from its center. If the gravitational field strength at the surface of S_1 is g_1, what is the gravitational field strength at the surface of S_2?

Picture the Problem The gravitational field strength at the surface of a sphere is given by $g = GM/R^2$, where R is the radius of the sphere and M is its mass.

Express the gravitational field strength on the surface of S_1:

$$g_1 = \frac{GM}{R^2}$$

Express the gravitational field strength on the surface of S_2:

$$g_2 = \frac{GM}{R^2}$$

Divide the second of these equations by the first and simplify to obtain:

$$\frac{g_2}{g_1} = \frac{\frac{GM}{R^2}}{\frac{GM}{R^2}} = 1 \Rightarrow \boxed{g_1 = g_2}$$

72 •• Two widely separated uniform solid spheres, S_1 and S_2, have equal masses but different radii, R_1 and R_2. If the gravitational field strength on the surface of S_1 is g_1, what is the gravitational field strength on the surface of S_2?

Picture the Problem The gravitational field strength at the surface of a sphere is given by $g = GM/R^2$, where R is the radius of the sphere and M is its mass.

Express the gravitational field strength on the surface of S_1:

$$g_1 = \frac{GM}{R_1^2}$$

Express the gravitational field strength on the surface of S_2:

$$g_2 = \frac{GM}{R_2^2}$$

Divide the second of these equations by the first and simplify to obtain:

$$\frac{g_2}{g_1}=\frac{\frac{GM}{R_2^2}}{\frac{GM}{R_1^2}}=\frac{R_1^2}{R_2^2}\Rightarrow g_2=\boxed{\frac{R_1^2}{R_2^2}g_1}$$

Remarks: The gravitational field strengths depend only on the masses and radii because the points of interest are outside spherically symmetric distributions of mass.

73 •• Two concentric uniform thin spherical shells have masses M_1 and M_2 and radii a and $2a$, as in Figure 11-26. What is the magnitude of the gravitational force on a point particle of mass m (not shown) located (*a*) a distance $3a$ from the center of the shells? (*b*) a distance $1.9a$ from the center of the shells? (*c*) a distance $0.9a$ from the center of the shells?

Picture the Problem The magnitude of the gravitational force is $F_g = mg$ where g inside a spherical shell is zero and outside is given by $g = GM/r^2$.

(*a*) The gravitational force on a particle of mass m is given by:

$$F_g = mg$$

At $r = 3a$, the masses of both spheres contribute to g:

$$F_g(3a)=m\frac{G(M_1+M_2)}{(3a)^2}=\boxed{\frac{Gm(M_1+M_2)}{9a^2}}$$

(*b*) At $r = 1.9a$, g due to M_2 is zero and:

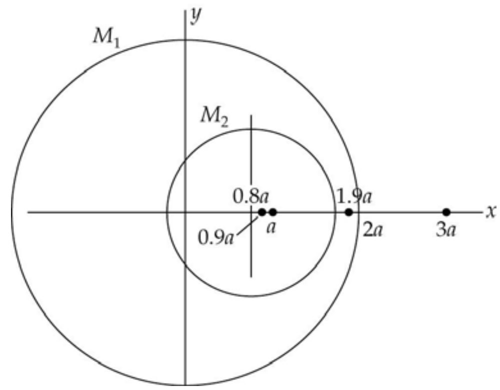
$$F_g(1.9a)=m\frac{GM_1}{(1.9a)^2}=\boxed{\frac{GmM_1}{3.61a^2}}$$

(*c*) At $r = 0.9a$, $g = 0$ and:

$$F_g(0.9a)=\boxed{0}$$

74 •• The inner spherical shell in Problem 73 is shifted so that its center is now on the x axis at $x = 0.8a$. What is the magnitude of the gravitational force on a particle of point mass m located on the x axis at (*a*) $x = 3a$, (*b*) $x = 1.9a$, (*c*) $x = 0.9a$?

Picture the Problem The configuration is shown on the right. The centers of the spheres are indicated by the center-lines. The x coordinates of the mass m for Parts (a), (b), and (c) are indicated along the x axis. The magnitude of the gravitational force is $F_g = mg$ where g inside a spherical shell is zero and outside is given by $g = \frac{GM}{r^2}$.



(a) Express the gravitational force acting on the object whose mass is m:

$$F_g = m(g_{1x} + g_{2x}) \tag{1}$$

Find g_{1x} at $x = 3a$:

$$g_{1x}(3a) = \frac{GM_1}{(3a)^2} = \frac{GM_1}{9a^2}$$

Find g_{2x} at $x = 3a$:

$$g_{2x}(3a) = \frac{GM_2}{(3a - 0.8a)^2} = \frac{GM_2}{4.84a^2}$$

Substitute for $g_{1x}(3a)$ and $g_{2x}(3a)$ in equation (1) and simplify to obtain:

$$F(3a) = m\left(\frac{GM_1}{9a^2} + \frac{GM_2}{4.84a^2}\right) = \boxed{\frac{Gm}{a^2}\left(\frac{M_1}{9} + \frac{M_2}{4.84}\right)}$$

(b) Find g_{2x} at $x = 1.9a$:

$$g_{2x}(1.9a) = \frac{GM_2}{(1.9a - 0.8a)^2} = \frac{GM_2}{1.21a^2}$$

Find g_{1x} at $x = 1.9a$:

$$g_{1x}(1.9a) = 0$$

Substitute for $g_{1x}(1.9a)$ and $g_{2x}(1.9a)$ and simplify to obtain:

$$F(1.9a) = mg = \boxed{\frac{GmM_2}{1.21a^2}}$$

(c) At $x = 0.9a$, $g_{1x} = g_{2x} = 0$ and:

$$F(0.9a) = \boxed{0}$$

75 •• **[SSM]** Suppose you are standing on a spring scale in an elevator that is descending at constant speed in a mine shaft located on the equator. (*a*) Show that the force on you due to Earth's gravity alone is proportional to your distance from the center of the planet. (*b*) Assume that the mine shaft located on the equator and is vertical. Do not neglect Earth's rotational motion. Show that the reading on the spring scale is proportional to your distance from the center of the planet.

Picture the Problem There are two forces acting on you as you descend in the elevator and are at a distance *r* from the center of Earth; an upward normal force (F_N) exerted by the scale, and a downward gravitational force (mg) exerted by Earth. Because you are in equilibrium (you are descending at constant speed) under the influence of these forces, the normal force exerted by the scale is equal in magnitude to the gravitational force acting on you. We can use Newton's law of gravity to express this gravitational force.

(*a*) Express the force of gravity acting on you when you are a distance *r* from the center of Earth:

$$F_g = \frac{GM(r)m}{r^2} \quad (1)$$

Using the definition of density, express the density of Earth between you and the center of Earth and the density of Earth as a whole:

$$\rho = \frac{M(r)}{V(r)} = \frac{M(r)}{\frac{4}{3}\pi r^3}$$

The density of Earth is also given by:

$$\rho = \frac{M_E}{V_E} = \frac{M_E}{\frac{4}{3}\pi R^3}$$

Equating these two expressions for ρ and solving for $M(r)$ yields:

$$M(r) = M_E\left(\frac{r}{R}\right)^3$$

Substitute for $M(r)$ in equation (1) and simplify to obtain:

$$F_g = \frac{GM_E\left(\frac{r}{R}\right)^3 m}{r^2} = \frac{GM_E m}{R^2}\frac{r}{R} \quad (2)$$

Apply Newton's law of gravity to yourself at the surface of Earth to obtain:

$$mg = \frac{GM_E m}{R^2} \Rightarrow g = \frac{GM_E}{R^2}$$

where *g* is the magnitude of the gravitational field at the surface of Earth.

Substitute for g in equation (2) to obtain:

$$F_g = \boxed{\left(\frac{mg}{R}\right)r}$$

That is, the force of gravity on you is proportional to your distance from the center of Earth.

(b) Apply Newton's 2nd law to your body to obtain:

$$F_{\mathrm{N}} - mg\frac{r}{R} = -mr\omega^2$$

where the net force $\left(-\boldsymbol{mr\omega}^2\right)$, directed toward the center of Earth, is the centripetal force acting on your body.

Solving for F_{N} yields:

$$F_{\mathrm{N}} = \left(\frac{mg}{R}\right)r - mr\omega^2$$

Note that this equation tells us that your effective weight increases linearly with distance from the center of Earth. However, due just to the effect of rotation, as you approach the center the centripetal force decreases linearly and, doing so, increases your effective weight.

76 •• Suppose Earth were a nonrotating uniform sphere. As a reward for earning the highest lab grade, your physics professor chooses your laboratory team to participate in a gravitational experiment at a deep mine on the equator. At this mine, there exists an elevator shaft going 15.0 km into Earth. Before making the measurement, you are asked to predict the decrease in the weight of a team member, who weighs 800 N at the surface of Earth, when she is at the bottom of the shaft. The density of Earth's crust actually increases with depth. Is your answer higher or lower than the actual experimental result?

Picture the Problem We can find the loss in weight at this depth by taking the difference between the weight of the student at the surface of Earth and her weight at a depth d = 15.0 km. To find the gravitational field at depth d, we'll use its definition and the mass of Earth that is between the bottom of the shaft and the center of Earth. We'll assume (incorrectly) that the density of Earth is constant.

The loss in weight of a team member is given by:

$$\Delta w = w(R_{\mathrm{E}}) - w(R) \qquad (1)$$

The mass M inside $R = R_{\mathrm{E}} - d$ is given by:

$$M = \rho V = \tfrac{4}{3}\rho\pi\left(R_{\mathrm{E}} - d\right)^3$$

Relate the mass of Earth to kits density and volume:

$$M_E = \rho V_E = \tfrac{4}{3}\rho\pi R_E^3$$

Divide the first of these equations by the second to obtain:

$$\frac{M}{M_E} = \frac{\tfrac{4}{3}\rho\pi(R_E - d)^3}{\tfrac{4}{3}\rho\pi R_E^3} = \frac{(R_E - d)^3}{R_E^3}$$

Solving for M yields:

$$M = M_E \frac{(R_E - d)^3}{R_E^3}$$

Express the gravitational field at $R = R_E - d$:

$$g = \frac{GM}{R^2} = \frac{GM_E(R_E - d)^3}{(R_E - d)^2 R_E^3} \qquad (2)$$

Express the gravitational field at $R = R_E$:

$$g_E = \frac{GM_E}{R_E^2} \qquad (3)$$

Divide equation (2) by equation (3) to obtain:

$$\frac{g}{g_E} = \frac{\dfrac{GM_E(R_E - d)^3}{(R_E - d)^2 R_E^3}}{\dfrac{GM_E}{R_E^2}} = \frac{R_E - d}{R_E}$$

Solving for g yields:

$$g = \frac{R_E - d}{R_E} g_E$$

The weight of the student at $R = R_E - d$ is given by:

$$w(R) = mg(R) = \frac{R_E - d}{R_E} mg_E$$

$$= \left(1 - \frac{d}{\mathrm{R}_E}\right) mg_E$$

Substitute for $w(R)$ in equation (1) and simplify to obtain:

$$\Delta w = mg_E - \left(1 - \frac{d}{\mathrm{R}_E}\right) mg_E = \frac{mg_E d}{R_E}$$

Substitute numerical values and evaluate Δw:

$$\Delta \boldsymbol{w} = \frac{(800\,\mathrm{N})(15.0\,\mathrm{km})}{6370\,\mathrm{km}} = \boxed{1.88\,\mathrm{N}}$$

If Earth's crustal density actually increased with depth, this increase with depth would partially compensate for the decrease in the fraction of Earth's mass between a descending team member and the center of Earth; with the result that the loss in weight would be lower than the actual experimental result.

77 •• **[SSM]** A solid sphere of radius R has its center at the origin. It has a uniform mass density ρ_0, except that there is a spherical cavity in it of radius $r = \frac{1}{2}R$ centered at $x = \frac{1}{2}R$ as in Figure 11-27. Find the gravitational field at points on the x axis for $|x| > R$. *Hint: The cavity may be thought of as a sphere of mass* $m = (4/3)\pi r^3\rho_0$ *plus a sphere of "negative" mass* $-m$.

Picture the Problem We can use the hint to find the gravitational field along the x axis.

Using the hint, express $g(x)$: $\quad g(x) = g_{\text{solid sphere}} + g_{\text{hollow sphere}}$

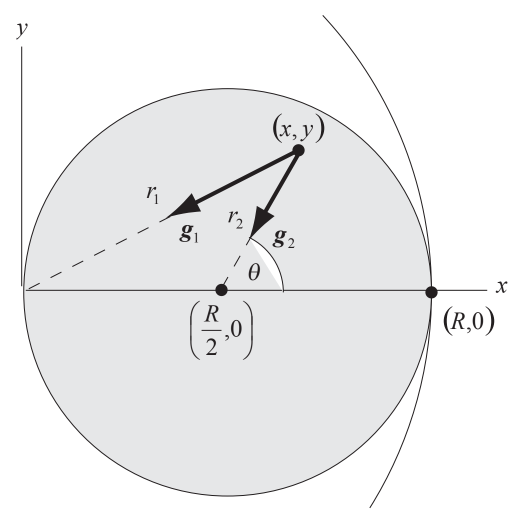
Substitute for $g_{\text{solid sphere}}$ and $g_{\text{hollow sphere}}$ and simplify to obtain:

$$g(x) = \frac{GM_{\text{solid sphere}}}{x^2} + \frac{GM_{\text{hollow sphere}}}{\left(x - \frac{1}{2}R\right)^2} = \frac{G\rho_0\left(\frac{4}{3}\pi R^3\right)}{x^2} + \frac{G\rho_0\left[-\frac{4}{3}\pi\left(\frac{1}{2}R\right)^3\right]}{\left(x - \frac{1}{2}R\right)^2}$$

$$= \boxed{G\left(\frac{4\pi\rho_0 R^3}{3}\right)\left[\frac{1}{x^2} - \frac{1}{8\left(x - \frac{1}{2}R\right)^2}\right]}$$

78 ••• For the sphere with the cavity in Problem 77, show that the gravitational field is uniform throughout the cavity, and find its magnitude and direction there.

Picture the Problem The diagram shows the portion of the solid sphere in which the hollow sphere is embedded. $\vec{g}_1$ is the field due to the solid sphere of radius R and density ρ_0 and $\vec{g}_2$ is the field due to the sphere of radius $\frac{1}{2}\boldsymbol{R}$ and negative density ρ_0 centered at $\frac{1}{2}\boldsymbol{R}$. We can find the resultant field by adding the x and y components of $\vec{g}_1$ and $\vec{g}_2$.



Use its definition to express $|\vec{g}_1|$:

$$|\vec{g}_1| = \frac{F_g}{m}$$

Substitute for the gravitational force to obtain:

$$|\vec{g}_1| = \frac{\frac{GMm}{r^2}}{m} = \frac{GM}{r^2}$$

Substituting the product of its density and volume for the mass of the sphere and simplifying yields:

$$|\vec{g}_1| = \frac{G\rho_0 V}{r^2} = \frac{4\pi\rho_0 r^3 G}{3r^2} = \frac{4\pi\rho_0 rG}{3}$$

Find the x and y components of $\vec{g}_1$:

$$g_{1x} = -g_1\cos\theta = -g_1\left(\frac{x}{r}\right) = -\frac{4\pi\rho_0 Gx}{3}$$

and

$$g_{1y} = -g_1\sin\theta = -g_1\left(\frac{y}{r}\right) = -\frac{4\pi\rho_0 Gy}{3}$$

where the negative signs indicate that the field points inward.

Proceed similarly to express $|\vec{g}_2|$:

$$|\vec{\mathbf{g}}_2| = \frac{4\pi\rho_0 r_2 G}{3}$$

Express the x and y components of $\vec{g}_2$:

$$g_{2x} = g_2\left(\frac{x - \frac{1}{2}R}{r_2}\right) = \frac{4\pi\rho_0 G\left(x - \frac{1}{2}R\right)}{3}$$

and

$$g_{2y} = g_2\left(\frac{y}{r_2}\right) = \frac{4\pi\rho_0 Gy}{3}$$

Add the x components and simplify to obtain the x component of the resultant field:

$$\begin{aligned} g_x &= g_{1x} + g_{2x} \\ &= -\frac{4\pi\rho_0 Gx}{3} + \frac{4\pi\rho_0 G\left(x - \frac{1}{2}R\right)}{3} \\ &= -\frac{2\pi\rho_0 GR}{3} \end{aligned}$$

where the negative sign indicates that the field points inward.

Add the y components and simplify to obtain the y component of the resultant field:

$$\begin{aligned} g_y &= g_{1y} + g_{2y} \\ &= -\frac{4\pi\rho_0 Gy}{3} + \frac{4\pi\rho_0 Gy}{3} = 0 \end{aligned}$$

Express $\vec{g}$ in vector form:

$$\vec{g} = g_x\hat{i} + g_y\hat{j} = \boxed{\left(-\frac{2\pi\rho_0 GR}{3}\right)\hat{i} + 0\hat{j}}$$

The magnitude of $\vec{g}$ is:

$$|\vec{g}| = \sqrt{g_x^2 + g_y^2} = \boxed{\frac{2\pi\rho_0 GR}{3}}$$

79 ••• A straight, smooth tunnel is dug through a uniform spherical planet of mass density ρ_0. The tunnel passes through the center of the planet and is perpendicular to the planet's axis of rotation, which is fixed in space. The planet rotates with a constant angular speed ω so objects in the tunnel have no apparent weight. Find the required angular speed of the planet ω.

Picture the Problem The gravitational field will exert an inward radial force on the objects in the tunnel. We can relate this force to the angular speed of the planet by using Newton's 2nd law of motion.

Letting r be the distance from the objects to the center of the planet, use Newton's 2nd law to relate the gravitational force acting on the objects to their angular speed:

$$F_{net} = F_g = mr\omega^2$$

or

$$mg = mr\omega^2 \Rightarrow \omega = \sqrt{\frac{g}{r}} \qquad (1)$$

Use its definition to express g:

$$g = \frac{F_g}{m}$$

Substitute for F_g to obtain:

$$g = \frac{F_g}{m} = \frac{\frac{GMm}{r^2}}{m} = \frac{GM}{r^2}$$

Substituting the product of its density and volume for the mass of planet and simplifying yields:

$$g = \frac{G\rho_0 V}{r^2} = \frac{4\pi\rho_0 r^3 G}{3r^2} = \frac{4\pi\rho_0 rG}{3}$$

Substituting for g in equation (1) and simplifying yields:

$$\omega = \sqrt{\frac{\frac{4\pi\rho_0 rG}{3}}{r}} = \boxed{\sqrt{\frac{4\pi\rho_0 G}{3}}}$$

80 ••• The density of a sphere is given by $\rho(r) = C/r$. The sphere has a radius of 5.0 m and a mass of 1.0×10^{11} kg. (*a*) Determine the constant *C*. (*b*) Obtain expressions for the gravitational field for the regions (1) $r > 5.0$ m and (2) $r < 5.0$ m.

Picture the Problem Because we're given the mass of the sphere, we can find C by expressing the mass of the sphere in terms of C. We can use its definition to find the gravitational field of the sphere both inside and outside its surface.

(a) Express the mass of a differential element of the sphere:

$$dm = \rho\, dV = \rho\left(4\pi r^2 dr\right)$$

Integrate to express the mass of the sphere in terms of C:

$$M = 4\pi C \int_0^{5.0\,\mathrm{m}} r dr = \left(50\,\mathrm{m}^2\right)\pi C$$

Solving for C yields:

$$C = \frac{M}{\left(50\,\mathrm{m}^2\right)\pi}$$

Substitute numerical values and evaluate C:

$$\boldsymbol{C} = \frac{1.0\times10^{11}\,\mathrm{kg}}{\left(50\,\mathrm{m}^2\right)\boldsymbol{\pi}} = 6.37\times10^{8}\,\mathrm{kg/m}^2$$

$$= \boxed{6.4\times10^{8}\,\mathrm{kg/m}^2}$$

(b) Use its definition to express the gravitational field of the sphere at a distance from its center greater than its radius:

$$g = \frac{GM}{r^2}$$

(1) For $r > 5.0$ m:

$$g(r > 5.0\,\mathrm{m}) = \frac{\left(6.673\times10^{-11}\,\mathrm{N\cdot m^2/kg^2}\right)\left(1.0\times10^{11}\,\mathrm{kg}\right)}{r^2} = \boxed{\frac{6.7\,\mathrm{N\cdot m^2/kg}}{r^2}}$$

Use its definition to express the gravitational field of the sphere at a distance from its center less than its radius:

$$g = G\frac{\int_0^r 4\pi r^2 \rho\, dr}{r^2} = G\frac{\int_0^r 4\pi r^2 \frac{C}{r} dr}{r^2}$$

$$= G\frac{4\pi C\int_0^r r\, dr}{r^2} = 2\pi GC$$

(2) For $r < 5.0$ m:

$$g(r < 5.0\,\mathrm{m}) = 2\pi\left(6.673\times10^{-11}\,\mathrm{N\cdot m^2/kg^2}\right)\left(6.37\times10^{8}\,\mathrm{kg/m^2}\right) = \boxed{0.27\,\mathrm{N/kg}}$$

81 ••• [SSM] A small diameter hole is drilled into the sphere of Problem 80 toward the center of the sphere to a depth of 2.0 m below the sphere's surface. A small mass is dropped from the surface into the hole. Determine the speed of the small mass as it strikes the bottom of the hole.

Picture the Problem We can use conservation of energy to relate the work done by the gravitational field to the speed of the small object as it strikes the bottom of the hole. Because we're given the mass of the sphere, we can find C by expressing the mass of the sphere in terms of C. We can then use the definition of the gravitational field to find the gravitational field of the sphere inside its surface. The work done by the field equals the negative of the change in the potential energy of the system as the small object falls in the hole.

Use conservation of energy to relate the work done by the gravitational field to the speed of the small object as it strikes the bottom of the hole:

$$K_f - K_i + \Delta U = 0$$

or, because $K_i = 0$ and $W = -\Delta U$,

$$W = \tfrac{1}{2}mv^2 \Rightarrow v = \sqrt{\frac{2W}{m}} \qquad (1)$$

where v is the speed with which the object strikes the bottom of the hole and W is the work done by the gravitational field.

Express the mass of a differential element of the sphere:

$$dm = \rho\, dV = \rho\left(4\pi r^2 dr\right)$$

Integrate to express the mass of the sphere in terms of C:

$$M = 4\pi C \int_0^{5.0\,\text{m}} r dr = \left(50\,\text{m}^2\right)\pi C$$

Solving for C yields:

$$C = \frac{M}{\left(50\,\text{m}^2\right)\pi}$$

Substitute numerical values and evaluate C:

$$\boldsymbol{C} = \frac{1.0\times10^{11}\,\text{kg}}{\left(50\,\text{m}^2\right)\boldsymbol{\pi}} = 6.37\times10^{8}\,\text{kg/m}^2$$

Use its definition to express the gravitational field of the sphere at a distance from its center less than its radius:

$$g = \frac{F_g}{m} = \frac{GM}{r^2} = G\frac{\int_0^r 4\pi r^2 \rho\, dr}{r^2} = G\frac{\int_0^r 4\pi r^2 \frac{C}{r} dr}{r^2} = G\frac{4\pi C\int_0^r r\, dr}{r^2} = 2\pi GC$$

Express the work done on the small object by the gravitational force acting on it:

$$W = -\int_{5.0\,\text{m}}^{3.0\,\text{m}} mg\,dr = (2\,\text{m})mg$$

Substitute in equation (1) and simplify to obtain:

$$v = \sqrt{\frac{2(2.0\,\text{m})m(2\pi GC)}{m}} = \sqrt{(8.0\,\text{m})\pi GC}$$

Substitute numerical values and evaluate v:

$$\boldsymbol{v} = \sqrt{(8.0\,\text{m})\boldsymbol{\pi}\left(6.673\times10^{-11}\,\text{N}\cdot\text{m}^2/\text{kg}^2\right)\left(6.37\times10^{8}\,\text{kg/m}^2\right)} = \boxed{1.0\,\text{m/s}}$$

82 ••• As a geologist for a mining company, you are working on a method for determining possible locations of underground ore deposits. Assume that where the company owns land the crust of Earth is 40.0 km thick and has a density of about 3000 kg/m^3. Suppose a spherical deposit of heavy metals with a density of 8000 kg/m^3 and radius of 1000 m was centered 2000 m below the surface. You propose to detect it by determining its affect on the local surface value of g. Find $\Delta g/g$ at the surface directly above this deposit, where Δg is the increase in the gravitational field due to the deposit.

Picture the Problem The spherical deposit of heavy metals will increase the gravitational field at the surface of Earth. We can express this increase in terms of the difference in densities of the deposit and Earth and then form the quotient $\Delta g/g$.

Express Δg due to the spherical deposit:

$$\Delta g = \frac{G\Delta M}{r^2} \qquad (1)$$

Express the mass of the spherical deposit:

$$M = \Delta\rho V = \Delta\rho\left(\tfrac{4}{3}\pi R^3\right) = \tfrac{4}{3}\pi\,\Delta\rho\,R^3$$

Substitute in equation (1):

$$\Delta g = \frac{\tfrac{4}{3}G\pi\,\Delta\rho\,R^3}{r^2}$$

Express $\Delta g/g$:

$$\frac{\Delta g}{g} = \frac{\dfrac{\tfrac{4}{3}G\pi\,\Delta\rho\,R^3}{r^2}}{g} = \frac{\tfrac{4}{3}G\pi\,\Delta\rho\,R^3}{gr^2}$$

Substitute numerical values and evaluate $\Delta g/g$:

$$\frac{\Delta g}{g}=\frac{\frac{4}{3}\pi\left(6.673\times10^{-11}\,\text{N}\cdot\text{m}^2/\text{kg}^2\right)\left(5000\,\text{kg/m}^3\right)\left(1000\,\text{m}\right)^3}{\left(9.81\,\text{N/kg}\right)\left(2000\,\text{m}\right)^2}=\boxed{3.56\times10^{-5}}$$

83 ••• **[SSM]** Two identical spherical cavities are made in a lead sphere of radius R. The cavities each have a radius $R/2$. They touch the outside surface of the sphere and its center as in Figure 11-28. The mass of a solid uniform lead sphere of radius R is M. Find the force of attraction on a point particle of mass m located at a distance d from the center of the lead sphere.

Picture the Problem The force of attraction of the small sphere of mass m to the lead sphere of mass M is the sum of the forces due to the solid sphere ($\vec{F}_\text{S}$) and the cavities ($\vec{F}_\text{C}$) of negative mass.

Express the force of attraction:

$$\vec{F}=\vec{F}_\text{S}+\vec{F}_\text{C} \qquad (1)$$

Use the law of gravity to express the force due to the solid sphere:

$$\vec{F}_\text{S}=-\frac{GMm}{d^2}\hat{i}$$

Express the magnitude of the force acting on the small sphere due to one cavity:

$$F_\text{C}=\frac{GM'm}{d^2+\left(\frac{R}{2}\right)^2}$$

where M' is the negative mass of a cavity.

Relate the negative mass of a cavity to the mass of the sphere before hollowing:

$$M'=-\rho V=-\rho\left[\tfrac{4}{3}\pi\left(\frac{R}{2}\right)^3\right]$$
$$=-\tfrac{1}{8}\left(\tfrac{4}{3}\pi\rho R^3\right)=-\tfrac{1}{8}M$$

Letting θ be the angle between the x axis and the line joining the center of the small sphere to the center of either cavity, use the law of gravity to express the force due to the two cavities:

$$\vec{F}_\text{C}=2\frac{GMm}{8\left(d^2+\frac{R^2}{4}\right)}\cos\theta\,\hat{i}$$

because, by symmetry, the y components add to zero.

Use the figure to express $\cos\theta$:

$$\cos\theta = \frac{d}{\sqrt{d^2 + \frac{R^2}{4}}}$$

Substitute for $\cos\theta$ and simplify to obtain:

$$\vec{F}_C = \frac{GMm}{4\left(d^2 + \frac{R^2}{4}\right)} \frac{d}{\sqrt{d^2 + \frac{R^2}{4}}}\hat{i} = \frac{GMmd}{4\left(d^2 + \frac{R^2}{4}\right)^{3/2}}\hat{i}$$

Substitute in equation (1) and simplify:

$$\vec{F} = -\frac{GMm}{d^2}\hat{i} + \frac{GMmd}{4\left(d^2 + \frac{R^2}{4}\right)^{3/2}}\hat{i}$$

$$= \boxed{-\frac{GMm}{d^2}\left[1 - \frac{\frac{d^3}{4}}{\left\{d^2 + \frac{R^2}{4}\right\}^{3/2}}\right]\hat{i}}$$

84 ••• A globular cluster is a roughly spherical collection of up to millions of stars bound together by the force of gravity. Astronomers can measure the velocities of stars in the cluster to study its composition and to get an idea of the mass distribution within the cluster. Assuming that all of the stars have about the same mass and are distributed uniformly within the cluster, show that the mean speed of a star in a circular orbit around the center of the cluster should increase linearly with its distance from the center.

Picture the Problem Let R be the size of the cluster, and N the total number of stars in it. We can apply Newton's law of gravity and the 2nd law of motion to relate the net force (which depends on the number of stars $N(r)$ in a sphere whose radius is equal to the distance between the star of interest and the center of the cluster) acting on a star at a distance r from the center of the cluster to its speed. We can use the definition of density, in conjunction with the assumption of uniform distribution of the starts within the cluster, to find $N(r)$ and, ultimately, express the orbital speed v of a star in terms of the total mass of the cluster.

Using Newton's law of gravity and 2nd law, express the force acting on a star at a distance r from the center of the cluster:

$$F(r) = \frac{GN(r)M^2}{r^2} = M\frac{v^2}{r} \qquad (1)$$

where $N(r)$ is the number of stars within a distance r of the center of the cluster and M is the mass of an individual star.

Using the uniform distribution assumption and the definition of density, relate the number of stars $N(r)$ within a distance r of the center of the cluster to the total number N of stars in the cluster:

$$\rho = \frac{N(r)M}{\frac{4}{3}\pi r^3} = \frac{NM}{\frac{4}{3}\pi R^3} \Rightarrow N(r) = N\frac{r^3}{R^3}$$

Substitute for $N(r)$ in equation (1) to obtain:

$$\frac{GNM^2}{r^2}\frac{r^3}{R^3} = M\frac{v^2}{r}$$

Solving for v yields:

$$v = r\sqrt{\frac{GNM}{R^3}} \Rightarrow v \propto r$$

The mean speed v of a star in a circular orbit about the center of the cluster increases linearly with distance r from the center.

General Problems

85 • The mean distance of Pluto from the Sun is 39.5 AU. Find the period of Pluto.

Picture the Problem We can use Kepler's 3rd law to relate Pluto's period to its mean distance from the sun.

Using Kepler's 3rd law, relate the period of Pluto to its mean distance from the sun:

$$\boldsymbol{T}_{\text{Pluto}}^2 = \boldsymbol{C}\boldsymbol{r}_{\text{Pluto}}^3 \Rightarrow \boldsymbol{T}_{\text{Pluto}} = \sqrt{\boldsymbol{C}\boldsymbol{r}_{\text{Pluto}}^3}$$

where $C = \frac{4\pi^2}{GM_s} = 2.973\times10^{-19}\ \text{s}^2/\text{m}^3$.

Substitute numerical values and evaluate T_{Pluto}:

$$T_{\text{Pluto}} = \sqrt{\left(2.973\times10^{-19}\ \text{s}^2/\text{m}^3\right)\left(39.5\ \text{AU}\times\frac{1.50\times10^{11}\ \text{m}}{\text{AU}}\right)^3}$$
$$= 7.864\times10^{9}\ \text{s}\times\frac{1\ \text{h}}{3600\ \text{s}}\times\frac{1\ \text{d}}{24\ \text{h}}\times\frac{1\ \text{y}}{365.25\ \text{d}} = \boxed{249\ \text{y}}$$

86 • Calculate the mass of Earth using the known values of G, g, and R_{E}.

Picture the Problem Consider an object of mass m at the surface of Earth. We can relate the weight of this object to the gravitational field of Earth and to the mass of Earth.

Using Newton's 2nd law, relate the weight of an object at the surface of Earth to the gravitational force acting on it:

$$w = mg = \frac{GM_{\text{E}}m}{R_{\text{E}}^2} \Rightarrow M_{\text{E}} = \frac{gR_{\text{E}}^2}{G}$$

Substitute numerical values and evaluate M_{E}:

$$M_{\text{E}} = \frac{(9.81\ \text{N/kg})\left(6.37\times10^{6}\ \text{m}\right)^2}{6.673\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2} = \boxed{5.97\times10^{24}\ \text{kg}}$$

87 •• The force exerted by Earth on a particle of mass m a distance r ($r > R_{\text{E}}$) from the center of Earth has the magnitude mgR_{E}^2/r^2, where $g = GM_{\text{E}}/R_{\text{E}}^2$. ($a$) Calculate the work you must do to move the particle from distance r_1 to distance r_2. (b) Show that when $r_1 = R_{\text{E}}$ and $r_2 = R_{\text{E}} + h$, the result can be written as $W = mgR_{\text{E}}^2\left[(1/R_{\text{E}}) - 1/(R_{\text{E}}+h)\right]$. ($c$) Show that when $h \ll R_{\text{E}}$, the work is given approximately by $W = mgh$.

Picture the Problem The work you must do against gravity to move the particle from a distance r_1 to r_2 is the negative of the change in the particle's gravitational potential energy.

(a) Relate the work you must do to the change in the gravitational potential energy of Earth-particle system:

$$W = -\Delta U = -\int_{r_1}^{r_2} \vec{F}_{\text{g}}\cdot d\vec{r}$$
$$= -\int_{r_1}^{r_2} F_{\text{g}}\,dr(\cos 180^\circ) = \int_{r_1}^{r_2} F_{\text{g}}\,dr$$

Substitute for F_g and evaluate the integral to obtain:

$$W = GM_Em\int_{r_1}^{r_2}\frac{dr}{r^2} = -GM_Em\left(\frac{1}{r_2}-\frac{1}{r_1}\right)$$

$$= \boxed{GM_Em\left(\frac{1}{r_1}-\frac{1}{r_2}\right)}$$

(*b*) Substitute gR_E^2 for GM_E, R_E for r_1, and $R_E + h$ for r_2 to obtain:

$$W = \boxed{mgR_E^2\left(\frac{1}{R_E}-\frac{1}{R_E+h}\right)} \quad (1)$$

(*c*) Rewrite equation (1) with a common denominator and simplify to obtain:

$$W = mgR_E^2\left(\frac{R_E+h-R_E}{R_E(R_E+h)}\right) = mgh\left(\frac{R_E}{R_E+h}\right) = mgh\left(\frac{1}{1+\frac{h}{R_E}}\right) \approx \boxed{mgh}$$

provided $h << R_E$.

88 •• The average density of the moon is $\rho = 3340\ \text{kg/m}^3$. Find the minimum possible period T of a spacecraft orbiting the moon.

Picture the Problem Let m represent the mass of the spacecraft. From Kepler's 3rd law we know that its period will be a minimum when it is in orbit just above the surface of the moon. We'll use Newton's 2nd law to relate the angular speed of the spacecraft to the gravitational force acting on it.

Relate the period of the spacecraft to its angular speed:

$$T = \frac{2\pi}{\omega} \quad (1)$$

Using Newton's 2nd law of motion, relate the gravitational force acting on the spacecraft when it is in orbit at the surface of the moon to the angular speed of the spacecraft:

$$\sum F_{\text{radial}} = \frac{GM_Mm}{R_M^2} = mR_M\omega^2$$

Solving for ω and simplifying yields:

$$\omega = \sqrt{\frac{GM_M}{R_M^3}} = \sqrt{\frac{G\left(\frac{4}{3}\pi\rho R_M^3\right)}{R_M^3}}$$

$$= \sqrt{\tfrac{4}{3}G\pi\rho}$$

Substitute for ω in equation (1) and simplify to obtain:

$$T_{\min} = \frac{2\pi}{\sqrt{\frac{4}{3}G\pi\rho}} = \sqrt{\frac{3\pi}{\rho G}}$$

Substitute numerical values and evaluate $T_{\min}$:

$$\boldsymbol{T}_{\min} = \sqrt{\frac{3\boldsymbol{\pi}}{\left(6.673\times10^{-11}\ \mathrm{N\cdot m^2/kg^2}\right)\left(3340\ \mathrm{kg/m^3}\right)}} = 6503\ \mathrm{s} = \boxed{1\ \mathrm{h}\ 48\ \mathrm{min}}$$

89 •• **[SSM]** A *neutron star* is a highly condensed remnant of a massive star in the last phase of its evolution. It is composed of neutrons (hence the name) because the star's gravitational force causes electrons and protons to ″coalesce″ into the neutrons. Suppose at the end of its current phase, the Sun collapsed into a neutron star (it can't in actuality because it does not have enough mass) of radius 12.0 km, without losing any mass in the process. (*a*) Calculate the ratio of the gravitational acceleration at the surface of the Sun following its collapse compared to its value at the surface of the Sun today. (*b*) Calculate the ratio of the escape speed from the surface of the neutron-Sun to its value today.

Picture the Problem We can apply Newton's 2nd law and the law of gravity to an object of mass *m* at the surface of the Sun and the neutron-Sun to find the ratio of the gravitational accelerations at their surfaces. Similarly, we can express the ratio of the corresponding expressions for the escape speeds from the two suns to determine their ratio.

(*a*) Express the gravitational force acting on an object of mass *m* at the surface of the Sun:

$$F_g = ma_g = \frac{GM_{\mathrm{Sun}}m}{R_{\mathrm{Sun}}^2}$$

Solving for a_g yields:

$$a_g = \frac{GM_{\mathrm{Sun}}}{R_{\mathrm{Sun}}^2} \qquad (1)$$

The gravitational force acting on an object of mass *m* at the surface of a neutron-Sun is:

$$F_g = ma'_g = \frac{GM_{\text{neutron-Sun}}m}{R_{\text{neutron-Sun}}^2}$$

Solving for a'_g yields:

$$a'_g = \frac{GM_{\text{neutron-Sun}}}{R_{\text{neutron-Sun}}^2} \qquad (2)$$

Divide equation (2) by equation (1) to obtain:

$$\frac{a'_{\text{g}}}{a_{\text{g}}}=\frac{\dfrac{GM_{\text{neutron-Sun}}}{R^2_{\text{neutron-Sun}}}}{\dfrac{GM_{\text{Sun}}}{R^2_{\text{Sun}}}}=\frac{\dfrac{M_{\text{neutron-Sun}}}{R^2_{\text{neutron-Sun}}}}{\dfrac{M_{\text{Sun}}}{R^2_{\text{Sun}}}}$$

Because $M_{\text{neutron-Sun}}=M_{\text{Sun}}$:

$$\frac{a'_{\text{g}}}{a_{\text{g}}}=\frac{R^2_{\text{Sun}}}{R^2_{\text{neutron-Sun}}}$$

Substitute numerical values and evaluate the ratio $a'_{\text{g}}/a_{\text{g}}$:

$$\frac{a'_{\text{g}}}{a_{\text{g}}}=\left(\frac{6.96\times10^{8}\text{ m}}{12.0\times10^{3}\text{ m}}\right)^2=\boxed{3.36\times10^{9}}$$

(*b*) The escape speed from the neutron-Sun is given by:

$$v'_{\text{e}}=\sqrt{\frac{GM_{\text{neutron-Sun}}}{R_{\text{neutron-Sun}}}}$$

The escape speed from the Sun is given by:

$$v_{\text{e}}=\sqrt{\frac{GM_{\text{Sun}}}{R_{\text{Sun}}}}$$

Dividing the first of these equations by the second and simplifying yields:

$$\frac{v'_{\text{e}}}{v_{\text{e}}}=\sqrt{\frac{R_{\text{Sun}}}{R_{\text{neutron-Sun}}}}$$

Substitute numerical values and evaluate $v'_{\text{e}}/v_{\text{e}}$:

$$\frac{v'_{\text{e}}}{v_{\text{e}}}=\sqrt{\frac{6.96\times10^{8}\text{ m}}{12.0\times10^{3}\text{ m}}}=\boxed{241}$$

90 •• Suppose the Sun could collapse into a neutron star of radius 12.0 km as in Problem 89. Your research team is in charge of sending a probe from Earth to study the transformed Sun, and the probe needs to end up in a circular orbit 4500 km from the neutron-Sun's center. (*a*) Calculate the orbital speed of the probe. (*b*) Later on plans call for construction of a permanent spaceport in that same orbit to study the neutron-Sun in great detail. To transport equipment and supplies, scientists on Earth need you to determine the escape speed for rockets launched from the spaceport (relative to the spaceport) in the direction of the spaceport's orbital velocity at takeoff time. What is that speed and how does it compare to the escape speed at the surface of Earth?

Picture the Problem We can apply Newton's 2nd law to the probe orbiting the Sun to determine its orbital speed. Using the escape-speed equation will allow us to find the escape speed for rockets launched from the spaceport.

(*a*) Apply Newton's 2nd law to the probe of mass *m* in orbit about the Sun:

$$\sum F_{\text{radial}}=\frac{GmM_{\text{Sun}}}{r^2}=m\frac{v^2_{\text{orbital}}}{r}$$

where *r* is the orbital radius.

Solving for v_{orbital} yields:

$$v_{\text{orbital}} = \sqrt{\frac{GM_{\text{Sun}}}{r}}$$

Substitute numerical values and evaluate v_{orbital}:

$$\boldsymbol{v}_{\text{orbital}} = \sqrt{\frac{(6.673\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(1.99\times10^{30}\ \text{kg})}{4.50\times10^{6}\ \text{m}}} = 5.431\times10^{6}\ \text{m/s}$$
$$= \boxed{5.43\times10^{6}\ \text{m/s}}$$

(*b*) The escape speed (relative to the spaceport) for rockets launched from the spaceport is given by:

$$\boldsymbol{v}_{\substack{\text{rel to}\\\text{spaceport}}} = \boldsymbol{v}_{\text{e}} - \boldsymbol{v}_{\text{orbital}} \qquad (1)$$

The escape speed at a distance *r* from the center of the neutron-Sun is given by:

$$v_{\text{e}} = \sqrt{\frac{2GM_{\text{neutron-Sun}}}{r}} = \sqrt{2}\sqrt{\frac{GM_{\text{neutron-Sun}}}{r}}$$
$$= \sqrt{2}v_{\text{orbital}}$$

Substituting for v_{e} in equation (1) yields:

$$v_{\substack{\text{rel to}\\\text{spaceport}}} = \sqrt{2}v_{\text{orbital}} - v_{\text{orbital}}$$
$$= \left(\sqrt{2}-1\right)v_{\text{orbital}}$$

Substitute numerical values and evaluate $\boldsymbol{v}_{\substack{\text{rel to}\\\text{spaceport}}}$:

$$\boldsymbol{v}_{\substack{\text{rel to}\\\text{spaceport}}} = \left(\sqrt{2}-1\right)\left(5.431\times10^{6}\ \text{m/s}\right) = \boxed{2.25\times10^{6}\ \text{m/s}}$$

Express the ratio of $\boldsymbol{v}_{\substack{\text{rel to}\\\text{spaceport}}}$ to $v_{\text{e,Earth}}$:

$$\frac{v_{\substack{\text{rel to}\\\text{spaceport}}}}{v_{\text{e,Earth}}} = \frac{2.25\times10^{6}\ \text{m/s}}{11.2\ \text{km/s}} \approx \boxed{201}$$

91 •• A satellite is circling the moon (radius 1700 km) close to the surface at a speed *v*. A projectile is launched from the moon vertically up at the same initial speed *v*. How high will it rise?

Picture the Problem We can use conservation of energy to establish a relationship between the height *h* to which the projectile will rise and its initial speed. The application of Newton's 2nd law will relate the orbital speed, which is equal to the initial speed of the projectile, to the mass and radius of the moon.

Use conservation of energy to relate the initial energies of the projectile to its final energy:

$$K_f - K_i + U_f - U_i = 0$$

or, because $K_f = 0$,

$$-\tfrac{1}{2}mv^2 - \frac{GM_{moon}m}{R_{moon}+h} + \frac{GM_{moon}m}{R_{moon}} = 0$$

Solving for h yields:

$$h = R_M\left(\frac{1}{1-\dfrac{v^2R_{moon}}{2GM_{moon}}} - 1\right) \qquad (1)$$

Use Newton's 2nd law to relate the orbital speed of the satellite to the gravitational force acting on it:

$$\sum F_{radial} = \frac{GM_Mm}{R_M^2} = m\frac{v^2}{R_M}$$

Solve for v^2 to obtain:

$$v^2 = \frac{GM_M}{R_M}$$

Substitute for v^2 in equation (1) and simplify to obtain:

$$h = R_M\left(\frac{1}{1-\dfrac{1}{2}} - 1\right) = R = \boxed{1.70\,\text{Mm}}$$

92 •• *Black holes* are objects whose gravitational field is so strong that not even light can escape. One way of thinking about this is to consider a spherical object whose density is so large that the escape speed at its surface is greater than the speed of light c. It turns out that if a star's radius is smaller than a value called the *Schwarzchild radius* R_S, then the star will be a black hole, that is, light originating from its surface cannot escape. (*a*) For a non-rotating black hole, the Schwarzchild radius depends only upon the mass of the black hole. Show that it is related to that mass M by $R_S = 2GM/c^2$. (*b*) For a black hole whose mass is ten solar masses, calculate the value of the Schwarzchild radius.

Picture the Problem We can use the escape-speed equation, with $v_e = c$, to derive the expression for the Schwarzchild radius of a non-rotating black hole.

(*a*) The escape speed at a distance r from the center of a spherical object of mass M is given by:

$$v_e = \sqrt{\frac{2GM}{r}}$$

Setting $v_e = c$ yields:

$$c = \sqrt{\frac{2GM}{R_S}} \Rightarrow R_S = \boxed{\frac{2GM}{c^2}}$$

(*b*) For a black hole whose mass is ten solar masses:

$$R_S = \frac{2G(10M_{Sun})}{c^2} = \frac{20GM_{Sun}}{c^2}$$

Substitute numerical values and evaluate R_S:

$$R_S = \frac{20(6.673\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(1.99\times10^{30}\ \text{kg})}{(2.998\times10^{8}\ \text{m/s})^2} = \boxed{29.5\ \text{km}}$$

93 •• In a binary star system, two stars follow circular orbits about their common center of mass. If the stars have masses m_1 and m_2 and are separated by a distance r, show that the period of rotation is related to r by $T^2 = \dfrac{4\pi^2 r^3}{G(m_1+m_2)}$.

Picture the Problem Let the origin of our coordinate system be at the center of mass of the binary star system and let the distances of the stars from their center of mass be r_1 and r_2. The period of rotation is related to the angular speed of the star system and we can use Newton's 2nd law of motion to relate this speed to the separation of the stars.

Relate the square of the period of the motion of the stars to their angular speed:

$$T^2 = \frac{4\pi^2}{\omega^2} \qquad (1)$$

Using Newton's 2nd law of motion, relate the gravitational force acting on the star whose mass is m_2 to the angular speed of the system:

$$\sum F_{radial} = \frac{Gm_1m_2}{(r_1+r_2)^2} = m_2r_2\omega^2$$

Solving for ω^2 yields:

$$\omega^2 = \frac{Gm_1}{r_2(r_1+r_2)^2} \qquad (2)$$

From the definition of the center of mass we have:

$$m_1r_1 = m_2r_2 \qquad (3)$$

where $r = r_1 + r_2 \qquad (4)$

Eliminate r_1 from equations (3) and (4) and solve for r_2 to obtain:

$$r_2 = \frac{rm_1}{m_1+m_2}$$

Eliminate r_2 from equations (3) and (4) and solve for r_1 to obtain:

$$r_1 = \frac{rm_2}{m_1+m_2}$$

Substituting for r_1 and r_2 in equation (2) yields:

$$\omega^2 = \frac{G(m_1 + m_2)}{r^3}$$

Finally, substitute for ω^2 in equation (1) and simplify:

$$T^2 = \frac{4\pi^2}{\frac{G(m_1 + m_2)}{r^3}} = \boxed{\frac{4\pi^2 r^3}{G(m_1 + m_2)}}$$

94 •• Two particles of masses m_1 and m_2 are released from rest at a large separation distance. Find their speeds v_1 and v_2 when their separation distance is r. The initial separation distance is given as large, but large is a relative term. Relative to what distance is it large?

Picture the Problem Because the two-particle system has zero initial energy and zero initial linear momentum; we can use energy and momentum conservation to obtain simultaneous equations in the variables r, v_1 and v_2. We'll assume that initial separation distance of the particles and their final separation r *is large compared to the size of the particles* so that we can treat them as though they are point particles.

Use conservation of energy to relate the speeds of the particles when their separation distance is r:

$$E_\text{i} = E_\text{f}$$

or

$$0 = \tfrac{1}{2}m_1 v_1^2 + \tfrac{1}{2}m_2 v_2^2 - \frac{Gm_1 m_2}{r} \quad (1)$$

Use conservation of linear momentum to obtain a second relationship between the speeds of the particles and their masses:

$$p_\text{i} = p_\text{f}$$

or

$$0 = m_1 v_1 + m_2 v_2 \quad (2)$$

Solve equation (2) for v_1 and substitute in equation (1) to obtain:

$$v_2^2\left(m_2 + \frac{m_2^2}{m_1}\right) = \frac{2Gm_1 m_2}{r} \quad (3)$$

Solve equation (3) for v_2:

$$v_2 = \boxed{\sqrt{\frac{2Gm_1^2}{r(m_1 + m_2)}}}$$

Solve equation (2) for v_1 and substitute for v_2 to obtain:

$$v_1 = \boxed{\sqrt{\frac{2Gm_2^2}{r(m_1 + m_2)}}}$$

95 •• **[SSM]** Uranus, the seventh planet in the Solar System, was first observed in 1781 by William Herschel. Its orbit was then analyzed in terms of

Kepler's Laws. By the 1840's, observations of Uranus clearly indicated that its true orbit was different from the Keplerian calculation by an amount that could not be accounted for by observational uncertainty. The conclusion was that there must be another influence other than the Sun and the known planets lying inside Uranus's orbit. This influence was hypothesized to be due to an eighth planet, whose predicted orbit was described in 1845 independently by two astronomers: John Adams (no relation to our president) and Urbain LeVerrier. In September of 1846, John Galle, searching in the sky at the place predicted by Adams and LeVerrier, made the first observation of Neptune. Uranus and Neptune are in orbit about the Sun with periods of 84.0 and 164.8 years, respectively. To see the effect that Neptune had on Uranus, determine the ratio of the gravitational force between Neptune and Uranus to that between Uranus and the Sun, when Neptune and Uranus are at their closest approach to one another (i.e. when aligned with the Sun). The masses of the Sun, Uranus and Neptune are 333,000, 14.5 and 17.1 times that of Earth, respectively.

Picture the Problem We can use the law of gravity and Kepler's 3rd law to express the ratio of the gravitational force between Neptune and Uranus to that between Uranus and the Sun, when Neptune and Uranus are at their closest approach to one another.

The ratio of the gravitational force between Neptune and Uranus to that between Uranus and the Sun, when Neptune and Uranus are at their closest approach to one another is given by:

$$\frac{F_{\text{g,N-U}}}{F_{\text{g,U-S}}}=\frac{\dfrac{GM_{\text{N}}M_{\text{U}}}{(r_{\text{N}}-r_{\text{U}})^2}}{\dfrac{GM_{\text{U}}M_{\text{S}}}{r_{\text{U}}^2}}=\frac{M_{\text{N}}r_{\text{U}}^2}{M_{\text{S}}(r_{\text{N}}-r_{\text{U}})^2}\quad(1)$$

Applying Kepler's 3rd law to Uranus yields:

$$T_{\text{U}}^2=Cr_{\text{U}}^3\quad(2)$$

Applying Kepler's 3rd law to Neptune yields:

$$T_{\text{N}}^2=Cr_{\text{N}}^3\quad(3)$$

Divide equation (3) by equation (2) to obtain:

$$\frac{T_{\text{N}}^2}{T_{\text{U}}^2}=\frac{Cr_{\text{N}}^3}{Cr_{\text{U}}^3}=\frac{r_{\text{N}}^3}{r_{\text{U}}^3}\Rightarrow r_{\text{N}}=r_{\text{U}}\left(\frac{T_{\text{N}}}{T_{\text{U}}}\right)^{2/3}$$

Substitute for r_{N} in equation (1) to obtain:

$$\frac{F_{\text{g,N-U}}}{F_{\text{g,U-S}}}=\frac{M_{\text{N}}r_{\text{U}}^2}{M_{\text{S}}\left(r_{\text{U}}\left(\dfrac{T_{\text{N}}}{T_{\text{U}}}\right)^{2/3}-r_{\text{U}}\right)^2}$$

Simplifying this expression yields:

$$\frac{F_{\text{g,N-U}}}{F_{\text{g,U-S}}}=\frac{M_{\text{N}}}{M_{\text{S}}\left(\left(\frac{T_{\text{N}}}{T_{\text{U}}}\right)^{2/3}-1\right)^2}$$

Because $M_N = 17.1M_E$ and $M_S = 333{,}000M_E$:

$$\frac{F_{\text{g,N-U}}}{F_{\text{g,U-S}}}=\frac{17.1M_{\text{E}}}{3.33\times10^5 M_{\text{E}}\left(\left(\frac{T_{\text{N}}}{T_{\text{U}}}\right)^{2/3}-1\right)^2}=\frac{17.1}{3.33\times10^5\left(\left(\frac{T_{\text{N}}}{T_{\text{U}}}\right)^{2/3}-1\right)^2}$$

Substitute numerical values and evaluate $\frac{F_{\text{g,N-U}}}{F_{\text{g,U-S}}}$:

$$\frac{F_{\text{g,N-U}}}{F_{\text{g,U-S}}}=\frac{17.1}{3.33\times10^5\left(\left(\frac{164.8\text{ y}}{84.0\text{ y}}\right)^{2/3}-1\right)^2}\approx 2\times10^{-4}$$

Because this ratio is so small, during the time at which Neptune is closest to Uranus, the force exerted on Uranus by Neptune is much less than the force exerted on Uranus by the Sun.

96 •• It is believed that, at the center of our galaxy, is a ″super-massive″ black hole. One datum that leads to this conclusion is the important recent observation of stellar motion in the vicinity of the galactic center. If one such star moves in an elliptical orbit with a period of 15.2 years and has a semi-major axis of 5.5 light-days (the distance light travels in 5.5 days), what is the mass around which the star moves in its Keplerian orbit?

Picture the Problem We can apply Kepler's 3rd law to the orbital motion of the star to find the effective mass around which it is moving.

Using Kepler's 3rd law, relate the orbital period of the star to the semi-major axis of its orbit:

$$T^2=\frac{4\pi^2}{GM}a^3\Rightarrow M=\frac{4\pi^2a^3}{GT^2}$$

where M is the mass around which the star moves in its Keplerian orbit.

Substitute numerical values and evaluate M:

$$M=\frac{4\pi^2\left(5.5\,\text{d}\times\dfrac{86400\,\text{s}}{\text{d}}\times 2.998\times10^8\ \text{m/s}\right)^3}{\left(6.673\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2\right)\left(15.2\ \text{y}\times\dfrac{3.156\times10^7\ \text{s}}{\text{y}}\right)^2}=7.434\times10^{36}\ \text{kg}$$

$$=7.434\times10^{36}\ \text{kg}\times\frac{1\,M_{\text{Sun}}}{1.99\times10^{30}\ \text{kg}}=\boxed{3.7\times10^6\ M_{\text{Sun}}}$$

97 •• **[SSM]** Four identical planets are arranged in a square as shown in Figure 11-29. If the mass of each planet is M and the edge length of the square is a, what must be their speed if they are to orbit their common center under the influence of their mutual attraction?

Picture the Problem We can find the orbital speeds of the planets from their distance from the center of mass of the system and the period of their motion. Application of Kepler's 3rd law will allow us to express the period of their motion T in terms of the effective mass of the system; which we can find from its definition.

Express the orbital speeds of the planets in terms of their period T:

$$v=\frac{2\pi R}{T}\qquad(1)$$

where R is the distance to the center of mass of the four-planet system.

Apply Kepler's 3rd law to express the period of the planets:

$$T=\sqrt{\frac{4\pi^2}{GM_{\text{eff}}}R^3}$$

where M_{eff} is the effective mass of the four planets.

Substitute for T in equation (1) to obtain:

$$v=\frac{2\pi R}{\sqrt{\dfrac{4\pi^2}{GM_{\text{eff}}}R^3}}=\sqrt{\frac{GM_{\text{eff}}}{R}}\qquad(2)$$

The distance of each planet from the effective mass is:

$$R=\frac{a}{\sqrt{2}}$$

Find M_{eff} from its definition:

$$\frac{1}{M_{\text{eff}}}=\frac{1}{M}+\frac{1}{M}+\frac{1}{M}+\frac{1}{M}$$

and

$$M_{\text{eff}}=\tfrac{1}{4}M$$

Substitute for R and M_{eff} in equation (2) and simplify to obtain:

$$v = \boxed{\sqrt{\frac{\sqrt{2}GM}{4a}}}$$

98 •• A hole is drilled from the surface of Earth to its center as in Figure 11-30. Ignore Earth's rotation and any effects due to air resistance, and model Earth as a uniform sphere. (*a*) How much work is required to lift a particle of mass *m* from the center of Earth to Earth's surface? (*b*) If the particle is dropped from rest at the surface of Earth, what is its speed when it reaches the center of Earth? (*c*) What is the escape speed for a particle projected from the center of Earth? Express your answers in terms of *m, g,* and R_E.

Picture the Problem Let *r* represent the separation of the particle from the center of Earth and assume a uniform density for Earth. The work required to lift the particle from the center of Earth to its surface is the line integral of the gravitational force function. This function can be found from the law of gravity and by relating the mass of Earth between the particle and the center of Earth to Earth's mass. We can use the work-kinetic energy theorem to find the speed with which the particle, when released from the surface of Earth, will strike the center of Earth. Finally, the energy required for the particle to escape Earth from the center of Earth is the sum of the energy required to get it to the surface of Earth and the kinetic energy it must have to escape from the surface of Earth.

(*a*) Express the work required to lift the particle from the center of Earth to Earth's surface:

$$\boldsymbol{W} = \int_0^{\boldsymbol{R}} \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}} = -\int_0^{\boldsymbol{R}} \vec{\boldsymbol{F}}_g \cdot d\vec{\boldsymbol{r}} = \int_0^{\boldsymbol{R}_E} \boldsymbol{F}_g d\boldsymbol{r} \quad (1)$$

where F_g is the gravitational force acting on the particle.

Using the law of gravity, express the force acting on the particle as a function of its distance from the center of Earth:

$$F_g = \frac{GmM}{r^2} \quad (2)$$

where M is the mass of a sphere whose radius is r.

Express the ratio of M to M_E and simplify to obtain:

$$\frac{M}{M_E} = \frac{\rho\left(\frac{4}{3}\pi r^3\right)}{\rho\left(\frac{4}{3}\pi R_E^3\right)} = \frac{r^3}{R_E^3} \Rightarrow M = M_E \frac{r^3}{R_E^3}$$

Substitute for M in equation (2) to obtain:

$$F_g = \frac{GmM_E}{R_E^3} r = \frac{mgR_E^2}{R_E^3} r = \frac{mg}{R_E} r$$

Substitute for F_g in equation (1) and evaluate the integral:

$$W = \frac{mg}{R_E}\int_0^{R_E} r dr = \boxed{\tfrac{1}{2} gmR_E}$$

(*b*) Use the work-kinetic energy theorem to relate the kinetic energy of the particle as it reaches the center of Earth to the work done on it in moving it to the surface of Earth:

$$W = \Delta K = \tfrac{1}{2}mv^2$$

Substituting for W yields:

$$\tfrac{1}{2}gmR_E = \tfrac{1}{2}mv^2 \Rightarrow v = \boxed{\sqrt{gR_E}}$$

(*c*) Express the total energy required for the particle to escape when projected from the center of Earth:

$$E_{esc} = W + \tfrac{1}{2}mv_e^2 = \tfrac{1}{2}mv_{esc}^2$$

where v_e is the escape speed from the surface of Earth.

Substituting for W yields:

$$\tfrac{1}{2}gmR_E + \tfrac{1}{2}mv_e^2 = \tfrac{1}{2}mv_{esc}^2$$

or, simplifying,

$$gR_E + v_e^2 = v_{esc}^2$$

Because $v_e^2 = \dfrac{2GM}{R_E}$:

$$gR_E + \frac{2GM}{R_E} = v_{esc}^2 \qquad (3)$$

Apply Newton's 2nd law to an object of mass m at the surface of Earth to obtain:

$$mg = \frac{GMm}{R_E^2} \Rightarrow \frac{GM}{R_E} = gR_E$$

Substitute for GM/R_E in equation (3) to obtain:

$$gR_E + 2gR_E = v_{esc}^2 \Rightarrow v_{esc} = \boxed{\sqrt{3gR_E}}$$

Remarks: This escape speed is approximately 122% of the escape speed from the surface of Earth.

99 •• A thick spherical shell of mass M and uniform density has an inner radius R_1 and an outer radius R_2. Find the gravitational field g_r as a function of r for $0 < r < \infty$. Sketch a graph of g_r versus r.

Picture the Problem We need to find the gravitational field in three regions: $r < R_1$, $R_1 < r < R_2$, and $r > R_2$.

For $r < R_1$:

$$g(r < R_1) = \boxed{0}$$

For $r > R_2$, $g(r)$ is the field due to the thick spherical shell of mass M centered at the origin:

$$g(r > R_2) = \boxed{\frac{GM}{r^2}}$$

For $R_1 < r < R_2$, $g(r)$ is determined by the mass within the shell of radius r:

$$g(R_1 < r < R_2) = \frac{Gm}{r^2} \quad (1)$$

where $m = \frac{4}{3}\pi\rho\left(r^3 - R_1^3\right)$ (2)

Express the density of the spherical shell:

$$\rho = \frac{M}{V} = \frac{M}{\frac{4}{3}\pi\left(R_2^3 - R_1^3\right)}$$

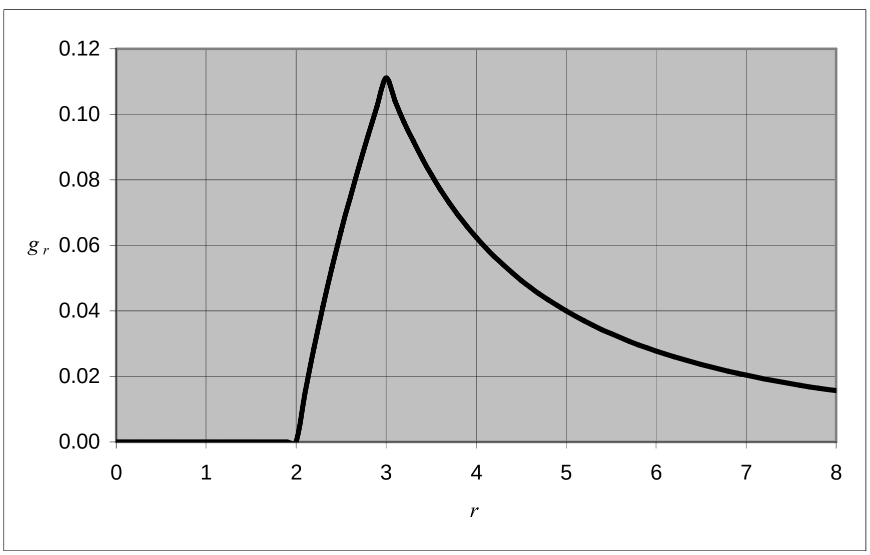
Substitute for ρ in equation (2) and simplify to obtain:

$$m = \frac{M\left(r^3 - R_1^3\right)}{R_2^3 - R_1^3}$$

Substitute for m in equation (1) to obtain:

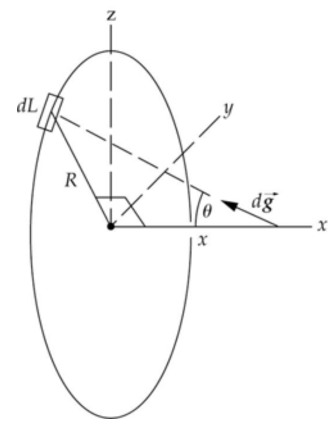
$$g(r) = \boxed{\frac{GM\left(r^3 - R_1^3\right)}{r^2\left(R_2^3 - R_1^3\right)}}$$

A graph of g_r with $R_1 = 2$, $R_2 = 3$, and $GM = 1$ follows:



100 •• (*a*) A thin uniform ring of mass M and radius R lies in the $x = 0$ plane and is centered at the origin. Sketch a plot of the gravitational field g_x versus x for all points on the x axis. (*b*) At what point, or points, on the axis is the magnitude of g_x a maximum?

Picture the Problem A ring of radius R is shown to the right. Choose a coordinate system in which the origin is at the center of the ring and x axis is as shown. An element of length dL and mass dm is responsible for the field dg at a distance x from the center of the ring. We can express the x component of dg and then integrate over the circumference of the ring to find the total field as a function of x.



(a) Express the differential gravitational field at a distance x from the center of the ring in terms of the mass of an elemental segment of length dL:

$$dg = \frac{Gdm}{R^2 + x^2}$$

Relate the mass of the element to its length:

$$dm = \lambda\, dL$$

where λ is the linear density of the ring.

Substitute for dm to obtain:

$$dg = \frac{G\lambda\, dL}{R^2 + x^2}$$

By symmetry, the y and z components of g vanish. The x component of dg is:

$$dg_x = dg\cos\theta = \frac{G\lambda\, dL}{R^2 + x^2}\cos\theta$$

Refer to the figure to obtain:

$$\cos\theta = \frac{x}{\sqrt{R^2 + x^2}}$$

Substituting for $\cos\theta$ yields:

$$dg_x = \frac{G\lambda\, dL}{R^2 + x^2}\frac{x}{\sqrt{R^2 + x^2}} = \frac{G\lambda\, xdL}{\left(R^2 + x^2\right)^{3/2}}$$

Because $\lambda = \dfrac{M}{2\pi R}$:

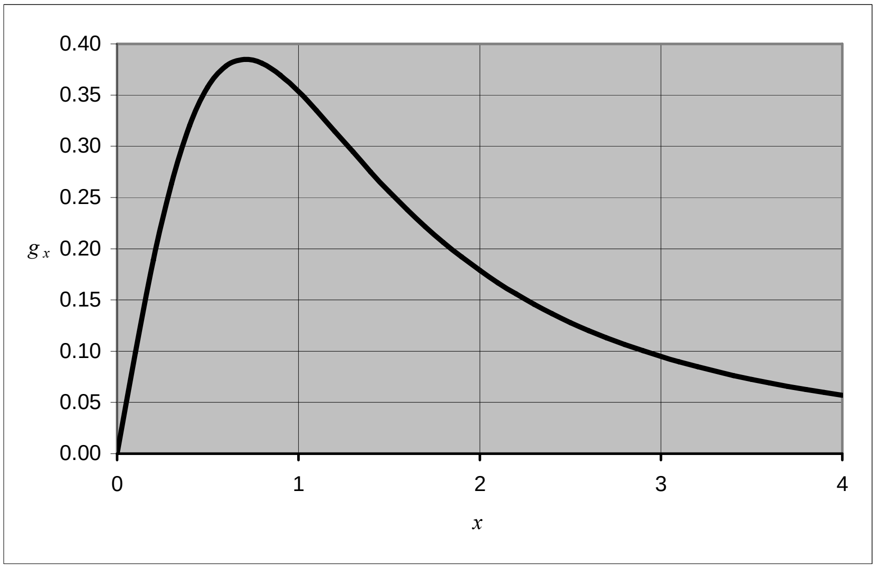
$$dg_x = \frac{GM\, xdL}{2\pi R\left(R^2 + x^2\right)^{3/2}}$$

Integrate to find $g(x)$:

$$g(x)=\frac{GM\,x}{2\pi\,R\left(R^2+x^2\right)^{3/2}}\int_0^{2\pi R}dL$$

$$=\boxed{\frac{GM}{\left(R^2+x^2\right)^{3/2}}\,x}$$

A graph of g_x follows. The curve is normalized with $R = 1$ and $GM = 1$.



(*b*) Differentiate $g(x)$ with respect to x and set the derivative equal to zero to identify extreme values:

$$\frac{dg}{dx}=GM\left[\frac{\left(x^2+R^2\right)^{3/2}-x\left(\frac{3}{2}\right)\left(x^2+R^2\right)^{1/2}(2x)}{\left(R^2+x^2\right)^3}\right]=0\text{ for extrema}$$

Simplify to obtain:

$$\left(x^2+R^2\right)^{3/2}-3x^2\left(x^2+R^2\right)^{1/2}=0$$

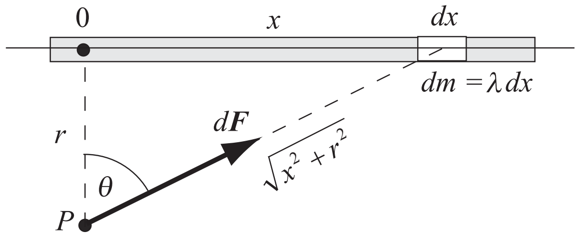
Solving for x yields:

$$x=\boxed{\pm\frac{R}{\sqrt{2}}}$$

Because the curve is concave downward, we can conclude that this result corresponds to a maximum. Note that this result agrees with our graphical maximum.

101 ••• Find the magnitude of the gravitational field a distance r from an infinitely long uniform thin rod whose mass per unit length is λ.

Picture the Problem The diagram shows a segment of the wire of length dx and mass $dm = \lambda dx$ at a distance x from the origin of our coordinate system. We can find the magnitude of the gravitational field at a distance r from the wire from the resultant gravitational force acting on a particle of mass m' located at point P and then integrating over the length of the wire.



Express the gravitational force acting on a particle of mass m' at a distance r from the wire due to the segment of the wire of length dx:

$$dF = m'dg \text{ or } dg = \frac{dF}{m'}$$

Using Newton's law of gravity, express dF:

$$dF = \frac{Gm'\lambda dx}{R^2}$$

or, because $R^2 = x^2 + r^2$,

$$dF = \frac{Gm'\lambda dx}{x^2 + r^2}$$

Substitute and simplify to express the gravitational field due to the segment of the wire of length dx:

$$dg = \frac{G\lambda dx}{x^2 + r^2}$$

By symmetry, the segment on the opposite side of the origin at the same distance from the origin will cancel out all but the radial component of the field, so the gravitational field will be given by:

$$dg = \frac{G\lambda dx}{x^2 + r^2}\cos\theta = \frac{G\lambda dx}{x^2 + r^2}\frac{r}{\sqrt{x^2 + r^2}} = \frac{G\lambda r}{\left(x^2 + r^2\right)^{3/2}}dx$$

Integrate dg from $x' = -\infty$ to $x' = +\infty$ to obtain:

$$g = \int_{-\infty}^{\infty}\frac{G\lambda r}{\left(x'^2 + r^2\right)^{3/2}}dx' = 2G\lambda\int_{0}^{\infty}\frac{r}{\left(x'^2 + r^2\right)^{3/2}}dx' = \frac{2G\lambda}{r}\left[\frac{x}{\sqrt{x'^2 + r^2}}\right]_0^{\infty} = \boxed{\frac{2G\lambda}{r}}$$

102 ••• One question in early planetary science was whether each of the rings of Saturn was solid or was, instead, composed of individual chunks, each in its own orbit. There was a simple ring speed observation that resolved this issue. Here was the idea: astronomers would measure the speed of the inner and outer portion of the ring. If the inner portion of the ring moved more slowly than the outer portion, then the ring was solid; if the opposite was true, then it was actually composed of separate chunks. Let's see how this results from a theoretical viewpoint. Let the radial width of a given ring (there are many) be Δr, the average distance of that ring from the center of Saturn be represented by R, and designate the average speed of that ring by v_{avg}. (*a*) If the ring is solid, show that the *difference* in speed between its outermost and innermost portions, Δv, is given by the approximate expression $\boldsymbol{\Delta v} = \boldsymbol{v}_{out} - \boldsymbol{v}_{in} \approx \boldsymbol{v}_{avg}\dfrac{\boldsymbol{\Delta r}}{\boldsymbol{R}}$. Here, v_{out} is the speed of the outermost portion of the ring, v_{in} is the speed of the innermost portion. (*b*) If, however, the ring is composed of many small chunks, show that $\Delta v \approx -\dfrac{1}{2}\left(v_{avg}\dfrac{\Delta r}{R}\right)$. (Assume that $\Delta r \ll R$.)

Picture the Problem We can use the relationship between the angular speed of an orbiting object and its tangential velocity to express the speeds v_{in} and v_{out} of the innermost and outermost portions of the ring. In Part (*b*) we can use Newton's law of gravity, in conjunction with the 2nd law of motion, to relate the tangential speed of a chunk of the ring to the gravitational force acting on it. As in Part (*a*), once we know v_{in} and v_{out}, we can express the difference between them to obtain the desired results.

(*a*) The difference between v_{out} and v_{in} is:

$$\boldsymbol{\Delta v} = \boldsymbol{v}_{out} - \boldsymbol{v}_{in} \qquad (1)$$

The speed of a point in the ring at the average distance R from the center of Saturn under the assumption that the ring is solid and rotates with an angular speed ω is given by:

$$v(R) = \omega R$$

Express the speeds v_{in} and v_{out} of the innermost and outermost portions of the ring:

$$v_{in} = \left(R - \tfrac{1}{2}\Delta r\right)\omega$$
and
$$v_{out} = \left(R + \tfrac{1}{2}\Delta r\right)\omega$$

Substituting for v_{in} and v_{out} in equation (1) and simplifying yields:

$$\begin{aligned}\Delta v &= \left(R + \tfrac{1}{2}\Delta r\right)\omega - \left(R - \tfrac{1}{2}\Delta r\right)\omega \\ &= \omega\Delta r = \frac{v_{avg}}{R}\Delta r = \boxed{v_{avg}\frac{\Delta r}{R}}\end{aligned}$$

(*b*) Assume that a chunk of the ring is moving in a circular orbit around the center of Saturn under the force of gravity and apply Newton's 2nd law to obtain:

$$\sum F_{radial} = \frac{GMm}{R'^2} = m\frac{v^2}{R'} \Rightarrow v = \sqrt{\frac{GM}{R'}}$$

where M is the mass of Saturn and R' the distance from its center.

Express v_{out} by substituting for $R+\frac{1}{2}\Delta r$ for $\boldsymbol{R'}$ and simplifying:

$$v_{out}=\sqrt{\frac{GM}{R+\frac{1}{2}\Delta r}}=\sqrt{\frac{GM}{R\left(1+\frac{1}{2}\frac{\Delta r}{R}\right)}}$$

$$=\sqrt{\frac{GM}{R}}\left(1+\frac{1}{2}\frac{\Delta r}{R}\right)^{-1/2}$$

Expanding $\left(1+\frac{1}{2}\frac{\Delta r}{R}\right)^{-1/2}$ binomially, discarding higher-order terms, and simplifying yields:

$$v_{out}=\sqrt{\frac{GM}{R}}\left(1-\frac{1}{2}\left(\frac{1}{2}\frac{\Delta r}{R}\right)\right.$$
$$\left.+\text{higher order terms}\right)$$

$$\approx\sqrt{\frac{GM}{R}}\left(1-\frac{1}{4}\frac{\Delta r}{R}\right)$$

Proceed similarly to obtain, for v_{in}:

$$v_{in}\approx\sqrt{\frac{GM}{R}}\left(1+\frac{1}{4}\frac{\Delta r}{R}\right)$$

Express the difference between v_{out} and v_{in} and simplify to obtain:

$$\Delta v=v_{out}-v_{in}\approx\sqrt{\frac{GM}{R}}\left(1-\frac{1}{4}\frac{\Delta r}{R}\right)-\sqrt{\frac{GM}{R}}\left(1+\frac{1}{4}\frac{\Delta r}{R}\right)=\sqrt{\frac{GM}{R}}\left(-\frac{1}{2}\frac{\Delta r}{R}\right)$$

Because $v_{avg}=\sqrt{\frac{GM}{R}}$:

$$\Delta v\approx\boxed{-\frac{1}{2}\left(v_{avg}\frac{\Delta r}{R}\right)}$$

103 ••• **[SSM]** In this problem you are to find the gravitational potential energy of the thin rod in Example 11-8 and a point particle of mass m_0 that is on the x axis at $x = x_0$. (*a*) Show that the potential energy shared by an element of the rod of mass dm (shown in Figure 11-14) and the point particle of mass m_0 located at $x_0\geq\frac{1}{2}L$ is given by

$$dU=-\frac{Gm_0dm}{x_0-x_s}=\frac{GMm_0}{L(x_0-x_s)}dx_s$$

where $U = 0$ at $x_0 = \infty$. (*b*) Integrate your result for Part (*a*) over the length of the rod to find the total potential energy for the system. Generalize your function $U(x_0)$ to any place on the x axis in the region $x > L/2$ by replacing x_0 by a general coordinate x and write it as $U(x)$. (*c*) Compute the force on m_0 at a general point x using $F_x = -dU/dx$ and compare your result with m_0g, where g is the field at x_0 calculated in Example 11-8.

Picture the Problem Let $U = 0$ at $x = \infty$. The potential energy of an element of the stick dm and the point mass m_0 is given by the definition of gravitational potential energy: $dU=-Gm_0dm/r$ where r is the separation of dm and m_0.

(*a*) Express the potential energy of the masses m_0 and dm:

$$dU = -\frac{Gm_0 dm}{x_0 - x_s}$$

The mass dm is proportional to the size of the element dx_s:

$$dm = \lambda\, dx_s$$

where $\lambda = \frac{M}{L}$.

Substitute for dm and λ to express dU in terms of x_s:

$$dU = -\frac{Gm_0 \lambda\, dx_s}{x_0 - x_s} = \boxed{-\frac{GMm_0\, dx_s}{L(x_0 - x_s)}}$$

(*b*) Integrate dU to find the total potential energy of the system:

$$U = -\frac{GMm_0}{L}\int_{-L/2}^{L/2} \frac{dx_s}{x_0 - x_s} = \frac{GMm_0}{L}\left[\ln\left(x_0 - \frac{L}{2}\right) - \ln\left(x_0 + \frac{L}{2}\right)\right]$$

$$= \boxed{-\frac{GMm_0}{L}\ln\left(\frac{x_0 + L/2}{x_0 - L/2}\right)}$$

(*c*) Because x_0 is a general point along the x axis:

$$F(x_0) = -\frac{dU}{dx_0} = \frac{GMm_0}{L}\left[\frac{1}{x_0 + \frac{L}{2}} - \frac{1}{x_0 - \frac{L}{2}}\right]$$

Further simplification yields:

$$F(x_0) = -\frac{Gmm_0}{x^2 - L^2/4}$$

This answer and the answer given in Example 11-8 are the same.

104 ••• A uniform sphere of mass M is located near a thin, uniform rod of mass m and length L as in Figure 11-31. Find the gravitational force of attraction exerted by the sphere on the rod.

Picture the Problem Choose a mass element dm of the rod of thickness dx at a distance x from the origin. All such elements of the rod experience a gravitational force dF due to presence of the sphere centered at the origin. We can find the total gravitational force of attraction experienced by the rod by integrating dF from $x = a$ to $x = a + L$.

Express the gravitational force dF acting on the element of the rod of mass dm:

$$dF = \frac{GMdm}{x^2}$$

Express dm in terms of the mass m and length L of the rod:

$$dm = \frac{m}{L}dx$$

Substitute for dm to obtain:

$$dF = \frac{GMm}{L}\frac{dx}{x^2}$$

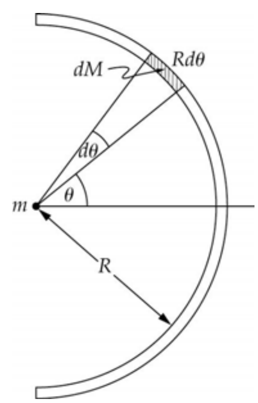
Integrate dF from $x = a$ to $x = a + L$ to find the total gravitational force acting on the rod:

$$F = \frac{GMm}{L}\int_a^{a+L} x^{-2}dx = -\frac{GMm}{L}\left[\frac{1}{x}\right]_a^{a+L}$$

$$= \boxed{\frac{GMm}{a(a+L)}}$$

105 ••• A thin uniform 20-kg rod with a length equal to 5.0 m is bent into a semicircle. What is the gravitational force exerted by the rod on a 0.10-kg point mass located at the center of curvature of the circular arc?

Picture the Problem The semicircular rod is shown in the figure. We'll use an element of length $Rd\theta = (L/\pi)d\theta$ whose mass dM is $(M/\pi)d\theta$. By symmetry, $F_y = 0$. We'll first find dF_x and then integrate over θ from $-\pi/2$ to $\pi/2$.



Express dF_x:

$$dF_x = \frac{GmdM}{R^2} = \frac{GMm}{\pi\left(\frac{L}{\pi}\right)^2}d\theta\cos\theta$$

Integrate dF_x over θ from $-\pi/2$ to $\pi/2$:

$$F_x = \frac{\pi\, GMm}{L^2}\int_{-\pi/2}^{\pi/2}\cos\theta\, d\theta = \frac{2\pi\, GMm}{L^2}$$

Substitute numerical values and evaluate F_x:

$$\boldsymbol{F_x} = \frac{2\boldsymbol{\pi}(6.673\times10^{-11}\,\text{N}\cdot\text{m}^2/\text{kg}^2)(20\,\text{kg})(0.10\,\text{kg})}{(5.0\,\text{m})^2} = \boxed{34\,\text{pN}}$$

106 ••• Both the Sun and the moon exert gravitational forces on the oceans of Earth, causing tides. (*a*) Show that the ratio of the force exerted on a point particle on the surface of Earth by the Sun to that exerted by the moon is $M_s r_m^2 / M_m r_s^2$. Here M_s and M_m represent the masses of the Sun and moon and r_s and r_m are the distances of the particle from Earth to the Sun and Earth to the moon, respectively. Evaluate this ratio numerically. (*b*) Even though the Sun exerts a much greater force on the oceans than does the moon, the moon has a greater effect on the tides because it is the *difference* in the force from one side of Earth to the other that is important. Differentiate the expression $F = G\,m_1m_2/r^2$ to calculate the change in F due to a small change in r. Show that $dF/F = (-2\,dr)/r$. (*c*) The oceanic *tidal bulge* (that is, the elongation of the liquid water of the oceans causing two opposite high and two opposite low spots) is caused by the difference in gravitational force on the oceans from one side of Earth to the other. Show that for a small difference in distance compared to the average distance, the ratio of the differential gravitational force exerted by the Sun to the differential gravitational force exerted by the moon on Earth's oceans is given by $\Delta F_s / \Delta F_m \approx \left(M_s r_m^3\right) / \left(M_m r_s^3\right)$. Calculate this ratio. What is your conclusion? Which object, the moon or the Sun, is the main cause of the tidal stretching of the oceans on Earth?

Picture the Problem We can begin by expressing the forces exerted by the Sun and the moon on a body of water of mass m and taking the ratio of these forces. In (*b*) we'll simply follow the given directions and in (*c*) we can approximate differential quantities with finite quantities to establish the given ratio.

(*a*) Express the force exerted by the sun on a body of water of mass m:

$$F_S = \frac{GM_S m}{r_S^2}$$

Express the force exerted by the moon on a body of water of mass m:

$$F_m = \frac{GM_m m}{r_m^2}$$

Divide the first of these equations by the second and simplify to obtain:

$$\frac{F_S}{F_m} = \boxed{\frac{M_S r_m^2}{M_m r_S^2}}$$

Substitute numerical values and evaluate this ratio:

$$\frac{F_S}{F_m} = \frac{\left(1.99\times10^{30}\text{ kg}\right)\left(3.84\times10^{8}\text{ m}\right)^2}{\left(7.36\times10^{22}\text{ kg}\right)\left(1.50\times10^{11}\text{ m}\right)^2} = \boxed{177}$$

(*b*) Find $\dfrac{dF}{dr}$:

$$\frac{dF}{dr} = -\frac{2Gm_1m_2}{r^3} = -2\frac{F}{r}$$

Solve for the ratio $\frac{dF}{F}$:

$$\frac{dF}{F} = \boxed{-2\frac{dr}{r}}$$

(*c*) Express the change in force ΔF for a small change in distance Δr:

$$\Delta F = -2\frac{F}{r}\Delta r$$

Express ΔF_S:

$$\Delta F_S = -2\frac{\frac{GmM_S}{r_S^2}}{r_S}\Delta r_S = -2\frac{GmM_S}{r_S^3}\Delta r_S$$

Express ΔF_m:

$$\Delta F_m = -2\frac{GmM_m}{r_m^3}\Delta r_m$$

Divide the first of these equations by the second and simplify:

$$\frac{\Delta F_S}{\Delta F_m} = \frac{\frac{M_S}{r_S^3}\Delta r_S}{\frac{M_m}{r_m^3}\Delta r_m} = \frac{M_S r_m^3}{M_m r_S^3}\frac{\Delta r_S}{\Delta r_m}$$

Because $\frac{\Delta r_S}{\Delta r_m} = 1$:

$$\frac{\Delta F_S}{\Delta F_m} = \boxed{\frac{M_S r_m^3}{M_m r_S^3}}$$

Substitute numerical values and evaluate this ratio:

$$\frac{\Delta F_S}{\Delta F_m} = \frac{(1.99\times10^{30}\text{ kg})(3.84\times10^{8}\text{ m})^3}{(7.36\times10^{22}\text{ kg})(1.50\times10^{11}\text{ m})^3} = \boxed{0.454}$$

Because the ratio of the forces is less than one, the moon is the main cause of the tidal stretching of the oceans on Earth.

107 ••• United Federation Starship *Excelsior* is dropping two small robotic probes towards the surface of a neutron star for exploration. The mass of the star is the same as that of the Sun, but the star's diameter is only 10 km. The robotic probes are linked together by a 1.0-m-long steel cord (which includes communication lines between the two probes), and are dropped vertically (that is, one always above the other). The ship hovers at rest above the star's surface. As the Chief of Materials Engineering on the ship, you are concerned that the communication between the two probes, a crucial aspect of the mission, will not survive. (*a*) Outline your briefing session to the mission commander and explain the existence of a "stretching force" that will try to pull the robots apart as they fall toward the planet. (See Problems 105 and 106 for hints.) (*b*) Assume that the cord in use has a breaking tension of 25 kN and that the robots each have a mass of 1.0 kg. How close will the robots be to the surface of the star before the cord breaks?

Picture the Problem Let M_{NS} be the mass of the neutron star and m the mass of each robot. We can use Newton's law of gravity to express the difference in the tidal-like forces acting on the coupled robots. Expanding the expression for the force on the robot further from the neutron star binomially will lead us to an expression for the distance at which the breaking tension in the connecting cord will be exceeded.

(*a*) The gravitational force is greater on the lower robot, so if it were not for the cable its acceleration would be greater than that of the upper robot and they would separate. In opposing this separation the cable is stressed.

(*b*) Letting the separation of the two robots be Δr, and the distance from the center of the star to the lower robot be r, use Newton's law of gravity to express the difference in the forces acting on the robots:

$$F_{\text{tide}} = \frac{GM_{NS}m}{r^2} - \frac{GM_{NS}m}{(r+\Delta r)^2}$$

$$= GM_{NS}m\left[\frac{1}{r^2} - \frac{1}{r^2\left(1+\frac{\Delta r}{r}\right)^2}\right]$$

$$= \frac{GM_{NS}m}{r^2}\left[1-\left(1+\frac{\Delta r}{r}\right)^{-2}\right]$$

Expand the expression in the square brackets binomially and simplify to obtain:

$$1-\left(1+\frac{\Delta r}{r}\right)^{-2} \approx 1-\left(1-2\frac{\Delta r}{r}\right)$$

$$= 2\frac{\Delta r}{r}$$

Substituting yields:

$$F_{\text{tide}} \approx \frac{2GM_{NS}m}{r^3}\Delta r$$

Letting F_B be the breaking tension of the cord, substitute for F_{tide} and solve for the value of r corresponding to the breaking strain being exceeded:

$$r = \sqrt[3]{\frac{2GM_{NS}m}{F_B}\Delta r}$$

Substitute numerical values and evaluate r:

$$\boldsymbol{r} = \sqrt[3]{\frac{2(6.673\times10^{-11}\,\text{N}\cdot\text{m}^2/\text{kg}^2)(1.99\times10^{30}\,\text{kg})(1.0\,\text{kg})}{25\,\text{kN}}(1.0\,\text{m})} = \boxed{2.2\times10^5\ \text{m}}$$