Chapter 14
Oscillations

Conceptual Problems

1 • True or false:
(*a*) For a simple harmonic oscillator, the period is proportional to the square of the amplitude.
(*b*) For a simple harmonic oscillator, the frequency does not depend on the amplitude.
(*c*) If the net force on a particle undergoing one-dimensional motion is proportional to, and oppositely directed from, the displacement from equilibrium, the motion is simple harmonic.

(*a*) False. In simple harmonic motion, the period is independent of the amplitude.

(*b*) True. In simple harmonic motion, the frequency is the reciprocal of the period which, in turn, is independent of the amplitude.

(*c*) True. This is the condition for simple harmonic motion

2 • If the amplitude of a simple harmonic oscillator is tripled, by what factor is the energy changed?

Determine the Concept The energy of a simple harmonic oscillator varies as the square of the amplitude of its motion. Hence, tripling the amplitude increases the energy by a factor of 9.

3 •• **[SSM]** An object attached to a spring exhibits simple harmonic motion with an amplitude of 4.0 cm. When the object is 2.0 cm from the equilibrium position, what percentage of its total mechanical energy is in the form of potential energy? (*a*) One-quarter. (*b*) One-third. (*c*) One-half. (*d*) Two-thirds. (*e*) Three-quarters.

Picture the Problem The total energy of an object undergoing simple harmonic motion is given by $E_{\text{tot}} = \frac{1}{2}kA^2$, where k is the force constant and A is the amplitude of the motion. The potential energy of the oscillator when it is a distance x from its equilibrium position is $U(x) = \frac{1}{2}kx^2$.

Express the ratio of the potential energy of the object when it is 2.0 cm from the equilibrium position to its total energy:

$$\frac{U(x)}{E_{\text{tot}}} = \frac{\frac{1}{2}kx^2}{\frac{1}{2}kA^2} = \frac{x^2}{A^2}$$

Evaluate this ratio for $x = 2.0$ cm and $A = 4.0$ cm:

$$\frac{U(2\,\text{cm})}{E_{\text{tot}}} = \frac{(2.0\,\text{cm})^2}{(4.0\,\text{cm})^2} = \frac{1}{4}$$

and $\boxed{(a)}$ is correct.

4 • An object attached to a spring exhibits simple harmonic motion with an amplitude of 10.0 cm. How far from equilibrium will the object be when the system's potential energy is equal to its kinetic energy? (*a*) 5.00 cm. (*b*) 7.07 cm. (*c*) 9.00 cm. (*d*) The distance can't be determined from the data given.

Determine the Concept Because the object's total energy is the sum of its kinetic and potential energies, when its potential energy equals its kinetic energy, its potential energy (and its kinetic energy) equals one-half its total energy.

Equate the object's potential energy to one-half its total energy:

$$U_s = \tfrac{1}{2}E_{\text{total}}$$

Substituting for U_s and E_{total} yields:

$$\tfrac{1}{2}kx^2 = \tfrac{1}{2}\left(\tfrac{1}{2}kA^2\right) \Rightarrow x = \frac{A}{\sqrt{2}}$$

Substitute the numerical value of A and evaluate x to obtain:

$$x = \frac{10.0\,\text{cm}}{\sqrt{2}} = 7.07\,\text{cm}$$

and $\boxed{(b)}$ is correct.

5 • Two identical systems each consist of a spring with one end attached to a block and the other end attached to a wall. The springs are horizontal, and the blocks are supported from below by a frictionless horizontal table. The blocks are oscillating in simple harmonic motions such that the amplitude of the motion of block A is four times as large as the amplitude of the motion of block B. How do their maximum speeds compare? (*a*) $v_{\text{A max}} = v_{\text{B max}}$, (*b*) $v_{\text{A max}} = 2v_{\text{B max}}$, (*c*) $v_{\text{A max}} = 4v_{\text{B max}}$, (*d*) This comparison cannot be done by using the data given.

Determine the Concept The maximum speed of a simple harmonic oscillator is the product of its angular frequency and its amplitude. The angular frequency of a simple harmonic oscillator is the square root of the quotient of the force constant of the spring and the mass of the oscillator.

Relate the maximum speed of system A to the amplitude of its motion:

$$v_{\text{A max}} = \omega_{\text{A}} A_{\text{A}}$$

Relate the maximum speed of system B to the amplitude of its motion:

$$v_{\text{B max}} = \omega_{\text{B}} A_{\text{B}}$$

Divide the first of these equations by the second to obtain:

$$\frac{v_{\text{A max}}}{v_{\text{B max}}} = \frac{\omega_{\text{A}} A_{\text{A}}}{\omega_{\text{B}} A_{\text{B}}}$$

Because the systems differ only in amplitude, $\boldsymbol{\omega}_{\text{A}} = \boldsymbol{\omega}_{\text{B}}$, and:

$$\frac{v_{\text{A max}}}{v_{\text{B max}}} = \frac{A_{\text{A}}}{A_{\text{B}}}$$

Substituting for A_{A} and simplifying yields:

$$\frac{v_{\text{A max}}}{v_{\text{B max}}} = \frac{4A_{\text{B}}}{A_{\text{B}}} = 4 \Rightarrow v_{\text{A max}} = 4v_{\text{B max}}$$

$\boxed{(c)}$ is correct.

6 • Two systems each consist of a spring with one end attached to a block and the other end attached to a wall. The springs are horizontal, and the blocks are supported from below by a frictionless horizontal table. The blocks are oscillating in simple harmonic motions with equal amplitudes. However, the force constant of spring A is four times as large as the force constant of spring B. How do their maximum speeds compare? (*a*) $v_{\text{A max}} = v_{\text{B max}}$, (*b*) $v_{\text{A max}} = 2v_{\text{B max}}$, (*c*) $v_{\text{A max}} = 4v_{\text{B max}}$, (*d*) This comparison cannot be done by using the data given.

Determine the Concept The maximum speed of a simple harmonic oscillator is the product of its angular frequency and its amplitude. The angular frequency of a simple harmonic oscillator is the square root of the quotient of the force constant of the spring and the mass of the oscillator.

Relate the maximum speed of system A to its force constant:

$$v_{\text{A max}} = \omega_{\text{A}} A_{\text{A}} = \sqrt{\frac{k_{\text{A}}}{m_{\text{A}}}} A_{\text{A}}$$

Relate the maximum speed of system B to its force constant:

$$v_{\text{B max}} = \omega_{\text{B}} A_{\text{B}} = \sqrt{\frac{k_{\text{B}}}{m_{\text{B}}}} A_{\text{B}}$$

Divide the first of these equations by the second and simplify to obtain:

$$\frac{v_{\text{A max}}}{v_{\text{B max}}} = \frac{\sqrt{\frac{k_{\text{A}}}{m_{\text{A}}}} A_{\text{A}}}{\sqrt{\frac{k_{\text{B}}}{m_{\text{B}}}} A_{\text{B}}} = \sqrt{\frac{m_{\text{B}}}{m_{\text{A}}} \frac{k_{\text{A}}}{k_{\text{B}}}} \frac{A_{\text{A}}}{A_{\text{B}}}$$

Because the systems differ only in their force constants:

$$\frac{v_{\text{A max}}}{v_{\text{B max}}} = \sqrt{\frac{k_{\text{A}}}{k_{\text{B}}}}$$

Substituting for k_A and simplifying yields:

$$\frac{v_{A\,max}}{v_{B\,max}} = \sqrt{\frac{4k_B}{k_B}} = 2 \Rightarrow v_{A\,max} = 2v_{B\,max}$$

$\boxed{(b)}$ is correct.

7 •• **[SSM]** Two systems each consist of a spring with one end attached to a block and the other end attached to a wall. The identical springs are horizontal, and the blocks are supported from below by a frictionless horizontal table. The blocks are oscillating in simple harmonic motions with equal amplitudes. However, the mass of block A is four times as large as the mass of block B. How do their maximum speeds compare? (*a*) $v_{A\,max} = v_{B\,max}$, (*b*) $v_{A\,max} = 2v_{B\,max}$, (*c*) $v_{A\,max} = \frac{1}{2}v_{B\,max}$, (*d*) This comparison cannot be done by using the data given.

Determine the Concept The maximum speed of a simple harmonic oscillator is the product of its angular frequency and its amplitude. The angular frequency of a simple harmonic oscillator is the square root of the quotient of the force constant of the spring and the mass of the oscillator.

Relate the maximum speed of system A to its force constant:

$$v_{A\,max} = \omega_A A_A = \sqrt{\frac{k_A}{m_A}}A_A$$

Relate the maximum speed of system B to its force constant:

$$v_{B\,max} = \omega_B A_B = \sqrt{\frac{k_B}{m_B}}A_B$$

Divide the first of these equations by the second and simplify to obtain:

$$\frac{v_{A\,max}}{v_{B\,max}} = \frac{\sqrt{\frac{k_A}{m_A}}A_A}{\sqrt{\frac{k_B}{m_B}}A_B} = \sqrt{\frac{m_B}{m_A}\frac{k_A}{k_B}}\frac{A_A}{A_B}$$

Because the systems differ only in the masses attached to the springs:

$$\frac{v_{A\,max}}{v_{B\,max}} = \sqrt{\frac{m_B}{m_A}}$$

Substituting for m_A and simplifying yields:

$$\frac{v_{A\,max}}{v_{B\,max}} = \sqrt{\frac{m_B}{4m_B}} = \tfrac{1}{2} \Rightarrow v_{A\,max} = \tfrac{1}{2}v_{B\,max}$$

$\boxed{(c)}$ is correct.

8 •• Two systems each consist of a spring with one end attached to a block and the other end attached to a wall. The identical springs are horizontal, and the blocks are supported from below by a frictionless horizontal table. The blocks are

oscillating in simple harmonic motions with equal amplitudes. However, the mass of block A is four times as large as the mass of block B. How do the magnitudes of their maximum acceleration compare? (*a*) $a_{\text{A max}} = a_{\text{B max}}$, (*b*) $a_{\text{A max}} = 2a_{\text{B max}}$, (*c*) $a_{\text{A max}} = \frac{1}{2}a_{\text{B max}}$, (*d*) $a_{\text{A max}} = \frac{1}{4}a_{\text{B max}}$, (*e*) This comparison cannot be done by using the data given.

Determine the Concept The maximum acceleration of a simple harmonic oscillator is the product of the square of its angular frequency and its amplitude. The angular frequency of a simple harmonic oscillator is the square root of the quotient of the force constant of the spring and the mass of the oscillator.

Relate the maximum acceleration of system A to its force constant:

$$a_{\text{A max}} = \omega_{\text{A}}^2 A_{\text{A}} = \frac{k_{\text{A}}}{m_{\text{A}}} A_{\text{A}}$$

Relate the maximum acceleration of system B to its force constant:

$$a_{\text{B max}} = \omega_{\text{B}}^2 A_{\text{B}} = \frac{k_{\text{B}}}{m_{\text{B}}} A_{\text{B}}$$

Divide the first of these equations by the second and simplify to obtain:

$$\frac{a_{\text{A max}}}{a_{,\text{max}}} = \frac{\frac{k_{\text{A}}}{m_{\text{A}}} A_{\text{A}}}{\frac{k_{\text{B}}}{m_{\text{B}}} A_{\text{B}}} = \frac{m_{\text{B}}}{m_{\text{A}}} \frac{k_{\text{A}}}{k_{\text{B}}} \frac{A_{\text{A}}}{A_{\text{B}}}$$

Because the systems differ only in the masses attached to the springs:

$$\frac{a_{\text{A max}}}{a_{\text{B max}}} = \frac{m_{\text{B}}}{m_{\text{A}}}$$

Substituting for m_{A} and simplifying yields:

$$\frac{a_{\text{A max}}}{a_{\text{B max}}} = \frac{m_{\text{B}}}{4m_{\text{B}}} = \tfrac{1}{4} \Rightarrow a_{\text{A max}} = \tfrac{1}{4}a_{\text{B max}}$$

$\boxed{(d)}$ is correct.

9 •• **[SSM]** In general physics courses, the mass of the spring in simple harmonic motion is usually neglected because its mass is usually much smaller than the mass of the object attached to it. However, this is not always the case. If you neglect the mass of the spring when it is not negligible, how will your calculation of the system's period, frequency and total energy compare to the actual values of these parameters? Explain.

Determine the Concept Neglecting the mass of the spring, the period of a simple harmonic oscillator is given by $T = 2\pi/\omega = 2\pi\sqrt{m/k}$ where *m* is the mass of the oscillating system (spring plus object) and its total energy is given by $E_{\text{total}} = \frac{1}{2}kA^2$.

Neglecting the mass of the spring results in your using a value for the mass of the oscillating system that is smaller than its actual value. Hence your calculated value for the period will be smaller than the actual period of the system.

Because $\omega = \sqrt{k/m}$, neglecting the mass of the spring will result in your using a value for the mass of the oscillating system that is smaller than its actual value. Hence your calculated value for the frequency of the system will be larger than the actual frequency of the system.

Because the total energy of the oscillating system is the sum of its potential and kinetic energies, ignoring the mass of the spring will cause your calculation of the system's kinetic energy to be too small and, hence, your calculation of the total energy to be too small.

10 •• Two mass–spring systems oscillate with periods T_A and T_B. If $T_A = 2T_B$ and the systems' springs have identical force constants, it follows that the systems' masses are related by (*a*) $m_A = 4m_B$, (*b*) $m_A = m_B/\sqrt{2}$, (*c*) $m_A = m_B/2$, (*d*) $m_A = m_B/4$.

Picture the Problem We can use $T = 2\pi\sqrt{m/k}$ to express the periods of the two mass-spring systems in terms of their force constants. Dividing one of the equations by the other will allow us to express m_A in terms of m_B.

Express the period of system A:

$$T_A = 2\pi\sqrt{\frac{m_A}{k_A}} \Rightarrow m_A = \frac{k_A T_A^2}{4\pi^2}$$

Relate the mass of system B to its period:

$$m_B = \frac{k_B T_B^2}{4\pi^2}$$

Divide the first of these equations by the second and simplify to obtain:

$$\frac{m_A}{m_B} = \frac{\dfrac{k_A T_A^2}{4\pi^2}}{\dfrac{k_B T_B^2}{4\pi^2}} = \frac{k_A T_A^2}{k_B T_B^2}$$

Because the force constants of the two systems are the same:

$$\frac{m_A}{m_B} = \frac{T_A^2}{T_B^2} = \left(\frac{T_A}{T_B}\right)^2$$

Substituting for T_A and simplifying yields:

$$\frac{m_A}{m_B} = \left(\frac{2T_B}{T_B}\right)^2 = 4 \Rightarrow m_A = 4m_B$$

$\boxed{(a)}$ is correct.

11 •• Two mass–spring systems oscillate at frequencies f_A and f_B. If $f_A = 2f_B$ and the systems' springs have identical force constants, it follows that the systems' masses are related by (*a*) $m_A = 4m_B$, (*b*) $m_A = m_B/\sqrt{2}$, (*c*) $m_A = m_B/2$, (*d*) $m_A = m_B/4$.

Picture the Problem We can use $f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$ to express the frequencies of the two mass-spring systems in terms of their masses. Dividing one of the equations by the other will allow us to express m_A in terms of m_B.

Express the frequency of mass-spring system A as a function of its mass:

$$f_A = \frac{1}{2\pi}\sqrt{\frac{k}{m_A}}$$

Express the frequency of mass-spring system B as a function of its mass:

$$f_B = \frac{1}{2\pi}\sqrt{\frac{k}{m_B}}$$

Divide the second of these equations by the first to obtain:

$$\frac{f_B}{f_A} = \sqrt{\frac{m_A}{m_B}}$$

Solve for m_A:

$$m_A = \left(\frac{f_B}{f_A}\right)^2 m_B = \left(\frac{f_B}{2f_B}\right)^2 m_B = \tfrac{1}{4}m_B$$

$\boxed{(d)}$ is correct.

12 •• Two mass–spring systems *A* and *B* oscillate so that their total mechanical energies are equal. If $m_A = 2m_B$, which expression best relates their amplitudes? (*a*) $A_A = A_B/4$, (*b*) $A_A = A_B/\sqrt{2}$, (*c*) $A_A = A_B$, (*d*) Not enough information is given to determine the ratio of the amplitudes.

Picture the Problem We can relate the energies of the two mass-spring systems through either $E = \frac{1}{2}kA^2$ or $E = \frac{1}{2}m\omega^2A^2$ and investigate the relationship between their amplitudes by equating the expressions, substituting for m_A, and expressing A_A in terms of A_B.

Express the energy of mass-spring system A:

$$E_A = \tfrac{1}{2}k_A A_A^2 = \tfrac{1}{2}m_A\omega_A^2 A_A^2$$

Express the energy of mass-spring system B:

$$E_B = \tfrac{1}{2}k_B A_B^2 = \tfrac{1}{2}m_B\omega_B^2 A_B^2$$

Divide the first of these equations by the second to obtain:

$$\frac{E_A}{E_B} = 1 = \frac{\tfrac{1}{2}m_A\omega_A^2 A_A^2}{\tfrac{1}{2}m_B\omega_B^2 A_B^2}$$

Substitute for m_A and simplify:

$$1 = \frac{2m_B\omega_A^2 A_A^2}{m_B\omega_B^2 A_B^2} = \frac{2\omega_A^2 A_A^2}{\omega_B^2 A_B^2}$$

Solve for A_A:

$$A_A = \frac{\omega_B}{\sqrt{2}\omega_A}A_B$$

Without knowing how ω_A and ω_B, or k_A and k_B, are related, we cannot simplify this expression further. $\boxed{(\boldsymbol{d})}$ is correct.

13 •• **[SSM]** Two mass–spring systems *A* and *B* oscillate so that their total mechanical energies are equal. If the force constant of spring A is two times the force constant of spring B, then which expression best relates their amplitudes? (*a*) $A_A = A_B/4$, (*b*) $A_A = A_B/\sqrt{2}$, (*c*) $A_A = A_B$, (*d*) Not enough information is given to determine the ratio of the amplitudes.

Picture the Problem We can express the energy of each system using $E = \tfrac{1}{2}kA^2$ and, because the energies are equal, equate them and solve for A_A.

Express the energy of mass-spring system A in terms of the amplitude of its motion:

$$E_A = \tfrac{1}{2}k_A A_A^2$$

Express the energy of mass-spring system B in terms of the amplitude of its motion:

$$E_B = \tfrac{1}{2}k_B A_B^2$$

Because the energies of the two systems are equal we can equate them to obtain:

$$\tfrac{1}{2}k_A A_A^2 = \tfrac{1}{2}k_B A_B^2 \Rightarrow A_A = \sqrt{\frac{k_B}{k_A}}A_B$$

Substitute for k_A and simplify to obtain:

$$A_A = \sqrt{\frac{k_B}{2k_B}}A_B = \frac{A_B}{\sqrt{2}}$$

$\boxed{(\boldsymbol{b})}$ is correct.

14 •• The length of the string or wire supporting a pendulum bob increases slightly when the temperature of the string or wire is raised. How does this affect a clock operated by a simple pendulum?

Determine the Concept The period of a simple pendulum depends on the square root of the length of the pendulum. Increasing the length of the pendulum will lengthen its period and, hence, the clock will run slow.

15 •• A lamp hanging from the ceiling of the club car in a train oscillates with period T_0 when the train is at rest. The period will be (match left and right columns)

1. greater than T_0 when	A. The train moves horizontally at constant velocity.
2. less than T_0 when	B. The train rounds a curve at constant speed.
3. equal to T_0 when	C. The train climbs a hill at constant speed.
	D. The train goes over the crest of a hill at constant speed.

Determine the Concept The period of the lamp varies inversely with the square root of the effective value of the local gravitational field.

1-B. The period will be greater than T_0 when the train rounds a curve of radius R with speed v.

2-D. The period will be less than T_0 when the train goes over the crest of a hill of radius of curvature R with constant speed.

3-A. The period will be equal to T_0 when the train moves horizontally with constant velocity.

16 •• Two simple pendulums are related as follows. Pendulum A has a length L_A and a bob of mass m_A; pendulum B has a length L_B and a bob of mass m_B. If the period of A is twice that of B, then (*a*) $L_A = 2L_B$ and $m_A = 2m_B$, (*b*) $L_A = 4L_B$ and $m_A = m_B$, (*c*) $L_A = 4L_B$ whatever the ratio m_A/m_B, (*d*) $L_A = \sqrt{2}L_B$ whatever the ratio m_A/m_B.

Picture the Problem The period of a simple pendulum is independent of the mass of its bob and is given by $T = 2\pi\sqrt{L/g}$.

Express the period of pendulum A:

$$T_A = 2\pi\sqrt{\frac{L_A}{g}}$$

Express the period of pendulum B:

$$T_B = 2\pi\sqrt{\frac{L_B}{g}}$$

Divide the first of these equations by the second and solve for L_A/L_B:

$$\frac{L_A}{L_B} = \left(\frac{T_A}{T_B}\right)^2$$

Substitute for T_A and solve for L_B to obtain:

$$L_A = \left(\frac{2T_B}{T_B}\right)^2 L_B = 4L_B$$

$\boxed{(c)}$ is correct.

17 •• **[SSM]** Two simple pendulums are related as follows. Pendulum A has a length L_A and a bob of mass m_A; pendulum B has a length L_B and a bob of mass m_B. If the frequency of A is one-third that of B, then (*a*) $L_A = 3L_B$ and $m_A = 3m_B$, (*b*) $L_A = 9L_B$ and $m_A = m_B$, (*c*) $L_A = 9L_B$ regardless of the ratio m_A/m_B, (*d*) $L_A = \sqrt{3}L_B$ regardless of the ratio m_A/m_B.

Picture the Problem The frequency of a simple pendulum is independent of the mass of its bob and is given by $f = \frac{1}{2\pi}\sqrt{g/L}$.

Express the frequency of pendulum A:

$$f_A = \frac{1}{2\pi}\sqrt{\frac{g}{L_A}} \Rightarrow L_A = \frac{g}{4\pi^2 f_A^2}$$

Similarly, the length of pendulum B is given by:

$$L_B = \frac{g}{4\pi^2 f_B^2}$$

Divide the first of these equations by the second and simplify to obtain:

$$\frac{L_A}{L_B}=\frac{\dfrac{g}{4\pi^2 f_A^2}}{\dfrac{g}{4\pi^2 f_B^2}}=\frac{f_B^2}{f_A^2}=\left(\frac{f_B}{f_A}\right)^2$$

Substitute for f_A to obtain:

$$\frac{L_A}{L_B}=\left(\frac{f_B}{\frac{1}{3}f_B}\right)^2=9\Rightarrow L_A=9L_B$$

$\boxed{(c)}$ is correct.

18 •• Two simple pendulums are related as follows. Pendulum A has a length L_A and a bob of mass m_A; pendulum B has a length L_B a bob of mass m_B. They have the same period. The only thing different between their motions is that the amplitude of A's motion is twice that of B's motion, then (*a*) $L_A = L_B$ and $m_A = m_B$, (*b*) $L_A = 2L_B$ and $m_A = m_B$, (*c*) $L_A = L_B$ whatever the ratio m_A/m_B, (*d*) $L_A = \frac{1}{2}L_B$ whatever the ratio m_A/m_B.

Picture the Problem The period of a simple pendulum is independent of the mass of its bob and is given by $T = 2\pi\sqrt{L/g}$. For small amplitudes, the period is independent of the amplitude.

Express the period of pendulum A:

$$T_A=2\pi\sqrt{\frac{L_A}{g}}$$

Express the period of pendulum B:

$$T_B=2\pi\sqrt{\frac{L_B}{g}}$$

Divide the first of these equations by the second and solve for L_A/L_B:

$$\frac{L_A}{L_B}=\left(\frac{T_A}{T_B}\right)^2$$

Because their periods are the same:

$$\frac{L_A}{L_B}=\left(\frac{T_B}{T_B}\right)^2=1\Rightarrow L_A=L_B$$

$\boxed{(c)}$ is correct.

19 •• True or false:

(*a*) The mechanical energy of a damped, undriven oscillator decreases exponentially with time.
(*b*) Resonance for a damped, driven oscillator occurs when the driving frequency exactly equals the natural frequency.

(*c*) If the *Q* factor of a damped oscillator is high, then its resonance curve will be narrow.
(*d*) The decay time τ for a spring-mass oscillator with linear damping is independent of its mass.
(*e*) The *Q* factor for a driven spring-mass oscillator with linear damping is independent of its mass.

(*a*) True. Because the energy of an oscillator is proportional to the square of its amplitude, and the amplitude of a damped, undriven oscillator decreases exponentially with time, so does its energy.

(*b*) False. For a damped driven oscillator, the resonant frequency ω' is given by $\omega' = \omega_0\sqrt{1-\left(\frac{b}{2m\omega_0}\right)^2}$, where ω_0 is the natural frequency of the oscillator.

(*c*) True. The ratio of the width of a resonance curve to the resonant frequency equals the reciprocal of the *Q* factor ($\Delta\boldsymbol{\omega}/\boldsymbol{\omega}_0 = 1/\boldsymbol{Q}$). Hence, the larger *Q* is, the narrower the resonance curve.

(*d*) False. The decay time for a damped but undriven spring-mass oscillator is directly proportional to its mass.

(*e*) True. From $\Delta\omega/\omega_0 = 1/Q$ one can see that *Q* is independent of *m*.

20 •• Two damped spring-mass oscillating systems have identical spring and damping constants. However, system A's mass m_A is four times system B's. How do their decay times compare? (*a*) $\tau_A = 4\tau_B$, (*b*) $\tau_A = 2\tau_B$, (*c*) $\tau_A = \tau_B$, (*d*) Their decay times cannot be compared, given the information provided.

Picture the Problem The decay time τ of a damped oscillator is related to the mass *m* of the oscillator and the damping constant *b* for the motion according to $\tau = m/b$.

Express the decay time of System A:

$$\tau_A = \frac{m_A}{b_A}$$

The decay time for System B is given by:

$$\tau_B = \frac{m_B}{b_B}$$

Dividing the first of these equations by the second and simplifying yields:

$$\frac{\tau_A}{\tau_B} = \frac{\frac{m_A}{b_A}}{\frac{m_B}{b_B}} = \frac{m_A}{m_B}\frac{b_B}{b_A}$$

Because their damping constants are the same:

$$\frac{\tau_A}{\tau_B} = \frac{m_A}{m_B}$$

Substituting for m_A yields:

$$\frac{\tau_A}{\tau_B} = \frac{4m_B}{m_B} = 4 \Rightarrow \boldsymbol{\tau_A = 4\tau_B}$$

$\boxed{(\boldsymbol{a})}$ is correct.

21 •• Two damped spring-mass oscillating systems have identical spring constants and decay times. However, system A's mass m_A is system B's mass m_B. They are tweaked into oscillation and their decay times are measured to be the same. How do their damping constants, b, compare? (*a*) $b_A = 4b_B$, (*b*) $b_A = 2b_B$, (*c*) $b_A = b_B$, (*d*) $b_A = \frac{1}{2}b_B$, (*e*) Their decay times cannot be compared, given the information provided.

Picture the Problem The decay time τ of a damped oscillator is related to the mass m of the oscillator and the damping constant b for the motion according to $\tau = m/b$.

Express the damping constant of System A:

$$b_A = \frac{m_A}{\tau_A}$$

The damping constant for System B is given by:

$$b_B = \frac{m_B}{\tau_B}$$

Dividing the first of these equations by the second and simplifying yields:

$$\frac{b_A}{b_B} = \frac{\dfrac{m_A}{\tau_A}}{\dfrac{m_B}{\tau_B}} = \frac{m_A}{m_B}\frac{\tau_B}{\tau_A}$$

Because their decay times are the same:

$$\frac{b_A}{b_B} = \frac{m_A}{m_B}$$

Substituting for m_A yields:

$$\frac{b_A}{b_B} = \frac{2m_B}{m_B} = 2 \Rightarrow b_A = 2b_B$$

$\boxed{(\boldsymbol{b})}$ is correct.

22 •• Two damped, driven spring-mass oscillating systems have identical driving forces as well as identical spring and damping constants. However, the mass of system A is four times the mass of system B. Assume both systems are very weakly damped. How do their resonant frequencies compare?

(*a*) $\omega_A = \omega_B$, (*b*) $\omega_A = 2\omega_B$, (*c*) $\omega_A = \frac{1}{2}\omega_B$, (*d*) $\omega_A = \frac{1}{4}\omega_B$, (*e*) Their resonant frequencies cannot be compared, given the information provided.

Picture the Problem For very weak damping, the resonant frequency of a spring-mass oscillator is the same as its natural frequency and is given by $\omega_0 = \sqrt{k/m}$, where *m* is the oscillator's mass and *k* is the force constant of the spring.

Express the resonant frequency of System A:

$$\omega_A = \sqrt{\frac{k_A}{m_A}}$$

The resonant frequency of System B is given by:

$$\omega_B = \sqrt{\frac{k_B}{m_B}}$$

Dividing the first of these equations by the second and simplifying yields:

$$\frac{\omega_A}{\omega_B} = \frac{\sqrt{\frac{k_A}{m_A}}}{\sqrt{\frac{k_B}{m_B}}} = \sqrt{\frac{k_A}{k_B}\frac{m_B}{m_A}}$$

Because their force constants are the same:

$$\frac{\omega_A}{\omega_B} = \sqrt{\frac{m_B}{m_A}}$$

Substituting for m_A yields:

$$\frac{\omega_A}{\omega_B} = \sqrt{\frac{m_B}{4m_B}} = \tfrac{1}{2} \Rightarrow \omega_A = \tfrac{1}{2}\omega_B$$

$\boxed{(c)}$ is correct.

23 •• **[SSM]** Two damped, driven spring-mass oscillating systems have identical masses, driving forces, and damping constants. However, system A's force constant k_A is four times system B's force constant k_B. Assume they are both very weakly damped. How do their resonant frequencies compare?
(*a*) $\omega_A = \omega_B$, (*b*) $\omega_A = 2\omega_B$, (*c*) $\omega_A = \frac{1}{2}\omega_B$, (*d*) $\omega_A = \frac{1}{4}\omega_B$, (*e*) Their resonant frequencies cannot be compared, given the information provided.

Picture the Problem For very weak damping, the resonant frequency of a spring-mass oscillator is the same as its natural frequency and is given by $\omega_0 = \sqrt{k/m}$, where *m* is the oscillator's mass and *k* is the force constant of the spring.

Express the resonant frequency of System A:

$$\omega_A = \sqrt{\frac{k_A}{m_A}}$$

The resonant frequency of System B is given by:

$$\omega_B = \sqrt{\frac{k_B}{m_B}}$$

Dividing the first of these equations by the second and simplifying yields:

$$\frac{\omega_A}{\omega_B} = \frac{\sqrt{\frac{k_A}{m_A}}}{\sqrt{\frac{k_B}{m_B}}} = \sqrt{\frac{k_A}{k_B}\frac{m_B}{m_A}}$$

Because their masses are the same:

$$\frac{\omega_A}{\omega_B} = \sqrt{\frac{k_A}{k_B}}$$

Substituting for k_A yields:

$$\frac{\omega_A}{\omega_B} = \sqrt{\frac{4k_B}{k_B}} = 2 \Rightarrow \omega_A = 2\omega_B$$

$\boxed{(b)}$ is correct.

24 •• Two damped, driven simple-pendulum systems have identical masses, driving forces, and damping constants. However, system A's length is four times system B's length. Assume they are both very weakly damped. How do their resonant frequencies compare? (*a*) $\omega_A = \omega_B$, (*b*) $\omega_A = 2\omega_B$, (*c*) $\omega_A = \frac{1}{2}\omega_B$, (*d*) $\omega_A = \frac{1}{4}\omega_B$, (*e*) Their resonant frequencies cannot be compared, given the information provided.

Picture the Problem For very weak damping, the resonant frequency of a simple pendulum is the same as its natural frequency and is given by $\omega_0 = \sqrt{g/L}$, where L is the length of the simple pendulum and g is the gravitational field.

Express the resonant frequency of System A:

$$\omega_A = \sqrt{\frac{g}{L_A}}$$

The resonant frequency of System B is given by:

$$\omega_B = \sqrt{\frac{g}{L_B}}$$

Dividing the first of these equations by the second and simplifying yields:

$$\frac{\omega_A}{\omega_B}=\frac{\sqrt{\frac{g}{L_A}}}{\sqrt{\frac{g}{L_B}}}=\sqrt{\frac{L_B}{L_A}}$$

Substituting for L_A yields:

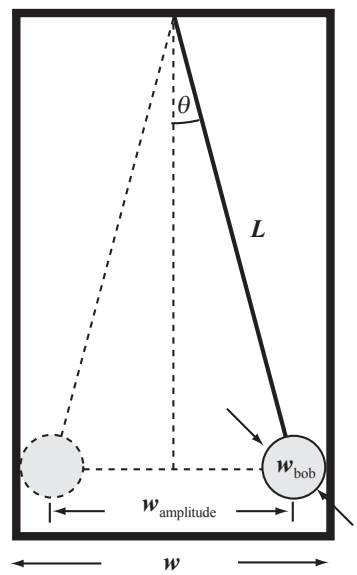
$$\frac{\omega_A}{\omega_B}=\sqrt{\frac{L_B}{4L_B}}=\tfrac{1}{2}\Rightarrow\omega_A=\tfrac{1}{2}\omega_B$$

$\boxed{(c)}$ is correct.

Estimation and Approximation

25 • **[SSM]** Estimate the width of a typical grandfather clocks' cabinet relative to the width of the pendulum bob, presuming the desired motion of the pendulum is simple harmonic.

Picture the Problem If the motion of the pendulum in a grandfather clock is to be simple harmonic motion, then its period must be independent of the angular amplitude of its oscillations. The period of the motion for large-amplitude oscillations is given by Equation 14-30 and we can use this expression to obtain a maximum value for the amplitude of swinging pendulum in the clock. We can then use this value and an assumed value for the length of the pendulum to estimate the width of the grandfather clocks' cabinet.



Referring to the diagram, we see that the minimum width of the cabinet is determined by the width of the bob and the width required to accommodate the swinging pendulum:

$$w=w_{bob}+w_{amplitude}$$

and

$$\frac{w}{w_{bob}}=1+\frac{w_{amplitude}}{w_{bob}} \qquad (1)$$

Express $w_{\text{amplitude}}$ in terms of the angular amplitude θ and the length of the pendulum L:

$$w_{\text{amplitude}} = 2L\sin\theta$$

Substituting for $w_{\text{amplitude}}$ in equation (1) yields:

$$\frac{w}{w_{\text{bob}}} = 1 + \frac{2L\sin\theta}{w_{\text{bob}}} \qquad (2)$$

Equation 14-30 gives us the period of a simple pendulum as a function of its angular amplitude:

$$T = T_0\left[1 + \frac{1}{2^2}\sin^2\frac{1}{2}\theta + \ldots\right]$$

If T is to be approximately equal to T_0, the second term in the brackets must be small compared to the first term. Suppose that:

$$\frac{1}{4}\sin^2\frac{1}{2}\theta \le 0.001$$

Solving for θ yields:

$$\theta \le 2\sin^{-1}(0.0632) \approx 7.25°$$

If we assume that the length of a grandfather clock's pendulum is about 1.5 m and that the width of the bob is about 10 cm, then equation (2) yields:

$$\frac{w}{w_{\text{bob}}} = 1 + \frac{2(1.5\text{ m})\sin 7.25°}{0.10\text{ m}} \approx \boxed{5}$$

26 • A small punching bag for boxing workouts is approximately the size and weight of a person's head and is suspended from a very short rope or chain. Estimate the natural frequency of oscillations of such a punching bag.

Picture the Problem For the purposes of this estimation, model the punching bag as a sphere of radius R and assume that the spindle about which it rotates to be 1.5 times the radius of the sphere. The natural frequency of oscillations of this physical pendulum is given by $f_0 = \omega_0 = \frac{1}{2\pi}\sqrt{\frac{MgD}{I}}$ where M is the mass of the pendulum, D is the distance from the point of support to the center of mass of the punching bag, and I is its moment of inertia about an axis through the spindle from which it is supported and about which it swivels.

Express the natural frequency of oscillation of the punching bag:

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{MgD}{I_{\text{spindle}}}} \qquad (1)$$

From the parallel-axis theorem we have:

$$I_{\text{spindle}} = I_{\text{cm}} + Mh^2$$

where $h = 1.5R + 0.5R = 2R$

Substituting for I_{cm} and h yields:

$$I_{\text{spindle}} = \tfrac{2}{5}MR^2 + M(2R)^2 = 4.4MR^2$$

Substitute for I_{spindle} in equation (1) to obtain:

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{Mg(2R)}{4.4MR^2}} = \frac{1}{2\pi}\sqrt{\frac{g}{2.2R}}$$

Assume that the radius of the punching bag is 10 cm, substitute numerical values and evaluate f_0:

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{9.81\,\text{m/s}^2}{2.2(0.10\,\text{m})}} \approx \boxed{1\,\text{Hz}}$$

27 • For a child on a swing, the amplitude drops by a factor of $1/e$ in about eight periods if no additional mechanical energy is given to the system. Estimate the Q factor for this system.

Picture the Problem The Q factor for this system is related to the decay constant τ through $Q = \omega_0\tau = 2\pi\tau/T$ and the amplitude of the child's damped motion varies with time according to $A = A_0e^{-t/2\tau}$. We can set the ratio of two displacements separated by eight periods equal to $1/e$ to determine τ in terms of T.

Express Q as a function of τ:

$$Q = \omega_0\tau = \frac{2\pi\tau}{T} \qquad (1)$$

The amplitude of the oscillations varies with time according to:

$$A = A_0e^{-t/2\tau}$$

The amplitude after eight periods is:

$$A_8 = A_0e^{-(t+8T)/2\tau}$$

Express and simplify the ratio A_8/A:

$$\frac{A_8}{A} = \frac{A_0e^{-(t+8T)/2\tau}}{A_0e^{-t/2\tau}} = e^{-4T/\tau}$$

Set this ratio equal to $1/e$ and solve for τ:

$$e^{-4T/\tau} = e^{-1} \Rightarrow \tau = 4T$$

Substitute in equation (1) and evaluate Q:

$$Q = \frac{2\pi(4T)}{T} = \boxed{8\pi}$$

28 •• (*a*) Estimate the natural period of oscillation for swinging your arms as you walk, when your hands are empty. (*b*) Now estimate the natural period of

oscillation when you are carrying a heavy briefcase. (*c*) Observe other people as they walk. Do your estimates seem reasonable?

Picture the Problem Assume that an average length for an arm is about 80 cm, and that it can be treated as a uniform rod, pivoted at one end. We can use the expression for the period of a physical pendulum to derive an expression for the period of the swinging arm. When carrying a heavy briefcase, the mass is concentrated mostly at the end of the rod (that is, in the briefcase), so we can treat the arm-plus-briefcase system as a simple pendulum.

(*a*) Express the period of a uniform rod pivoted at one end:

$$T = 2\pi\sqrt{\frac{I}{MgD}}$$

where I is the moment of inertia of the stick about an axis through one end, M is the mass of the stick, and D (= $L/2$) is the distance from the end of the stick to its center of mass.

Express the moment of inertia of a rod about an axis through its end:

$$I = \tfrac{1}{3}ML^2$$

Substitute the values for I and D in the expression for T and simplify to obtain:

$$T = 2\pi\sqrt{\frac{\frac{1}{3}ML^2}{Mg\left(\frac{1}{2}L\right)}} = 2\pi\sqrt{\frac{2L}{3g}}$$

Substitute numerical values and evaluate T:

$$T = 2\pi\sqrt{\frac{2(0.80\,\text{m})}{3(9.81\,\text{m/s}^2)}} = \boxed{1.5\,\text{s}}$$

(*b*) Express the period of a simple pendulum:

$$T' = 2\pi\sqrt{\frac{L'}{g}}$$

where L' is slightly longer than the arm length due to the size of the briefcase.

Assuming L' = 1.0 m, evaluate the period of the simple pendulum:

$$\boldsymbol{T'} = 2\boldsymbol{\pi}\sqrt{\frac{1.0\,\text{m}}{9.81\,\text{m/s}^2}} = \boxed{2.0\,\text{s}}$$

(*c*) From observation of people as they walk, these estimates seem reasonable.

Simple Harmonic Motion

29 • The position of a particle is given by x = (7.0 cm) cos $6\pi t$, where t is in seconds. What are (*a*) the frequency, (*b*) the period, and (*c*) the amplitude of the

particle's motion? (*d*) What is the first time after $t = 0$ that the particle is at its equilibrium position? In what direction is it moving at that time?

Picture the Problem The position of the particle is given by $x = A\cos(\omega t + \delta)$ where A is the amplitude of the motion, ω is the angular frequency, and δ is a phase constant. The frequency of the motion is given by $f = \omega/2\pi$ and the period of the motion is the reciprocal of its frequency.

(*a*) Use the definition of ω to determine f:

$$f = \frac{\omega}{2\pi} = \frac{6\pi\,\text{s}^{-1}}{2\pi} = \boxed{3.00\,\text{Hz}}$$

(*b*) Evaluate the reciprocal of the frequency:

$$T = \frac{1}{f} = \frac{1}{3.00\,\text{Hz}} = \boxed{0.333\,\text{s}}$$

(*c*) Compare $x = (7.0\text{ cm})\cos 6\pi t$ to $x = A\cos(\omega t + \delta)$ to conclude that:

$$A = \boxed{7.0\,\text{cm}}$$

(*d*) Express the condition that must be satisfied when the particle is at its equilibrium position:

$$\cos\omega t = 0 \Rightarrow \omega t = \frac{\pi}{2} \Rightarrow t = \frac{\pi}{2\omega}$$

Substituting for ω yields:

$$t = \frac{\pi}{2(6\pi)} = \boxed{0.0833\,\text{s}}$$

Differentiate x to find $v(t)$:

$$v = \frac{d}{dt}\left[(7.0\,\text{cm})\cos 6\pi t\right] = -(42\pi\,\text{cm/s})\sin 6\pi t$$

Evaluate $v(0.0833\text{ s})$:

$$v(0.0833\,\text{s}) = -(42\pi\,\text{cm/s})\sin 6\pi(0.0833\,\text{s}) < 0$$

Because $v < 0$, the particle is moving in the negative direction at $t = 0.0833$ s.

30 • What is the phase constant δ in $x = A\cos(\omega t + \delta)$ (Equation 14-4) if the position of the oscillating particle at time $t = 0$ is (*a*) 0, (*b*) $-A$, (*c*) A, (*d*) $A/2$?

Picture the Problem The initial position of the oscillating particle is related to the amplitude and phase constant of the motion by $x_0 = A\cos\delta$ where $0 \le \delta < 2\pi$.

(*a*) For $x_0 = 0$:

$$\cos\delta = 0 \Rightarrow \delta = \cos^{-1}(0) = \boxed{\frac{\pi}{2}, \frac{3\pi}{2}}$$

(*b*) For $x_0 = -A$:

$$-A = A\cos\delta \Rightarrow \delta = \cos^{-1}(-1) = \boxed{\pi}$$

(*c*) For $x_0 = A$:

$$A = A\cos\delta \Rightarrow \delta = \cos^{-1}(1) = \boxed{0}$$

(*d*) When $x = A/2$:

$$\frac{A}{2} = A\cos\delta \Rightarrow \delta = \cos^{-1}\left(\frac{1}{2}\right) = \boxed{\frac{\pi}{3}}$$

31 • **[SSM]** A particle of mass *m* begins at rest from $x = +25$ cm and oscillates about its equilibrium position at $x = 0$ with a period of 1.5 s. Write expressions for (*a*) the position *x* as a function of *t*, (*b*) the velocity v_x as a function of *t*, and (*c*) the acceleration a_x as a function of *t*.

Picture the Problem The position of the particle as a function of time is given by $x = A\cos(\omega t + \delta)$. Its velocity as a function of time is $v_x = -A\omega\sin(\omega t + \delta)$ and its acceleration is $a_x = -A\omega^2\cos(\omega t + \delta)$. The initial position and velocity give us two equations from which to determine the amplitude *A* and phase constant δ.

(*a*) Express the position, velocity, and acceleration of the particle as a function of *t*:

$$x = A\cos(\omega t + \delta) \quad (1)$$
$$v_x = -A\omega\sin(\omega t + \delta) \quad (2)$$
$$a_x = -A\omega^2\cos(\omega t + \delta) \quad (3)$$

Find the angular frequency of the particle's motion:

$$\omega = \frac{2\pi}{T} = \frac{4\pi}{3}\,\text{s}^{-1} = 4.19\,\text{s}^{-1}$$

Relate the initial position and velocity to the amplitude and phase constant:

$$x_0 = A\cos\delta$$

and

$$v_0 = -\omega A\sin\delta$$

Divide the equation for v_0 by the equation for x_0 to eliminate *A*:

$$\frac{v_0}{x_0} = \frac{-\omega A\sin\delta}{A\cos\delta} = -\omega\tan\delta$$

Solving for δ yields:

$$\delta = \tan^{-1}\left(-\frac{v_0}{x_0\omega}\right) = \tan^{-1}\left(-\frac{0}{x_0\omega}\right) = 0$$

Substitute in equation (1) to obtain:

$$x = (25\,\text{cm})\cos\left[\left(\frac{4\pi}{3}\text{s}^{-1}\right)t\right] = \boxed{(0.25\,\text{m})\cos\left[\left(4.2\,\text{s}^{-1}\right)t\right]}$$

(*b*) Substitute in equation (2) to obtain:

$$v_x = -(25\,\text{cm})\left(\frac{4\pi}{3}\text{s}^{-1}\right)\sin\left[\left(\frac{4\pi}{3}\text{s}^{-1}\right)t\right] = \boxed{-(1.0\,\text{m/s})\sin\left[\left(4.2\,\text{s}^{-1}\right)t\right]}$$

(*c*) Substitute in equation (3) to obtain:

$$a_x = -(25\,\text{cm})\left(\frac{4\pi}{3}\text{s}^{-1}\right)^2\cos\left[\left(\frac{4\pi}{3}\text{s}^{-1}\right)t\right] = \boxed{-\left(4.4\,\text{m/s}^2\right)\cos\left[\left(4.2\,\text{s}^{-1}\right)t\right]}$$

32 •• Find (*a*) the maximum speed and (*b*) the maximum acceleration of the particle in Problem 31. (*c*) What is the first time that the particle is at $x = 0$ and moving to the right?

Picture the Problem The maximum speed and maximum acceleration of the particle in are given by $v_{\max} = A\omega$ and $a_{\max} = A\omega^2$. The particle's position is given by $x = A\cos(\omega t + \delta)$ where A = 7.0 cm, $\omega = 6\pi\ \text{s}^{-1}$, and $\delta = 0$, and its velocity is given by $v = -A\omega\sin(\omega t + \delta)$.

(*a*) Express $v_{\max}$ in terms of A and ω:

$$v_{\max} = A\omega = (7.0\,\text{cm})\left(6\pi\,\text{s}^{-1}\right) = 42\pi\,\text{cm/s} = \boxed{1.3\,\text{m/s}}$$

(*b*) Express $a_{\max}$ in terms of A and ω:

$$a_{\max} = A\omega^2 = (7.0\,\text{cm})\left(6\pi\,\text{s}^{-1}\right)^2 = 252\pi^2\,\text{cm/s}^2 = \boxed{25\,\text{m/s}^2}$$

(*c*) When $x = 0$:

$$\cos\omega t = 0 \Rightarrow \omega t = \cos^{-1}(0) = \frac{\pi}{2}, \frac{3\pi}{2}$$

Evaluate v for $\omega t = \frac{\pi}{2}$:

$$v = -A\omega\sin\left(\frac{\pi}{2}\right) = -A\omega$$

That is, the particle is moving to the left.

Evaluate v for $\omega t = \frac{3\pi}{2}$:

$$v = -A\omega \sin\left(\frac{3\pi}{2}\right) = A\omega$$

That is, the particle is moving to the right.

Solve $\omega t = \frac{3\pi}{2}$ for t to obtain:

$$t = \frac{3\pi}{2\omega} = \frac{3\pi}{2\left(6\pi\,\text{s}^{-1}\right)} = \boxed{0.25\,\text{s}}$$

33 •• Work Problem 33 with the particle initially at x = 25 cm and moving with velocity v_0 = +50 cm/s.

Picture the Problem The position of the particle as a function of time is given by $x = A\cos(\omega t + \delta)$. Its velocity as a function of time is given by $v = -A\omega\sin(\omega t + \delta)$ and its acceleration by $a = -A\omega^2\cos(\omega t + \delta)$. The initial position and velocity give us two equations from which to determine the amplitude A and phase constant δ.

(*a*) Express the position, velocity, and acceleration of the particle as functions of t:

$$x = A\cos(\omega t + \delta) \qquad (1)$$
$$v_x = -A\omega\sin(\omega t + \delta) \qquad (2)$$
$$a_x = -A\omega^2\cos(\omega t + \delta) \qquad (3)$$

Find the angular frequency of the particle's motion:

$$\omega = \frac{2\pi}{T} = \frac{4\pi}{3}\,\text{s}^{-1} = 4.19\,\text{s}^{-1}$$

Relate the initial position and velocity to the amplitude and phase constant:

$$x_0 = A\cos\delta$$

and

$$v_0 = -\omega A\sin\delta$$

Divide these equations to eliminate A:

$$\frac{v_0}{x_0} = \frac{-\omega A\sin\delta}{A\cos\delta} = -\omega\tan\delta$$

Solving for δ yields:

$$\delta = \tan^{-1}\left(-\frac{v_0}{x_0\omega}\right)$$

Substitute numerical values and evaluate δ:

$$\delta = \tan^{-1}\left(-\frac{50\,\text{cm/s}}{(25\,\text{cm})\left(4.192\,\text{s}^{-1}\right)}\right)$$
$$= -0.445\,\text{rad}$$

Use either the x_0 or v_0 equation (x_0 is used here) to find the amplitude:

$$A = \frac{x_0}{\cos\delta} = \frac{25\,\text{cm}}{\cos(-0.445\,\text{rad})} = 27.7\,\text{cm}$$

Substitute in equation (1) to obtain:

$$x = \boxed{(0.28\,\text{m})\cos\left[\left(4.2\,\text{s}^{-1}\right)t - 0.45\right]}$$

(*b*) Substitute numerical values in equation (2) to obtain:

$$v_x = -(27.7\,\text{cm})\left(\frac{4\pi}{3}\text{s}^{-1}\right)\sin\left[\left(\frac{4\pi}{3}\text{s}^{-1}\right)t - 0.445\right] = \boxed{-(1.2\,\text{m/s})\sin\left[\left(4.2\,\text{s}^{-1}\right)t - 0.45\right]}$$

(*c*) Substitute numerical values in equation (3) to obtain:

$$a_x = -(27.7\,\text{cm})\left(\frac{4\pi}{3}\text{s}^{-1}\right)^2\cos\left[\left(\frac{4\pi}{3}\text{s}^{-1}\right)t - 0.445\right]$$
$$= \boxed{-\left(4.9\,\text{m/s}^2\right)\cos\left[\left(4.2\,\text{s}^{-1}\right)t - 0.45\right]}$$

34 •• The period of a particle that is oscillating in simple harmonic motion is 8.0 s and its amplitude is 12 cm. At $t = 0$ it is at its equilibrium position. Find the distance it travels during the intervals (*a*) $t = 0$ to $t = 2.0$ s, (*b*) $t = 2.0$ s to $t = 4.0$ s, (*c*) $t = 0$ to $t = 1.0$ s, and (*d*) $t = 1.0$ s to $t = 2.0$ s.

Picture the Problem The position of the particle as a function of time is given by $x = A\cos(\omega t + \delta)$. We're given the amplitude A of the motion and can use the initial position of the particle to determine the phase constant δ. Once we've determined these quantities, we can express the distance traveled Δx during any interval of time.

Express the position of the particle as a function of t:

$$x = (12\,\text{cm})\cos(\omega t + \delta) \qquad (1)$$

Find the angular frequency of the particle's motion:

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{8.0\,\text{s}} = \frac{\pi}{4}\text{s}^{-1}$$

Relate the initial position of the particle to the amplitude and phase constant:

$$x_0 = A\cos\delta$$

Solve for δ:

$$\delta = \cos^{-1}\left(\frac{x_0}{A}\right) = \cos^{-1}\left(\frac{0}{A}\right) = \frac{\pi}{2}$$

Substitute in equation (1) to obtain:

$$x = (12\,\text{cm})\cos\left[\left(\frac{\pi}{4}\text{s}^{-1}\right)t + \frac{\pi}{2}\right]$$

Express the distance the particle travels in terms of t_f and t_i:

$$\Delta x = \left|(12\,\text{cm})\cos\left[\left(\frac{\pi}{4}\text{s}^{-1}\right)t_f + \frac{\pi}{2}\right] - (12\,\text{cm})\cos\left[\left(\frac{\pi}{4}\text{s}^{-1}\right)t_i + \frac{\pi}{2}\right]\right|$$
$$= \left|(12\,\text{cm})\left\{\cos\left[\left(\frac{\pi}{4}\text{s}^{-1}\right)t_f + \frac{\pi}{2}\right] - \cos\left[\left(\frac{\pi}{4}\text{s}^{-1}\right)t_i + \frac{\pi}{2}\right]\right\}\right|$$

(*a*) Evaluate Δx for $t_f = 2.0$ s, $t_i = 0$ s:

$$\Delta x = \left|(12\,\text{cm})\left\{\cos\left[\left(\frac{\pi}{4}\text{s}^{-1}\right)(2.0\,\text{s}) + \frac{\pi}{2}\right] - \cos\left[\left(\frac{\pi}{4}\text{s}^{-1}\right)(0) + \frac{\pi}{2}\right]\right\}\right| = \boxed{12\,\text{cm}}$$

(*b*) Evaluate Δx for $t_f = 4.0$ s, $t_i = 2.0$ s:

$$\Delta x = \left|(12\,\text{cm})\left\{\cos\left[\left(\frac{\pi}{4}\text{s}^{-1}\right)(4.0\,\text{s}) + \frac{\pi}{2}\right] - \cos\left[\left(\frac{\pi}{4}\text{s}^{-1}\right)(2.0\,\text{s}) + \frac{\pi}{2}\right]\right\}\right| = \boxed{12\,\text{cm}}$$

(*c*) Evaluate Δx for $t_f = 1.0$ s, $t_i = 0$:

$$\Delta x = \left|(12\,\text{cm})\left\{\cos\left[\left(\frac{\pi}{4}\text{s}^{-1}\right)(1.0\,\text{s}) + \frac{\pi}{2}\right] - \cos\left[\left(\frac{\pi}{4}\text{s}^{-1}\right)(0) + \frac{\pi}{2}\right]\right\}\right|$$
$$= \left|(12\,\text{cm})\{-0.7071 - 0\}\right| = \boxed{8.5\,\text{cm}}$$

(*d*) Evaluate Δx for $t_f = 2.0$ s, $t_i = 1.0$ s:

$$\Delta x = \left|(12\,\text{cm})\left\{\cos\left[\left(\frac{\pi}{4}\text{s}^{-1}\right)(2.0\,\text{s}) + \frac{\pi}{2}\right] - \cos\left[\left(\frac{\pi}{4}\text{s}^{-1}\right)(1.0\,\text{s}) + \frac{\pi}{2}\right]\right\}\right| = \boxed{3.5\,\text{cm}}$$

35 •• The period of a particle oscillating in simple harmonic motion is 8.0 s. At $t = 0$, the particle is at rest at $x = A = 10$ cm. (*a*) Sketch x as a function of t. (*b*) Find the distance traveled in the first, second, third, and fourth second after $t = 0$.

Picture the Problem The position of the particle as a function of time is given by $x = (10\,\text{cm})\cos(\omega t + \delta)$. We can determine the angular frequency ω from the

period of the motion and the phase constant δ from the initial position and velocity. Once we've determined these quantities, we can express the distance traveled Δx during any interval of time.

Express the position of the particle as a function of t:

$$x = (10\,\text{cm})\cos(\omega t + \delta) \qquad (1)$$

Find the angular frequency of the particle's motion:

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{8.0\,\text{s}} = \frac{\pi}{4}\,\text{s}^{-1}$$

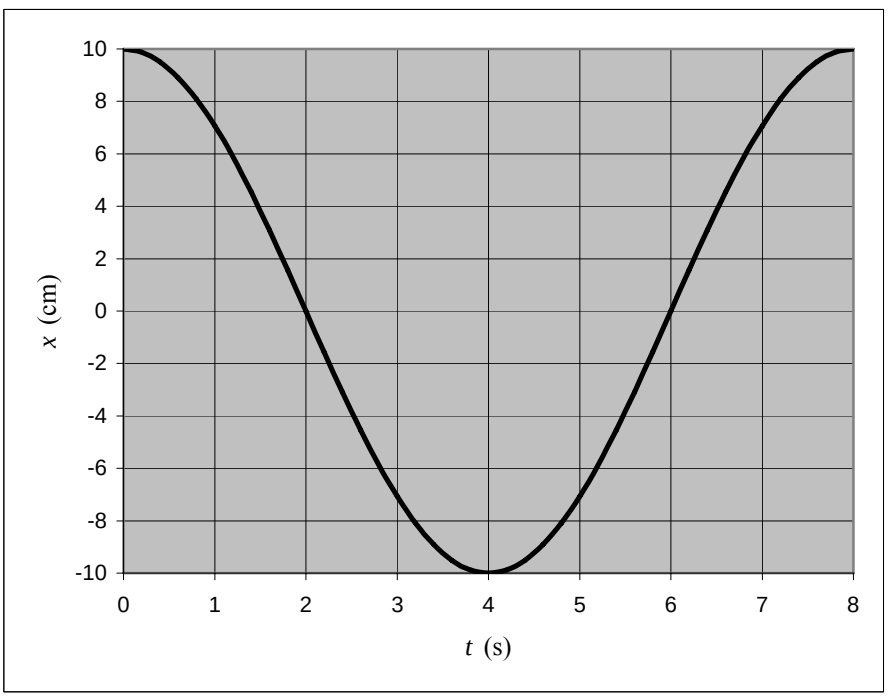
Find the phase constant of the motion:

$$\delta = \tan^{-1}\left(-\frac{v_0}{x_0\omega}\right) = \tan^{-1}\left(-\frac{0}{x_0\omega}\right) = 0$$

Substitute in equation (1) to obtain:

$$x = (10\,\text{cm})\cos\left[\left(\frac{\pi}{4}\,\text{s}^{-1}\right)t\right]$$

(*a*) A graph of $x = (10\,\text{cm})\cos\left[\left(\frac{\pi}{4}\,\text{s}^{-1}\right)t\right]$ follows:



(*b*) Express the distance the particle travels in terms of t_f and t_i:

$$\begin{aligned}\Delta x &= \left|(10\,\text{cm})\cos\left[\left(\frac{\pi}{4}\,\text{s}^{-1}\right)t_f\right] - (10\,\text{cm})\cos\left[\left(\frac{\pi}{4}\,\text{s}^{-1}\right)t_i\right]\right| \\ &= \left|(10\,\text{cm})\left\{\cos\left[\left(\frac{\pi}{4}\,\text{s}^{-1}\right)t_f\right] - \cos\left[\left(\frac{\pi}{4}\,\text{s}^{-1}\right)t_i\right]\right\}\right|\end{aligned} \qquad (2)$$

Substitute numerical values in equation (2) and evaluate Δx in each of the given time intervals to obtain:

t_f (s)	t_i (s)	Δx (cm)
1	0	$\boxed{2.9}$
2	1	$\boxed{7.1}$
3	2	$\boxed{7.1}$
4	3	$\boxed{2.9}$

36 •• Military specifications often call for electronic devices to be able to withstand accelerations of up to $10g$ ($10g = 98.1\ \text{m/s}^2$). To make sure that your company's products meet this specification, your manager has told you to use a "shaking table," which can vibrate a device at controlled and adjustable frequencies and amplitudes. If a device is placed on the table and made to oscillate at an amplitude of 1.5 cm, what should you adjust the frequency to in order to test for compliance with the $10g$ military specification?

Picture the Problem We can use the expression for the maximum acceleration of an oscillator to relate the $10g$ military specification to the compliance frequency.

Express the maximum acceleration of an oscillator:

$$a_{max} = A\omega^2$$

Express the relationship between the angular frequency and the frequency of the vibrations:

$$\omega = 2\pi f$$

Substitute for ω to obtain:

$$a_{max} = 4\pi^2 A f^2 \Rightarrow f = \frac{1}{2\pi}\sqrt{\frac{a_{max}}{A}}$$

Substitute numerical values and evaluate f:

$$f = \frac{1}{2\pi}\sqrt{\frac{98.1\,\text{m/s}^2}{1.5\times10^{-2}\,\text{m}}} = \boxed{13\,\text{Hz}}$$

37 •• **[SSM]** The position of a particle is given by $x = 2.5\cos\pi t$, where x is in meters and t is in seconds. (*a*) Find the maximum speed and maximum acceleration of the particle. (*b*) Find the speed and acceleration of the particle when $x = 1.5$ m.

Picture the Problem The position of the particle is given by $\boldsymbol{x = A\cos\omega t}$, where $A = 2.5$ m and $\omega = \pi$ rad/s. The velocity is the time derivative of the position and the acceleration is the time derivative of the velocity.

(*a*) The velocity is the time derivative of the position and the acceleration is the time derivative of the acceleration:

$$x = A\cos\omega t \Rightarrow v = \frac{dx}{dt} = -\omega A\sin\omega t$$

and $a = \frac{dv}{dt} = -\omega^2 A\cos\omega t$

The maximum value of $\sin\omega t$ is +1 and the minimum value of $\sin\omega t$ is −1. A and ω are positive constants:

$$v_{\max} = A\omega = (2.5\,\text{m})(\pi\,\text{s}^{-1}) = \boxed{7.9\,\text{m/s}}$$

The maximum value of $\cos\omega t$ is +1 and the minimum value of $\cos\omega t$ is −1:

$$a_{\max} = A\omega^2 = (2.5\,\text{m})(\pi\,\text{s}^{-1})^2 = \boxed{25\,\text{m/s}^2}$$

(*b*) Use the Pythagorean identity $\sin^2\omega t + \cos^2\omega t = 1$ to eliminate t from the equations for x and v:

$$\frac{v^2}{\omega^2 A^2} + \frac{x^2}{A^2} = 1 \Rightarrow |v| = \omega\sqrt{A^2 - x^2}$$

Substitute numerical values and evaluate $|v(1.5\,\text{m})|$:

$$|v(1.5\,\text{m})| = (\pi\,\text{rad/s})\sqrt{(2.5\,\text{m})^2 - (1.5\,\text{m})^2} = \boxed{6.3\,\text{m/s}}$$

Substitute x for $A\cos\omega t$ in the equation for a to obtain:

$$a = -\omega^2 x$$

Substitute numerical values and evaluate a:

$$a = -(\pi\,\text{rad/s})^2(1.5\,\text{m}) = \boxed{-15\,\text{m/s}^2}$$

38 ••• (*a*) Show that $A_0\cos(\omega t + \delta)$ can be written as $A_s\sin(\omega t) + A_c\cos(\omega t)$, and determine A_s and A_c in terms of A_0 and δ. (*b*) Relate A_c and A_s to the initial position and velocity of a particle undergoing simple harmonic motion.

Picture the Problem We can use the formula for the cosine of the sum of two angles to write $x = A_0\cos(\omega t + \delta)$ in the desired form. We can then evaluate x and dx/dt at $t = 0$ to relate A_c and A_s to the initial position and velocity of a particle undergoing simple harmonic motion.

(*a*) Apply the trigonometric identity $\cos(\omega t + \delta) = \cos\omega t\cos\delta - \sin\omega t\sin\delta$ to obtain:

$$\begin{aligned} x &= A_0\cos(\omega t + \delta) \\ &= A_0[\cos\omega t\cos\delta - \sin\omega t\sin\delta] \\ &= -A_0\sin\delta\sin\omega t + A_0\cos\delta\cos\omega t \\ &= \boxed{A_s\sin\omega t + A_c\cos\omega t} \end{aligned}$$

provided

$$A_s = -A_0\sin\delta \text{ and } A_c = A_0\cos\delta$$

(*b*) When $t = 0$:

$$x(0) = \boxed{A_0 \cos\delta = A_c}$$

Evaluate dx/dt:

$$v = \frac{dx}{dt} = \frac{d}{dt}\left[A_s \sin\omega t + A_c \cos\omega t\right]$$
$$= A_s\omega\cos\omega t - A_c\omega\sin\omega t$$

Evaluate $v(0)$ to obtain:

$$v(0) = \omega A_s = \boxed{-\omega A_0 \sin\delta}$$

Simple Harmonic Motion as Related to Circular Motion

39 • **[SSM]** A particle moves at a constant speed of 80 cm/s in a circle of radius 40 cm centered at the origin. (*a*) Find the frequency and period of the *x* component of its position. (*b*) Write an expression for the *x* component of its position as a function of time *t*, assuming that the particle is located on the +*y*-axis at time $t = 0$.

Picture the Problem We can find the period of the motion from the time required for the particle to travel completely around the circle. The frequency of the motion is the reciprocal of its period and the *x*-component of the particle's position is given by $x = A\cos(\omega t + \delta)$. We can use the initial position of the particle to determine the phase constant δ.

(*a*) Use the definition of speed to find the period of the motion:

$$T = \frac{2\pi r}{v} = \frac{2\pi(0.40\,\text{m})}{0.80\,\text{m/s}} = 3.14 = \boxed{3.1\,\text{s}}$$

Because the frequency and the period are reciprocals of each other:

$$f = \frac{1}{T} = \frac{1}{3.14\,\text{s}} = \boxed{0.32\,\text{Hz}}$$

(*b*) Express the *x* component of the position of the particle:

$$x = A\cos(\omega t + \delta) = A\cos(2\pi f t + \delta) \quad (1)$$

The initial condition on the particle's position is:

$$x(0) = 0$$

Substitute in the expression for *x* to obtain:

$$0 = A\cos\delta \Rightarrow \delta = \cos^{-1}(0) = \frac{\pi}{2}$$

Substitute for *A*, ω, and δ in equation (1) to obtain:

$$x = \boxed{(40\,\text{cm})\cos\left[\left(2.0\,\text{s}^{-1}\right)t + \frac{\pi}{2}\right]}$$

40 • A particle moves in a 15-cm-radius circle centered at the origin and completes 1.0 rev every 3.0 s. (*a*) Find the speed of the particle. (*b*) Find its angular speed ω. (*c*) Write an equation for the *x* component of the position of the particle as a function of time *t*, assuming that the particle is on the –*x* axis at time $t = 0$.

Picture the Problem We can find the period of the motion from the time required for the particle to travel completely around the circle. The angular frequency of the motion is 2π times the reciprocal of its period and the *x*-component of the particle's position is given by $x = A\cos(\omega t + \delta)$. We can use the initial position of the particle to determine the phase constant δ.

(*a*) Use the definition of speed to express and evaluate the speed of the particle:

$$v = \frac{2\pi r}{T} = \frac{2\pi(15\,\text{cm})}{3.0\,\text{s}} = \boxed{31\,\text{cm/s}}$$

(*b*) The angular speed of the particle is:

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{3.0\,\text{s}} = \boxed{\frac{2\pi}{3}\,\text{rad/s}}$$

(*c*) Express the *x* component of the position of the particle:

$$x = A\cos(\omega t + \delta) \qquad (1)$$

The initial condition on the particle's position is:

$$x(0) = -A$$

Substituting for *x* in equation (1) yields:

$$-A = A\cos\delta \Rightarrow \delta = \cos^{-1}(-1) = \pi$$

Substitute for *A*, ω, and δ in equation (1) to obtain:

$$x = \boxed{(15\,\text{cm})\cos\left(\left(\frac{2\pi}{3}\,\text{s}^{-1}\right)t + \pi\right)}$$

Energy in Simple Harmonic Motion

41 • A 2.4-kg object on a frictionless horizontal surface is attached to one end of a horizontal spring of force constant k = 4.5 kN/m. The other end of the spring is held stationary. The spring is stretched 10 cm from equilibrium and released. Find the system's total mechanical energy.

Picture the Problem The total mechanical energy of the object is given by $E_{\text{tot}} = \frac{1}{2}kA^2$, where *A* is the amplitude of the object's motion.

The total mechanical energy of the system is given by:

$$E_{\text{tot}} = \tfrac{1}{2}kA^2$$

Substitute numerical values and evaluate E_{tot}:

$$E_{\text{tot}} = \tfrac{1}{2}(4.5\,\text{kN/m})(0.10\,\text{m})^2 = \boxed{23\,\text{J}}$$

42 • Find the total energy of a system consisting of a 3.0-kg object on a frictionless horizontal surface oscillating with an amplitude of 10 cm and a frequency of 2.4 Hz at the end of a horizontal spring.

Picture the Problem The total energy of an oscillating object can be expressed in terms of its kinetic energy as it passes through its equilibrium position: $E_{\text{tot}} = \tfrac{1}{2}mv_{\text{max}}^2$. Its maximum speed, in turn, can be expressed in terms of its angular frequency and the amplitude of its motion.

Express the total energy of the object in terms of its maximum kinetic energy:

$$E = \tfrac{1}{2}mv_{\text{max}}^2$$

The maximum speed v_{max} of the oscillating object is given by:

$$v_{\text{max}} = A\omega = 2\pi Af$$

Substitute for v_{max} to obtain:

$$E = \tfrac{1}{2}m(2\pi Af)^2 = 2mA^2\pi^2 f^2$$

Substitute numerical values and evaluate E:

$$E = 2(3.0\,\text{kg})(0.10\,\text{m})^2\pi^2(2.4\,\text{s}^{-1})^2 = \boxed{3.4\,\text{J}}$$

43 • **[SSM]** A 1.50-kg object on a frictionless horizontal surface oscillates at the end of a spring of force constant k = 500 N/m. The object's maximum speed is 70.0 cm/s. (*a*) What is the system's total mechanical energy? (*b*) What is the amplitude of the motion?

Picture the Problem The total mechanical energy of the oscillating object can be expressed in terms of its kinetic energy as it passes through its equilibrium position: $E_{\text{tot}} = \tfrac{1}{2}mv_{\text{max}}^2$. Its total energy is also given by $E_{\text{tot}} = \tfrac{1}{2}kA^2$. We can equate these expressions to obtain an expression for A.

(*a*) Express the total mechanical energy of the object in terms of its maximum kinetic energy:

$$E = \tfrac{1}{2}mv_{\text{max}}^2$$

Substitute numerical values and evaluate E:

$$E = \tfrac{1}{2}(1.50\,\text{kg})(0.700\,\text{m/s})^2 = 0.3675\,\text{J} = \boxed{0.368\,\text{J}}$$

(*b*) Express the total mechanical energy of the object in terms of the amplitude of its motion:

$$E_{\text{tot}} = \tfrac{1}{2}kA^2 \Rightarrow A = \sqrt{\frac{2E_{\text{tot}}}{k}}$$

Substitute numerical values and evaluate A:

$$A = \sqrt{\frac{2(0.3675\,\text{J})}{500\,\text{N/m}}} = \boxed{3.83\,\text{cm}}$$

44 • A 3.0-kg object on a frictionless horizontal surface is oscillating on the end of a spring that has a force constant equal to 2.0 kN/m and a total mechanical energy of 0.90 J. (*a*) What is the amplitude of the motion? (*b*) What is the maximum speed?

Picture the Problem The total mechanical energy of the oscillating object can be expressed in terms of its kinetic energy as it passes through its equilibrium position: $E_{\text{tot}} = \tfrac{1}{2}mv_{\text{max}}^2$. Its total energy is also given by $E_{\text{tot}} = \tfrac{1}{2}kA^2$. We can solve the latter equation to find A and solve the former equation for v_{max}.

(*a*) Express the total mechanical energy of the object as a function of the amplitude of its motion:

$$E_{\text{tot}} = \tfrac{1}{2}kA^2 \Rightarrow A = \sqrt{\frac{2E_{\text{tot}}}{k}}$$

Substitute numerical values and evaluate A:

$$A = \sqrt{\frac{2(0.90\,\text{J})}{2000\,\text{N/m}}} = \boxed{3.0\,\text{cm}}$$

(*b*) Express the total mechanical energy of the object in terms of its maximum speed:

$$E_{\text{tot}} = \tfrac{1}{2}mv_{\text{max}}^2 \Rightarrow v_{\text{max}} = \sqrt{\frac{2E_{\text{tot}}}{m}}$$

Substitute numerical values and evaluate v_{max}:

$$v_{\text{max}} = \sqrt{\frac{2(0.90\,\text{J})}{3.0\,\text{kg}}} = \boxed{77\,\text{cm/s}}$$

45 • An object on a frictionless horizontal surface oscillates at the end of a spring with an amplitude of 4.5 cm. Its total mechanical energy is 1.4 J. What is the force constant of the spring?

Picture the Problem The total mechanical energy of the object is given by $E_{tot} = \frac{1}{2}kA^2$. We can solve this equation for the force constant k and substitute the numerical data to determine its value.

Express the total mechanical energy of the oscillator as a function of the amplitude of its motion:

$$E_{tot} = \tfrac{1}{2}kA^2 \Rightarrow k = \frac{2E_{tot}}{A^2}$$

Substitute numerical values and evaluate k:

$$\boldsymbol{k} = \frac{2(1.4\,\text{J})}{(0.045\,\text{m})^2} = \boxed{1.4\ \text{kN/m}}$$

46 •• A 3.0-kg object on a frictionless horizontal surface oscillates at the end of a spring with an amplitude of 8.0 cm. Its maximum acceleration is 3.5 m/s^2. Find the total mechanical energy.

Picture the Problem The total mechanical energy of the system is the sum of the potential and kinetic energies. That is, $E_{tot} = \frac{1}{2}kx^2 + \frac{1}{2}mv^2$. Newton's 2nd law relates the acceleration to the displacement. That is, $-kx = ma$. In addition, when $x = A$, $v = 0$. Use these equations to solve E_{tot} in terms of the given parameters m, A and a_{max}.

The total mechanical energy is the sum of the potential and kinetic energies. We don't know k so we need an equation relating k to one or more of the given parameters:

$$E_{tot} = \tfrac{1}{2}kx^2 + \tfrac{1}{2}mv^2$$

The force exerted by the spring equals the mass of the object multiplied by its acceleration:

$$-kx = ma \Rightarrow k = -\frac{ma}{x}$$

When $x = -A$, $a = a_{max}$. Thus,

$$k = -\frac{ma_{max}}{-A} = \frac{ma_{max}}{A}$$

Substitute to obtain:

$$E_{tot} = \tfrac{1}{2}\frac{ma_{max}}{A}x^2 + \tfrac{1}{2}mv^2$$

When $x = A$, $v = 0$. Substitute to obtain:

$$E_{tot} = \tfrac{1}{2}\frac{ma_{max}}{A}A^2 + 0 = \tfrac{1}{2}ma_{max}A$$

Substitute numerical values and evaluate E_{tot}:

$$E_{tot} = \tfrac{1}{2}(3.0\,\text{kg})(3.5\,\text{m/s}^2)(0.080\,\text{m}) = \boxed{0.42\,\text{J}}$$

Simple Harmonic Motion and Springs

47 • A 2.4-kg object on a frictionless horizontal surface is attached to a horizontal spring that has a force constant 4.5 kN/m. The spring is stretched 10 cm from equilibrium and released. What are (*a*) the frequency of the motion, (*b*) the period, (*c*) the amplitude, (*d*) the maximum speed, and (*e*) the maximum acceleration? (*f*) When does the object first reach its equilibrium position? What is its acceleration at this time?

Picture the Problem The frequency of the object's motion is given by $f = \frac{1}{2\pi}\sqrt{k/m}$ and its period is the reciprocal of *f*. The maximum velocity and acceleration of an object executing simple harmonic motion are $v_{max} = A\omega$ and $a_{max} = A\omega^2$, respectively.

(*a*) The frequency of the motion is given by:

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

Substitute numerical values and evaluate *f*:

$$f = \frac{1}{2\pi}\sqrt{\frac{4.5\,\text{kN/m}}{2.4\,\text{kg}}} = 6.89\,\text{Hz} = \boxed{6.9\,\text{Hz}}$$

(*b*) The period of the motion to is the reciprocal of its frequency:

$$T = \frac{1}{f} = \frac{1}{6.89\,\text{s}^{-1}} = 0.145\,\text{s} = \boxed{0.15\,\text{s}}$$

(*c*) Because the object is released from rest after the spring to which it is attached is stretched 10 cm:

$$A = \boxed{10\,\text{cm}}$$

(*d*) The object's maximum speed is given by:

$$v_{max} = A\omega = 2\pi fA$$

Substitute numerical values and evaluate v_{max}:

$$v_{max} = 2\pi\left(6.89\,\text{s}^{-1}\right)\left(0.10\,\text{m}\right) = 4.33\,\text{m/s} = \boxed{4.3\,\text{m/s}}$$

(*e*) The object's maximum acceleration is given by:

$$a_{max} = A\omega^2 = \omega v_{max} = 2\pi f v_{max}$$

Substitute numerical values and evaluate a_{max}:

$$\boldsymbol{a}_{max} = 2\boldsymbol{\pi}\left(6.89\,\text{s}^{-1}\right)\left(4.33\,\text{m/s}\right) = \boxed{1.9\times10^{2}\ \text{m/s}^{2}}$$

(*f*) The object first reaches its equilibrium when:

$$\boldsymbol{t} = \tfrac{1}{4}\boldsymbol{T} = \tfrac{1}{4}\left(0.145\,\text{s}\right) = \boxed{36\,\text{ms}}$$

Because the resultant force acting on the object as it passes through its equilibrium position is zero, the acceleration of the object is:

$$a = \boxed{0}$$

48 • A 5.00-kg object on a frictionless horizontal surface is attached to one end of a horizontal spring that has a force constant k = 700 N/m. The spring is stretched 8.00 cm from equilibrium and released. What are (*a*) the frequency of the motion, (*b*) the period, (*c*) the amplitude, (*d*) the maximum speed, and (*e*) the maximum acceleration? (*f*) When does the object first reach its equilibrium position? What is its acceleration at this time?

Picture the Problem The frequency of the object's motion is given by $f = \frac{1}{2\pi}\sqrt{k/m}$ and its period is the reciprocal of *f*. The maximum speed and acceleration of an object executing simple harmonic motion are $v_{max} = A\omega$ and $a_{max} = A\omega^2$, respectively.

(*a*) The frequency of the motion is given by:

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

Substitute numerical values and evaluate *f*:

$$f = \frac{1}{2\pi}\sqrt{\frac{700\,\text{N/m}}{5.00\,\text{kg}}} = 1.883\ \text{Hz} = \boxed{1.88\,\text{Hz}}$$

(*b*) The period of the motion is the reciprocal of its frequency:

$$T = \frac{1}{f} = \frac{1}{1.883\,\text{s}^{-1}} = 0.5310\ \text{s} = \boxed{0.531\,\text{s}}$$

(*c*) Because the object is released from rest after the spring to which it is attached is stretched 8.00 cm:

$$A = \boxed{8.00\,\text{cm}}$$

(*d*) The object's maximum speed is given by:

$$v_{max} = A\omega = 2\pi fA$$

Substitute numerical values and evaluate v_{max}:

$$v_{max} = 2\pi\left(1.883\,\text{s}^{-1}\right)\left(0.0800\,\text{m}\right) = 0.9465\,\text{m/s} = \boxed{0.947\,\text{m/s}}$$

(*e*) The object's maximum acceleration is given by:

$$a_{max} = A\omega^2 = \omega v_{max} = 2\pi f v_{max}$$

Substitute numerical values and evaluate a_{max}:

$$a_{max} = 2\pi\left(1.883\,\text{s}^{-1}\right)\left(0.9465\,\text{m/s}\right) = \boxed{11.2\,\text{m/s}^2}$$

(*f*) The object first reaches its equilibrium when:

$$\boldsymbol{t} = \tfrac{1}{4}\boldsymbol{T} = \tfrac{1}{4}\left(0.5310\,\text{s}\right) = \boxed{0.133\,\text{s}}$$

Because the resultant force acting on the object as it passes through its equilibrium point is zero, the acceleration of the object is $a = \boxed{0.}$

49 • **[SSM]** A 3.0-kg object on a frictionless horizontal surface is attached to one end of a horizontal spring, oscillates with an amplitude of 10 cm and a frequency of 2.4 Hz. (*a*) What is the force constant of the spring? (*b*) What is the period of the motion? (*c*) What is the maximum speed of the object? (*d*) What is the maximum acceleration of the object?

Picture the Problem (*a*) The angular frequency of the motion is related to the force constant of the spring through $\omega^2 = k/m$. (*b*) The period of the motion is the reciprocal of its frequency. (*c*) and (*d*) The maximum speed and acceleration of an object executing simple harmonic motion are $v_{max} = A\omega$ and $a_{max} = A\omega^2$, respectively.

(*a*) Relate the angular frequency of the motion to the force constant of the spring:

$$\omega^2 = \frac{k}{m} \Rightarrow k = m\omega^2 = 4\pi^2 f^2 m$$

Substitute numerical values to obtain:

$$k = 4\pi^2\left(2.4\,\text{s}^{-1}\right)^2\left(3.0\,\text{kg}\right) = 682\,\text{N/m} = \boxed{0.68\,\text{kN/m}}$$

(*b*) Relate the period of the motion to its frequency:

$$T = \frac{1}{f} = \frac{1}{2.4\,\text{s}^{-1}} = 0.417\,\text{s} = \boxed{0.42\,\text{s}}$$

(*c*) The maximum speed of the object is given by:

$$v_{\max} = A\omega = 2\pi fA$$

Substitute numerical values and evaluate $v_{\max}$:

$$v_{\max} = 2\pi\left(2.4\,\text{s}^{-1}\right)\left(0.10\,\text{m}\right) = 1.51\,\text{m/s} = \boxed{1.5\,\text{m/s}}$$

(*d*) The maximum acceleration of the object is given by:

$$a_{\max} = A\omega^2 = 4\pi^2 f^2 A$$

Substitute numerical values and evaluate $a_{\max}$:

$$a_{\max} = 4\pi^2\left(2.4\,\text{s}^{-1}\right)^2\left(0.10\,\text{m}\right) = \boxed{23\,\text{m/s}^2}$$

50 • An 85.0-kg person steps into a car of mass 2400 kg, causing it to sink 2.35 cm on its springs. If started into vertical oscillation, and assuming no damping, at what frequency will the car and passenger vibrate on these springs?

Picture the Problem We can find the frequency of vibration of the car-and-passenger system using $f = \frac{1}{2\pi}\sqrt{\frac{k}{M}}$, where M is the total mass of the system. The force constant of the spring can be determined from the compressing force and the amount of compression.

Express the frequency of the car-and-passenger system:

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{M}}$$

The force constant is given by:

$$k = \frac{F}{\Delta x} = \frac{mg}{\Delta x}$$

where m is the person's mass.

Substitute for k in the expression for f to obtain:

$$f = \frac{1}{2\pi}\sqrt{\frac{mg}{M\Delta x}}$$

Substitute numerical values and evaluate f:

$$f = \frac{1}{2\pi}\sqrt{\frac{\left(85.0\,\text{kg}\right)\left(9.81\,\text{m/s}^2\right)}{\left(2485\,\text{kg}\right)\left(2.35\times10^{-2}\,\text{m}\right)}} = \boxed{0.601\,\text{Hz}}$$

51 • A 4.50-kg object with an amplitude of 3.80 cm oscillates on a horizontal spring. The object's maximum acceleration is 26.0 m/s^2. Find (*a*) the force constant of the spring, (*b*) the frequency of the object, and (*c*) the period of the motion of the object.

Picture the Problem (*a*) We can relate the force constant *k* to the maximum acceleration by eliminating ω^2 between $\omega^2 = k/m$ and $a_{max} = A\omega^2$. (*b*) We can find the frequency *f* of the motion by substituting ma_{max}/A for *k* in $f = \frac{1}{2\pi}\sqrt{k/m}$. (*c*) The period of the motion is the reciprocal of its frequency. Assume that friction is negligible.

(*a*) Relate the angular frequency of the motion to the force constant and the mass of the oscillator:

$$\omega^2 = \frac{k}{m} \Rightarrow k = \omega^2 m$$

Relate the object's maximum acceleration to its angular frequency and amplitude and solve for the square of the angular frequency:

$$a_{max} = A\omega^2 \Rightarrow \omega^2 = \frac{a_{max}}{A} \qquad (1)$$

Substitute for ω^2 to obtain:

$$k = \frac{ma_{max}}{A}$$

Substitute numerical values and evaluate *k*:

$$k = \frac{(4.50\,\text{kg})(26.0\,\text{m/s}^2)}{3.80\times10^{-2}\,\text{m}} = \boxed{3.08\,\text{kN/m}}$$

(*b*) Replace ω in equation (1) by $2\pi f$ and solve for *f* to obtain:

$$f = \frac{1}{2\pi}\sqrt{\frac{a_{max}}{A}}$$

Substitute numerical values and evaluate *f*:

$$f = \frac{1}{2\pi}\sqrt{\frac{26.0\,\text{m/s}^2}{3.80\times10^{-2}\,\text{m}}} = 4.163\,\text{Hz} = \boxed{4.16\,\text{Hz}}$$

(*c*) The period of the motion is the reciprocal of its frequency:

$$T = \frac{1}{f} = \frac{1}{4.163\,\text{s}^{-1}} = \boxed{0.240\,\text{s}}$$

52 •• An object of mass *m* is suspended from a vertical spring of force constant 1800 N/m. When the object is pulled down 2.50 cm from equilibrium and released from rest, the object oscillates at 5.50 Hz. (*a*) Find *m*. (*b*) Find the amount the spring is stretched from its unstressed length when the object is in

equilibrium. (*c*) Write expressions for the displacement x, the velocity v_x, and the acceleration a_x as functions of time t.

Picture the Problem Choose a coordinate system in which upward is the $+y$ direction. We can find the mass of the object using $m = k/\omega^2$. We can apply a condition for translational equilibrium to the object when it is at its equilibrium position to determine the amount the spring has stretched from its natural length. Finally, we can use the initial conditions to determine A and δ and express $x(t)$ and then differentiate this expression to obtain $v_x(t)$ and $a_x(t)$.

(*a*) Express the angular frequency of the system in terms of the mass of the object fastened to the vertical spring and solve for the mass of the object:

$$\omega^2 = \frac{k}{m} \Rightarrow m = \frac{k}{\omega^2}$$

Express ω^2 in terms of f:

$$\omega^2 = 4\pi^2 f^2$$

Substitute for ω^2 to obtain:

$$m = \frac{k}{4\pi^2 f^2}$$

Substitute numerical values and evaluate m:

$$m = \frac{1800\,\text{N/m}}{4\pi^2\left(5.50\,\text{s}^{-1}\right)^2} = 1.507\,\text{kg} = \boxed{1.51\,\text{kg}}$$

(*b*) Letting Δx represent the amount the spring is stretched from its natural length when the object is in equilibrium, apply $\sum F_y = 0$ to the object when it is in equilibrium:

$$k\Delta x - mg = 0$$

Solve for m to obtain:

$$k\Delta x - \frac{kg}{4\pi^2 f^2} = 0 \Rightarrow \Delta x = \frac{g}{4\pi^2 f^2}$$

Substitute numerical values and evaluate Δx:

$$\Delta x = \frac{9.81\,\text{m/s}^2}{4\pi^2\left(5.50\,\text{s}^{-1}\right)^2} = \boxed{8.21\,\text{mm}}$$

(*c*) Express the position of the object as a function of time:

$$x = A\cos(\omega t + \delta)$$

Use the initial conditions $x_0 = -2.50$ cm and $v_0 = 0$ to find δ:

$$\delta = \tan^{-1}\left(-\frac{v_0}{\omega x_0}\right) = \tan^{-1}(0) = \pi$$

Evaluate ω:

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{1800\,\text{N/m}}{1.507\,\text{kg}}} = 34.56\,\text{rad/s}$$

Substitute to obtain:

$$x = (2.50\,\text{cm})\cos[(34.56\,\text{rad/s})t + \pi] = \boxed{-(2.50\,\text{cm})\cos[(34.6\,\text{rad/s})t]}$$

Differentiate $x(t)$ to obtain v_x:

$$v_x = (86.39\,\text{cm/s})\sin[(34.56\,\text{rad/s})t] = \boxed{(86.4\,\text{cm/s})\sin[(34.6\,\text{rad/s})t]}$$

Differentiate $v(t)$ to obtain a_x:

$$a_x = (29.86\,\text{m/s}^2)\cos[(34.56\,\text{rad/s})t] = \boxed{(29.9\,\text{m/s}^2)\cos[(34.6\,\text{rad/s})t]}$$

53 •• An object is hung on the end of a vertical spring and is released from rest with the spring unstressed. If the object falls 3.42 cm before first coming to rest, find the period of the resulting oscillatory motion.

Picture the Problem Let the system include the object and the spring. Then, the net external force acting on the system is zero. Choose $E_i = 0$ and apply the conservation of mechanical energy to the system.

Express the period of the motion in terms of its angular frequency:

$$T = \frac{2\pi}{\omega} \qquad (1)$$

Apply conservation of energy to the system:

$$E_i = E_f \Rightarrow 0 = U_g + U_{spring}$$

Substituting for U_g and U_{spring} yields:

$$0 = -mg\Delta x + \tfrac{1}{2}k(\Delta x)^2 \Rightarrow \omega = \frac{k}{m} = \sqrt{\frac{2g}{\Delta x}}$$

Substituting for ω in equation (1) yields:

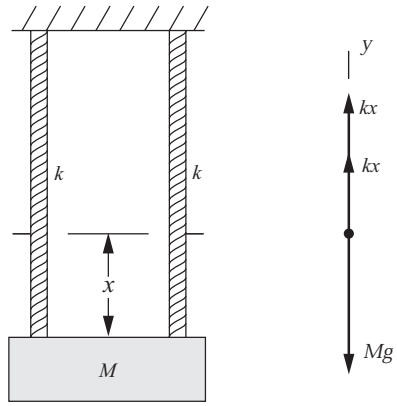
$$T = \frac{2\pi}{\sqrt{\frac{2g}{\Delta x}}} = 2\pi\sqrt{\frac{\Delta x}{2g}}$$

Substitute numerical values and evaluate T:

$$T = 2\pi\sqrt{\frac{3.42\,\text{cm}}{2(9.81\,\text{m/s}^2)}} = \boxed{0.262\,\text{s}}$$

54 •• A suitcase of mass 20 kg is hung from two bungee cords, as shown in Figure 14-27. Each cord is stretched 5.0 cm when the suitcase is in equilibrium. If the suitcase is pulled down a little and released, what will be its oscillation frequency?

Picture the Problem The diagram shows the stretched bungee cords supporting the suitcase under equilibrium conditions. We can use $f = \frac{1}{2\pi}\sqrt{\frac{k_{\text{eff}}}{M}}$ to express the frequency of the suitcase in terms of the effective "spring" constant k_{eff} and apply the condition for translational equilibrium to the suitcase to find k_{eff}.



Express the frequency of the suitcase oscillator:

$$f = \frac{1}{2\pi}\sqrt{\frac{k_{\text{eff}}}{M}} \qquad (1)$$

Apply $\sum F_y = 0$ to the suitcase to obtain:

$$kx + kx - Mg = 0$$

or

$$2kx - Mg = 0$$

or

$$k_{\text{eff}}x - Mg = 0 \Rightarrow k_{\text{eff}} = \frac{Mg}{x}$$

where $k_{\text{eff}} = 2k$

Substitute for k_{eff} in equation (1) to obtain:

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{x}}$$

Substitute numerical values and evaluate f:

$$f = \frac{1}{2\pi}\sqrt{\frac{9.81\,\text{m/s}^2}{0.050\,\text{m}}} = \boxed{2.2\,\text{Hz}}$$

55 •• A 0.120-kg block is suspended from a spring. When a small pebble of mass 30 g is placed on the block, the spring stretches an additional 5.0 cm. With the pebble on the block, the spring oscillates with an amplitude of 12 cm. (*a*) What is the frequency of the motion? (*b*) How long does the block take to travel from its lowest point to its highest point? (*c*) What is the net force on the pebble when it is at the point of maximum upward displacement?

Picture the Problem (*a*) The frequency of the motion of the stone and block depends on the force constant of the spring and the mass of the stone plus block. The force constant can be determined from the equilibrium of the system when

the spring is stretched additionally by the addition of the stone to the mass. (*b*) The time required for the block to travel from its lowest point to its highest point is half its period. (*c*) When the block is at the point of maximum upward displacement, it is momentarily at rest and the net force acting on it is its weight.

(*a*) Express the frequency of the motion in terms of k and m_{tot}:

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m_{tot}}} \qquad (1)$$

where m_{tot} is the total mass suspended from the spring.

Apply $\sum F_y = 0$ to the stone when it is at its equilibrium position:

$$k\Delta y - mg = 0 \Rightarrow k = \frac{mg}{\Delta y}$$

Substitute for k in equation (1) to obtain:

$$f = \frac{1}{2\pi}\sqrt{\frac{mg}{\Delta y m_{tot}}}$$

Substitute numerical values and evaluate f:

$$f = \frac{1}{2\pi}\sqrt{\frac{(0.030\text{ kg})(9.81\text{ m/s}^2)}{(0.050\text{ m})(0.15\text{ kg})}}$$
$$= 0.997\text{ Hz} = \boxed{1.0\text{ Hz}}$$

(*b*) The time to travel from its lowest point to its highest point is one-half its period:

$$t = \tfrac{1}{2}T = \frac{1}{2f} = \frac{1}{2(0.997\text{ s}^{-1})} = \boxed{0.50\text{ s}}$$

(*c*) When the stone is at a point of maximum upward displacement:

$$F_{net} = mg = (0.030\text{ kg})(9.81\text{ m/s}^2)$$
$$= \boxed{0.29\text{ N}}$$

56 •• Referring to Problem 69, find the maximum amplitude of oscillation at which the pebble will remain in contact with the block.

Picture the Problem We can use the maximum acceleration of the oscillator to express a_{max} in terms of A, k, and m. k can be determined from the equilibrium of the system when the spring is stretched additionally by the addition of the stone to the mass. If the stone is to remain in contact with the block, the block's maximum downward acceleration must not exceed g.

Express the maximum acceleration of the oscillator in terms of its angular frequency and amplitude of the motion:

$$a_{max} = A\omega^2$$

Relate ω^2 to the force constant of the spring and the mass of the block-plus-stone:

$$\omega^2 = \frac{k}{m_{tot}}$$

Substitute for ω^2 to obtain:

$$a_{max} = A\frac{k}{m_{tot}} \qquad (1)$$

Apply $\sum F_y = 0$ to the stone when it is at its equilibrium position:

$$k\Delta y - mg = 0 \Rightarrow k = \frac{mg}{\Delta y}$$

where Δy is the additional distance the spring stretched when the stone was placed on the block.

Substitute for k in equation (1) to obtain:

$$a_{max} = A\left(\frac{mg}{\Delta y m_{tot}}\right)$$

Set $a_{max} = g$ and solve for A_{max}:

$$A_{max} = \frac{\Delta y m_{tot}}{mg}g = \frac{m_{tot}}{m}\Delta y$$

Substitute numerical values and evaluate A_{max}:

$$A_{max} = \left(\frac{0.15\text{ kg}}{0.030\text{ kg}}\right)(0.050\text{ m}) = \boxed{25\text{ cm}}$$

57 •• An object of mass 2.0 kg is attached to the top of a vertical spring that is anchored to the floor. The unstressed length of the spring is 8.0 cm and the length of the spring when the object is in equilibrium is 5.0 cm. When the object is resting at its equilibrium position, it is given a sharp downward blow with a hammer so that its initial speed is 0.30 m/s. (*a*) To what maximum height above the floor does the object eventually rise? (*b*) How long does it take for the object to reach its maximum height for the first time? (*c*) Does the spring ever become unstressed? What minimum initial speed must be given to the object for the spring to be unstressed at some time?

Picture the Problem (*a*) The maximum height above the floor to which the object rises is the sum of its initial distance from the floor and the amplitude of its motion. We can find the amplitude of its motion by relating it to the object's maximum speed. (*b*) Because the object initially travels downward, it will be three-fourths of the way through its cycle when it first reaches its maximum

height. (*c*) We can find the minimum initial speed the object would need to be given in order for the spring to become uncompressed by applying conservation of mechanical energy.

(*a*) Relate *h*, the maximum height above the floor to which the object rises, to the amplitude of its motion:

$$h = A + 5.0\text{ cm} \qquad (1)$$

Relate the maximum speed of the object to the angular frequency and amplitude of its motion and solve for the amplitude:

$$v_{\max} = A\omega$$

or, because $\omega^2 = \dfrac{k}{m}$,

$$A = v_{\max}\sqrt{\frac{m}{k}} \qquad (2)$$

Apply $\sum \boldsymbol{F}_y = 0$ to the object when it is resting at its equilibrium position to obtain:

$$k\Delta y - mg = 0 \Rightarrow k = \frac{mg}{\Delta y}$$

Substitute for k in equation (2):

$$A = v_{\max}\sqrt{\frac{m\Delta y}{mg}} = v_{\max}\sqrt{\frac{\Delta y}{g}}$$

Substituting for A in equation (1) yields:

$$h = v_{\max}\sqrt{\frac{\Delta y}{g}} + 5.0\text{ cm}$$

Substitute numerical values and evaluate h:

$$h = 0.30\text{ m/s}^2\sqrt{\frac{0.030\text{ m}}{9.81\text{ m/s}^2}} + 5.0\text{ cm}$$

$$= \boxed{6.7\text{ cm}}$$

(*b*) The time required for the object to reach its maximum height the first time is three-fourths its period:

$$t = \tfrac{3}{4}T$$

Express the period of the motion of the oscillator:

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{m}{\frac{mg}{\Delta y}}} = 2\pi\sqrt{\frac{\Delta y}{g}}$$

Substitute for T in the expression for t to obtain:

$$t = \frac{3}{4}\left(2\pi\sqrt{\frac{\Delta y}{g}}\right) = \frac{3\pi}{2}\sqrt{\frac{\Delta y}{g}}$$

Substitute numerical values and evaluate t:

$$t = \frac{3\pi}{2}\sqrt{\frac{0.030\text{ m}}{9.81\text{ m/s}^2}} = \boxed{0.26\text{ s}}$$

(*c*) Because $h < 8.0$ cm, the spring is never uncompressed.

Using conservation of energy and letting U_g be zero 5 cm above the floor, relate the height to which the object rises, Δy, to its initial kinetic energy:

$$\Delta K + \Delta U_g + \Delta U_s = 0$$

or, because $K_f = U_i = 0$,

$$\tfrac{1}{2}mv_i^2 - mg\Delta y + \tfrac{1}{2}k(\Delta y)^2 - \tfrac{1}{2}k(L - y_i)^2 = 0$$

Because $\Delta y = L - y_i$:

$$\tfrac{1}{2}mv_i^2 - mg\Delta y + \tfrac{1}{2}k(\Delta y)^2 - \tfrac{1}{2}k(\Delta y)^2 = 0$$

and

$$\tfrac{1}{2}mv_i^2 - mg\Delta y = 0 \Rightarrow v_i = \sqrt{2g\Delta y}$$

Substitute numerical values and evaluate v_i:

$$v_i = \sqrt{2(9.81\text{ m/s}^2)(3.0\text{ cm})} = 77\text{ cm/s}$$

That is, the minimum initial speed that must be given to the object for the spring to be uncompressed at some time is $\boxed{77\text{ cm/s}}$

58 ••• A winch cable has a cross-sectional area of 1.5 cm^2 and a length of 2.5 m. Young's modulus for the cable is 150 GN/m^2. A 950-kg engine block is hung from the end of the cable. (*a*) By what length does the cable stretch? (*b*) Treating the cable as a simple spring, what is the oscillation frequency of the engine block at the end of the cable?

Picture the Problem We can relate the elongation of the cable to the load on it using the definition of Young's modulus and use the expression for the frequency of a spring-mass oscillator to find the oscillation frequency of the engine block at the end of the wire.

(*a*) Using the definition of Young's modulus, relate the elongation of the cable to the applied stress:

$$Y = \frac{\text{stress}}{\text{strain}} = \frac{F/A}{\Delta\ell/\ell} \Rightarrow \Delta\ell = \frac{F\ell}{AY} = \frac{Mg\ell}{AY}$$

Substitute numerical values and evaluate $\Delta\ell$:

$$\Delta\ell = \frac{(950\text{ kg})(9.81\text{ m/s}^2)(2.5\text{ m})}{(1.5\text{ cm}^2)(150\text{ GN/m}^2)} = 1.0355\text{ mm} = \boxed{1.0\text{ mm}}$$

(*b*) Express the oscillation frequency of the wire-engine block system:

$$f = \frac{1}{2\pi}\sqrt{\frac{k_{\text{eff}}}{M}}$$

Express the effective "spring" constant of the cable:

$$k_{\text{eff}} = \frac{F}{\Delta\ell} = \frac{Mg}{\Delta\ell}$$

Substitute for k_{eff} to obtain:

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{\Delta\ell}}$$

Substitute numerical values and evaluate *f*:

$$f = \frac{1}{2\pi}\sqrt{\frac{9.81\,\text{m/s}^2}{1.0355\,\text{mm}}} = \boxed{15\,\text{Hz}}$$

Simple Pendulum Systems

59 • **[SSM]** Find the length of a simple pendulum if its frequency for small amplitudes is 0.75 Hz.

Picture the Problem The frequency of a simple pendulum depends on its length and on the local gravitational field and is given by $f = \frac{1}{2\pi}\sqrt{\frac{g}{L}}$.

The frequency of a simple pendulum oscillating with small amplitude is given by:

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{L}} \Rightarrow L = \frac{g}{4\pi^2 f^2}$$

Substitute numerical values and evaluate *L*:

$$L = \frac{9.81\,\text{m/s}^2}{4\pi^2\left(0.75\,\text{s}^{-1}\right)^2} = \boxed{44\,\text{cm}}$$

60 • Find the length of a simple pendulum if its period for small amplitudes is 5.0 s.

Picture the Problem We can determine the required length of the pendulum from the expression for the period of a simple pendulum.

Express the period of a simple pendulum:

$$T = 2\pi\sqrt{\frac{L}{g}} \Rightarrow L = \frac{T^2 g}{4\pi^2}$$

Substitute numerical values and evaluate *L*:

$$L = \frac{(5.0\,\text{s})^2\left(9.81\,\text{m/s}^2\right)}{4\pi^2} = \boxed{6.2\,\text{m}}$$

61 • What would be the period of the pendulum in Problem 60 if the pendulum were on the moon, where the acceleration due to gravity is one-sixth that on Earth?

Picture the Problem We can find the period of the pendulum from $T = 2\pi\sqrt{L/g_{\text{moon}}}$ where $g_{\text{moon}} = \frac{1}{6}g$ and $L = 6.21$ m.

Express the period of a simple pendulum on the moon:

$$T = 2\pi\sqrt{\frac{L}{g_{\text{moon}}}}$$

Substitute numerical values and evaluate T:

$$T = 2\pi\sqrt{\frac{6.21\,\text{m}}{\frac{1}{6}\left(9.81\,\text{m/s}^2\right)}} = \boxed{12\,\text{s}}$$

62 • If the period of a 70.0-cm-long simple pendulum is 1.68 s, what is the value of g at the location of the pendulum?

Picture the Problem We can find the value of g at the location of the pendulum by solving the equation $T = 2\pi\sqrt{L/g}$ for g and evaluating it for the given length and period.

Express the period of a simple pendulum where the gravitational field is g:

$$T = 2\pi\sqrt{\frac{L}{g}} \Rightarrow g = \frac{4\pi^2 L}{T^2}$$

Substitute numerical values and evaluate g:

$$g = \frac{4\pi^2\left(0.700\,\text{m}\right)}{\left(1.68\,\text{s}\right)^2} = \boxed{9.79\,\text{m/s}^2}$$

63 • A simple pendulum set up in the stairwell of a 10-story building consists of a heavy weight suspended on a 34.0-m-long wire. What is the period of oscillation?

Picture the Problem We can use $T = 2\pi\sqrt{L/g}$ to find the period of this pendulum.

Express the period of a simple pendulum:

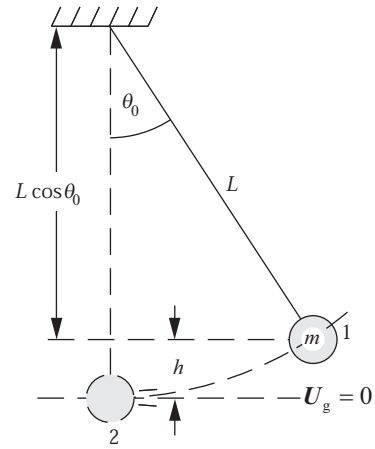
$$T = 2\pi\sqrt{\frac{L}{g}}$$

Substitute numerical values and evaluate T:

$$T = 2\pi\sqrt{\frac{34.0\,\text{m}}{9.81\,\text{m/s}^2}} = \boxed{11.7\,\text{s}}$$

64 •• Show that the total energy of a simple pendulum undergoing oscillations of small amplitude ϕ_0 (in radians) is $E \approx \frac{1}{2}mgL\phi_0^2$. *Hint: Use the approximation* $\cos\phi \approx 1 - \frac{1}{2}\phi^2$ *for small* ϕ.

Picture the Problem The figure shows the simple pendulum at maximum angular displacement ϕ_0. The total energy of the simple pendulum is equal to its initial gravitational potential energy. We can apply the definition of gravitational potential energy and use the small-angle approximation to show that $E \approx \frac{1}{2}mgL\phi_0^2$.



Express the total energy of the simple pendulum at maximum displacement:

$$E = U_{\text{max displacement}} = mgh$$

Referring to the diagram, express h in terms of L and ϕ_0:

$$h = L - L\cos\phi_0 = L(1 - \cos\phi_0)$$

Substituting for h yields:

$$E = mgL[1 - \cos\phi_0]$$

From the power series expansion for $\cos\phi$, for $\phi << 1$:

$$\cos\phi \approx 1 - \tfrac{1}{2}\phi^2$$

Substitute and simplify to obtain:

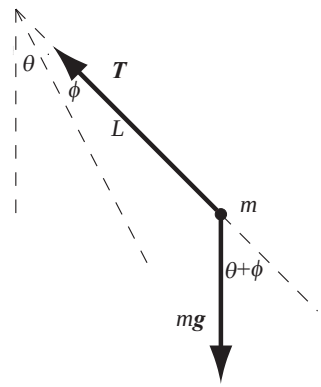
$$E = mgL\left[1 - \left(1 - \tfrac{1}{2}\phi_0^2\right)\right] = \boxed{\tfrac{1}{2}mgL\phi_0^2}$$

65 ••• **[SSM]** A simple pendulum of length L is attached to a massive cart that slides without friction down a plane inclined at angle θ with the horizontal, as shown in Figure 14-28. Find the period of oscillation for small oscillations of this pendulum.

Picture the Problem The cart accelerates down the ramp with a constant acceleration of $g\sin\theta$. This happens because the cart is much more massive than the bob, so the motion of the cart is unaffected by the motion of the bob oscillating back and forth. The path of the bob is quite complex in the reference frame of the ramp, but in the reference frame moving with the cart the path of the bob is much simpler—in this frame the bob moves back and forth along a circular arc. To solve this problem we first apply Newton's second law (to the bob) in the inertial reference frame of the ramp. Then we transform to the reference frame

moving with the cart in order to exploit the simplicity of the motion in that frame.

Draw the free-body diagram for the bob. Let ϕ denote the angle that the string makes with the normal to the ramp. The forces on the bob are the tension force and the force of gravity:



Apply Newton's 2nd law to the bob, labeling the acceleration of the bob relative to the ramp $\vec{a}_{\text{BR}}$:

$$\vec{T} + m\vec{g} = m\vec{a}_{\text{BR}}$$

The acceleration of the bob relative to the ramp is equal to the acceleration of the bob relative to the cart plus the acceleration of the cart relative to the ramp:

$$\vec{a}_{\text{BR}} = \vec{a}_{\text{BC}} + \vec{a}_{\text{CR}}$$

Substitute for $\vec{a}_{\text{BR}}$ in $\vec{T} + m\vec{g} = m\vec{a}_{\text{BR}}$:

$$\vec{T} + m\vec{g} = m\left(\vec{a}_{\text{BC}} + \vec{a}_{\text{CR}}\right)$$

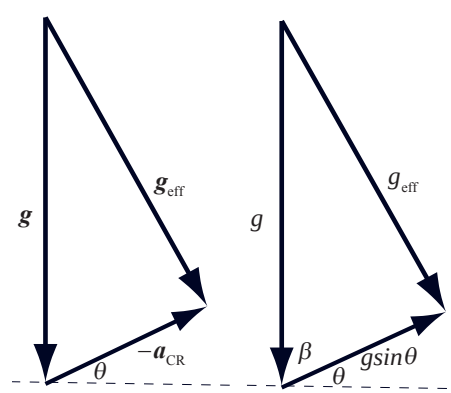
Rearrange terms and label $\vec{g} - \vec{a}_{\text{CR}}$ as $\vec{g}_{\text{eff}}$, where $\vec{g}_{\text{eff}}$ is the acceleration, relative to the cart, of an object in free fall. (If the tension force is set to zero the bob is in free fall.):

$$\vec{T} + m\left(\vec{g} - \vec{a}_{\text{CR}}\right) = m\vec{a}_{\text{BC}}$$

Label $\vec{g} - \vec{a}_{\text{CR}}$ as $\vec{g}_{\text{eff}}$ to obtain

$$\vec{T} + m\vec{g}_{\text{eff}} = m\vec{a}_{\text{BC}} \qquad (1)$$

To find the magnitude of $\vec{g}_{\text{eff}}$, first draw the vector addition diagram representing the equation $\vec{g}_{\text{eff}} = \vec{g} - \vec{a}_{\text{CR}}$. Recall that $a_{\text{CR}} = g \sin\theta$:



From the diagram, find the magnitude of $\vec{g}_{\text{eff}}$. Use the law of cosines:

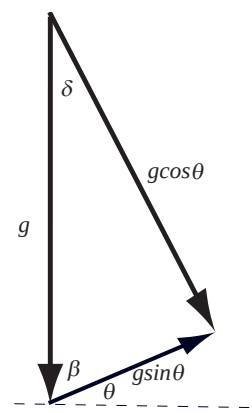
$$g_{\text{eff}}^2 = g^2 + g^2 \sin^2\theta - 2g\left(g\sin\theta\right)\cos\beta$$

But $\cos\beta = \sin\theta$, so

$$g_{\text{eff}}^2 = g^2 + g^2\sin^2\theta - 2g^2\sin^2\theta$$
$$= g^2\left(1-\sin^2\theta\right) = g^2\cos^2\theta$$

Thus $g_{\text{eff}} = g\cos\theta$

To find the direction of $\vec{g}_{\text{eff}}$, first redraw the vector addition diagram as shown:

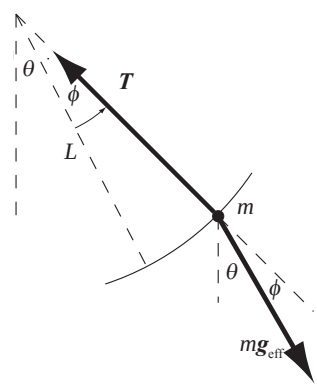


From the diagram find the direction of $\vec{g}_{\text{eff}}$. Use the law of cosines again and solve for δ:

$$g^2\sin^2\theta = g^2 + g^2\cos^2\theta - 2g^2\cos\theta\cos\delta$$

and so $\delta = \theta$

To find an equation for the motion of the bob draw the "free-body diagram" for the "forces" that appear in equation (1). Draw the path of the bob in the reference frame moving with the cart:



Take the tangential components of each vector in equation (1) in the frame of the cart yields. The tangential component of the acceleration is equal to the radius of the circle times the angular acceleration $\left(a_t = r\alpha\right)$:

$$0 - mg_{\text{eff}}\sin\phi = mL\frac{d^2\phi}{dt^2}$$

where L is the length of the string and $\frac{d^2\phi}{dt^2}$ is the angular acceleration of the bob. The positive tangential "direction" is counterclockwise.

Rearranging this equation yields:

$$mL\frac{d^2\phi}{dt^2}+mg_{\text{eff}}\sin\phi=0 \qquad (2)$$

For small oscillations of the pendulum:

$$|\phi|<<1 \text{ and } \sin\phi\approx\phi$$

Substituting for $\sin\phi$ in equation (2) yields:

$$mL\frac{d^2\phi}{dt^2}+mg_{\text{eff}}\phi=0$$

or

$$\frac{d^2\phi}{dt^2}+\frac{g_{\text{eff}}}{L}\phi=0 \qquad (3)$$

Equation (3) is the equation of motion for simple harmonic motion with angular frequency:

$$\omega=\sqrt{\frac{g_{\text{eff}}}{L}}$$

where ω is the angular frequency of the oscillations (and not the angular speed of the bob).

The period of this motion is:

$$T=\frac{2\pi}{\omega}=2\pi\sqrt{\frac{L}{g_{\text{eff}}}} \qquad (4)$$

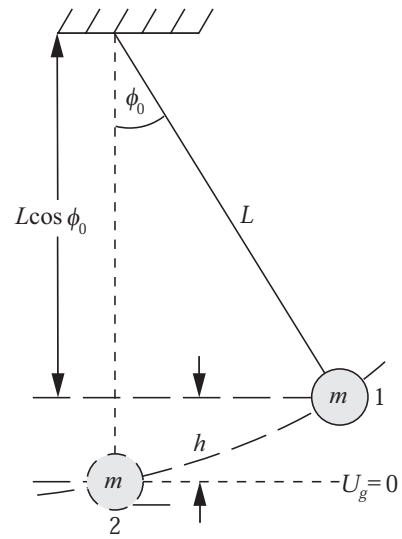
Substitute $g\cos\theta$ for g_{eff} in equation (4) to obtain:

$$\boldsymbol{T=2\pi\sqrt{\frac{L}{g_{\text{eff}}}}}=\boxed{\boldsymbol{2\pi\sqrt{\frac{L}{g\cos\theta}}}}$$

Remarks: Note that, in the limiting case $\theta = 0$, $T=2\pi\sqrt{L/g}$ and $T\rightarrow 0$. As $\theta\rightarrow 90°$, $T\rightarrow\infty$.

66 ••• The bob at the end of a simple pendulum of length L is released from rest from an angle ϕ_0. (*a*) Model the pendulum's motion as simple harmonic motion and find its speed as it passes through $\phi = 0$ by using the small angle approximation. (*b*) Using the conservation of energy, find this speed exactly for any angle (not just small angles). (*c*) Show that your result from Part (*b*) agrees with the approximate answer in Part (*a*) when ϕ_0 is small. (*d*) Find the difference between the approximate and exact results for $\phi_0 = 0.20$ rad and $L = 1.0$ m. (*e*) Find the difference between the approximate and exact results for $\phi_0 = 1.20$ rad and $L = 1.0$ m.

Picture the Problem The figure shows the simple pendulum at maximum angular displacement ϕ_0. We can express the angular position of the pendulum's bob in terms of its initial angular position and time and differentiate this expression to find the maximum speed of the bob. We can use conservation of energy to find an exact value for v_{max} and the approximation $\cos\phi \approx 1 - \frac{1}{2}\phi^2$ to show that this value reduces to the former value for small ϕ.



(*a*) Relate the speed of the pendulum's bob to its angular speed:

$$v = L\frac{d\phi}{dt} \qquad (1)$$

The angular position of the pendulum as a function of time is given by:

$$\phi = \phi_0 \cos\omega t$$

Differentiate this expression to express the angular speed of the pendulum:

$$\frac{d\phi}{dt} = -\phi_0\omega\sin\omega t$$

Substitute in equation (1) to obtain:

$$v = -L\phi_0\omega\sin\omega t = -v_{max}\sin\omega t$$

Simplify v_{max} to obtain:

$$v_{max} = L\phi_0\sqrt{\frac{g}{L}} = \boxed{\phi_0\sqrt{gL}}$$

(*b*) Use conservation of energy to relate the potential energy of the pendulum at point 1 to its kinetic energy at point 2:

$$\Delta K + \Delta U = 0$$

or, because $K_1 = U_2 = 0$,

$$K_2 - U_1 = 0$$

Substitute for K_2 and U_1:

$$\tfrac{1}{2}mv_2^2 - mgh = 0$$

Express h in terms of L and ϕ_0:

$$h = L(1 - \cos\phi_0)$$

Substituting for h yields:

$$\tfrac{1}{2}mv_2^2 - mgL(1 - \cos\phi_0) = 0$$

Solve for $v_2 = v_{max}$ to obtain:

$$v_{max} = \boxed{\sqrt{2gL(1-\cos\phi_0)}} \qquad (2)$$

(*c*) For $\phi_0 << 1$:

$$1-\cos\phi_0 \approx \tfrac{1}{2}\phi_0^2$$

Substitute in equation (2) to obtain:

$$v_{max} = \sqrt{2gL\left(\tfrac{1}{2}\phi_0^2\right)} = \boxed{\phi_0\sqrt{gL}}$$

in agreement with our result in part (*a*).

(*d*) Express the difference in the results from (*a*) and (*b*):

$$\Delta v = v_{max,a} - v_{max,b}$$

Substitute for $v_{max,a}$ and $v_{max,b}$ and simplify to obtain:

$$\begin{aligned}\Delta \boldsymbol{v} &= \boldsymbol{\phi}_0\sqrt{\boldsymbol{gL}} - \sqrt{2\boldsymbol{gL}(1-\cos\boldsymbol{\phi}_0)} \\ &= \sqrt{\boldsymbol{gL}}\left(\boldsymbol{\phi}_0 - \sqrt{2(1-\cos\boldsymbol{\phi}_0)}\right)\end{aligned} \qquad (3)$$

Substitute numerical values and evaluate Δv:

$$\Delta \boldsymbol{v} = \sqrt{(9.81\ \text{m/s}^2)(1.0\ \text{m})}\left(0.20\ \text{rad} - \sqrt{2(1-\cos(0.20\ \text{rad}))}\right) \approx \boxed{1\ \text{mm/s}}$$

(*e*) Evaluate equation (3) for $\phi_0 = 1.20$ rad and $L = 1.0$ m:

$$\Delta \boldsymbol{v} = \sqrt{(9.81\ \text{m/s}^2)(1.0\ \text{m})}\left(1.20\ \text{rad} - \sqrt{2(1-\cos(1.20\ \text{rad}))}\right) \approx \boxed{0.2\ \text{m/s}}$$

*Physical Pendulums

67 • **[SSM]** A thin 5.0-kg disk with a 20-cm radius is free to rotate about a fixed horizontal axis perpendicular to the disk and passing through its rim. The disk is displaced slightly from equilibrium and released. Find the period of the subsequent simple harmonic motion.

Picture the Problem The period of this physical pendulum is given by $T = 2\pi\sqrt{I/MgD}$ where I is the moment of inertia of the thin disk about the fixed horizontal axis passing through its rim. We can use the parallel-axis theorem to express I in terms of the moment of inertia of the disk about an axis through its center of mass and the distance from its center of mass to its pivot point.

Express the period of a physical pendulum:

$$T = 2\pi\sqrt{\frac{I}{MgD}}$$

Using the parallel-axis theorem, find the moment of inertia of the thin disk about an axis through the pivot point:

$$I = I_{cm} + MR^2 = \tfrac{1}{2}MR^2 + MR^2 = \tfrac{3}{2}MR^2$$

Substituting for I and simplifying yields:

$$T = 2\pi\sqrt{\frac{\frac{3}{2}MR^2}{MgR}} = 2\pi\sqrt{\frac{3R}{2g}}$$

Substitute numerical values and evaluate T:

$$T = 2\pi\sqrt{\frac{3(0.20\,\text{m})}{2(9.81\,\text{m/s}^2)}} = \boxed{1.1\,\text{s}}$$

68 • A circular hoop that has a 50-cm radius is hung on a narrow horizontal rod and allowed to swing in the plane of the hoop. What is the period of its oscillation, assuming that the amplitude is small?

Picture the Problem The period of this physical pendulum is given by $T = 2\pi\sqrt{I/MgD}$ where I is the moment of inertia of the circular hoop about an axis through its pivot point. We can use the parallel-axis theorem to express I in terms of the moment of inertia of the hoop about an axis through its center of mass and the distance from its center of mass to its pivot point.

Express the period of a physical pendulum:

$$T = 2\pi\sqrt{\frac{I}{MgD}}$$

Using the parallel-axis theorem, find the moment of inertia of the circular hoop about an axis through the pivot point:

$$I = I_{cm} + MR^2 = MR^2 + MR^2 = 2MR^2$$

Substitute for I and simplify to obtain:

$$T = 2\pi\sqrt{\frac{2MR^2}{MgR}} = 2\pi\sqrt{\frac{2R}{g}}$$

Substitute numerical values and evaluate T:

$$T = 2\pi\sqrt{\frac{2(0.50\,\text{m})}{9.81\,\text{m/s}^2}} = \boxed{2.0\,\text{s}}$$

69 • A 3.0-kg plane figure is suspended at a point 10 cm from its center of mass. When it is oscillating with small amplitude, the period of oscillation is 2.6 s. Find the moment of inertia I about an axis perpendicular to the plane of the figure through the pivot point.

Picture the Problem The period of a physical pendulum is given by $T = 2\pi\sqrt{I/MgD}$ where I is its moment of inertia about an axis through its pivot point. We can solve this equation for I and evaluate it using the given numerical data.

Express the period of a physical pendulum:

$$T = 2\pi\sqrt{\frac{I}{MgD}} \Rightarrow I = \frac{MgDT^2}{4\pi^2}$$

Substitute numerical values and evaluate I:

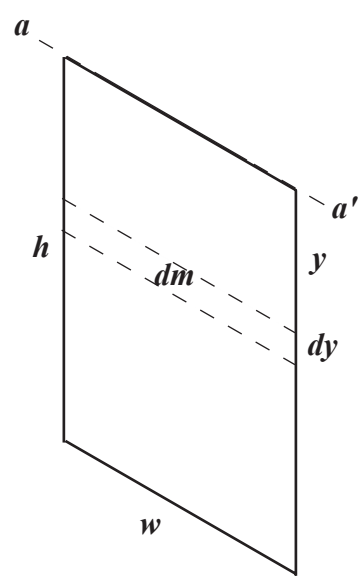
$$I = \frac{(3.0\,\text{kg})(9.81\,\text{m/s}^2)(0.10\,\text{m})(2.6\,\text{s})^2}{4\pi^2} = \boxed{0.50\,\text{kg}\cdot\text{m}^2}$$

70 •• You have designed a cat door that consists of a square piece of plywood that is 1.0 in. thick and 6.0 in. on a side, and is hinged at its top. To make sure the cat has enough time to get through it safely, the door should have a natural period of at least 1.0 s. Will your design work? If not, explain qualitatively what you would do to make it meet your requirements.

Picture the Problem The pictorial representation shows the cat door, of height h and width w, pivoted about an axis through a-a'. We can use

$$T = 2\pi\sqrt{\frac{I_{a-a'}}{mgD}}$$

to find the period of the door but first must find $I_{a-a'}$. The diagram also shows a differential strip of height dy and mass dm a distance y from the axis of rotation of the door. We can integrate the differential expression for the moment of inertia of this strip to determine the moment of inertia of the door.



The period of the cat door is given by:

$$T = 2\pi\sqrt{\frac{I_{a-a'}}{mgD}} \qquad (1)$$

where D is the distance from the center of mass of the door to the axis of rotation.

Express the moment of inertia, about the axis *a*-*a*′, of the cat door:

$$dI_{a-a'} = y^2 dm$$

or, because $dm = \rho dV = \rho t dA = \rho w t dy$,

$$dI_{a-a'} = \rho w t\, y^2 dy$$

Integrating this expression between $y = 0$ and $y = h$ yields:

$$I_{a-a'} = \rho w t \int_0^h y^2 dy = \tfrac{1}{3}\rho w t h^3$$

Because $\rho = \dfrac{m}{V} = \dfrac{m}{wht}$:

$$I_{a-a'} = \tfrac{1}{3}\left(\frac{m}{wht}\right) wth^3 = \tfrac{1}{3} mh^2$$

Substituting for D and $I_{a-a'}$ in equation (1) yields:

$$T = 2\pi\sqrt{\frac{\frac{1}{3}mh^2}{mg\left(\frac{1}{2}h\right)}} = 2\pi\sqrt{\frac{2h}{3g}}$$

Substitute numerical values and evaluate T:

$$T = 2\pi\sqrt{\frac{2\left(6.0\text{ in}\times\dfrac{2.540\text{ cm}}{\text{in}}\right)}{3\left(9.81\text{ m/s}^2\right)}} = 0.64\text{ s}$$

Thus the door's period is too short. The only way to increase it is to increase the height of the door.

71 •• You are given a meterstick and asked to drill a narrow hole through it so that, when the stick is pivoted about a horizontal axis through the hole, the period of the pendulum will be a minimum. Where should you drill the hole?

Picture the Problem Let x be the distance of the pivot from the center of the meter stick, m the mass of the meter stick, and L its length. We'll express the period of the meter stick as a function of the distance x and then differentiate this expression with respect to x to determine where the hole should be drilled to minimize the period.

Express the period of a physical pendulum:

$$T = 2\pi\sqrt{\frac{I}{MgD}} \qquad (1)$$

Express the moment of inertia of the meter stick about an axis through its center of mass:

$$I_{\text{cm}} = \tfrac{1}{12} mL^2$$

Using the parallel-axis theorem, express the moment of inertia of the meter stick about an axis through the pivot point:

$$I = I_{\text{cm}} + mx^2 = \tfrac{1}{12}mL^2 + mx^2$$

Substitute in equation (1) and simplify to obtain:

$$T = 2\pi\sqrt{\frac{\frac{1}{12}mL^2 + mx^2}{mgx}} = \frac{2\pi}{\sqrt{g}}\sqrt{\frac{L^2 + 12x^2}{12x}}$$

The condition for an extreme value of T is that $\frac{d}{dx}\left(\sqrt{\frac{L^2+12x^2}{12x}}\right) = 0.$ Evaluate this derivative to obtain:

$$\frac{12x^2 - L^2}{24x^2\sqrt{\frac{L^2+12x^2}{12x}}} = 0 \Rightarrow 12x^2 - L^2 = 0$$

Noting that only the positive solution is physically meaningful, solve for x:

$$x = \frac{L}{\sqrt{12}} = \frac{100\,\text{cm}}{\sqrt{12}} = 28.9\,\text{cm}$$

The hole should be drilled at a distance:

$$\boldsymbol{d} = 50.0\,\text{cm} - 28.9\,\text{cm} = \boxed{21.1\,\text{cm}}$$

from the center of the meter stick.

72 •• Figure 14-29 shows a uniform disk of radius R = 0.80 m, a mass of 6.00 kg, and a small hole a distance d from the disk's center that can serve as a pivot point. (*a*) What should be the distance d so that the period of this physical pendulum is 2.50 s? (*b*) What should be the distance d so that this physical pendulum will have the shortest possible period? What is this shortest possible period?

Picture the Problem (*a*) Let m represent the mass and R the radius of the uniform disk. The disk is a physical pendulum. We'll use the expression $T = 2\pi\sqrt{I/mgd}$ for the period of a physical pendulum. To find I we use the parallel-axis theorem ($I = I_{\text{cm}} + md^2$). (*b*) The period is a minimum when $dT/dx = 0$, where, to avoid notational difficulties, we have substituted x for d.

(*a*) Express the period of a physical pendulum:

$$T = 2\pi\sqrt{\frac{I}{mgd}}$$

Using the parallel-axis theorem ($\boldsymbol{I} = \boldsymbol{I}_{\text{cm}} + \boldsymbol{md}^2$), relate the moment of inertia about the axis through the hole to the moment of inertia I_{cm} about the parallel axis through the center of mass. Obtain I_{cm} from Table 9-1:

$$I = I_{\text{cm}} + md^2 = \tfrac{1}{2}mR^2 + md^2$$

Substituting for I yields:

$$T = 2\pi\sqrt{\frac{\frac{1}{2}mR^2 + md^2}{mgd}} = 2\pi\sqrt{\frac{\frac{1}{2}R^2 + d^2}{gd}} \qquad (1)$$

Square both sides of this equation, simplify, and substitute numerical values to obtain:

$$d^2 - \frac{gT^2}{4\pi^2}d + \frac{R^2}{2} = 0$$

or

$$d^2 - (1.553\,\text{m})d + 0.320\,\text{m}^2 = 0$$

Use the quadratic formula or your graphing calculator to obtain:

$$d = 0.238\,\text{m} = \boxed{24\,\text{cm}}$$

The second root, $d = 1.31$ m, is greater than R, so it is too large to be physically meaningful.

(*b*) The period T is related to the distance d by equation (1). T will be a minimum when $\left(\frac{1}{2}R^2 + d^2\right)/d$ is a minimum. Set the derivative of this expression equal to zero to find relative maxima and minima. We'll replace d with x to avoid the notational challenge of differentiating with respect to d. Evaluating $\frac{d}{dx}\left(\sqrt{\frac{\frac{1}{2}R^2 + x^2}{x}}\right) = 0$ yields:

$$\frac{2d^2 - \left(\frac{1}{2}R^2 + d^2\right)}{d^2} = 0 \Rightarrow 2d^2 - \left(\tfrac{1}{2}R^2 + d^2\right) = 0$$

where we have changed x back to d.

Solving for d yields:

$$d = \boxed{\frac{R}{\sqrt{2}}}$$

Evaluate equation (1) with $d = R/\sqrt{2}$ to obtain an expression for the shortest possible period:

$$T = 2\pi\sqrt{\frac{\frac{1}{2}R^2 + \frac{1}{2}R^2}{g\frac{R}{\sqrt{2}}}} = 2\pi\sqrt{\frac{\sqrt{2}R}{g}}$$

Substitute numerical values and evaluate T:

$$T = 2\pi\sqrt{\frac{\sqrt{2}(0.80\,\text{m})}{9.81\,\text{m/s}^2}} = \boxed{2.1\,\text{s}}$$

Remarks: We've shown that $d = R/\sqrt{2}$ corresponds to an *extreme* value; that is, to a maximum, a minimum, or an inflection point. To verify that this value of d corresponds to a minimum, we can either (1) show that d^2T/dx^2 evaluated at $x = R/\sqrt{2}$ (where $x = d$) is positive, or (2) graph T as a function of d and note that the graph is a minimum at $d = R/\sqrt{2}$.

73 ••• **[SSM]** Points P_1 and P_2 on a plane object (Figure 14-30) are distances h_1 and h_2, respectively, from the center of mass. The object oscillates with the same period T when it is free to rotate about an axis through P_1 and when it is free to rotate about an axis through P_2. Both of these axes are perpendicular to the plane of the object. Show that $h_1 + h_2 = gT^2/(4\pi)^2$, where $h_1 \neq h_2$.

Picture the Problem We can use the equation for the period of a physical pendulum and the parallel-axis theorem to show that $h_1 + h_2 = gT^2/4\pi^2$.

Express the period of the physical pendulum:

$$T = 2\pi\sqrt{\frac{I}{mgd}}$$

Using the parallel-axis theorem, relate the moment of inertia about an axis through P_1 to the moment of inertia about an axis through the plane's center of mass:

$$I = I_{\text{cm}} + mh_1^2$$

Substitute for I to obtain:

$$T = 2\pi\sqrt{\frac{I_{\text{cm}} + mh_1^2}{mgh_1}}$$

Square both sides of this equation and rearrange terms to obtain:

$$\frac{mgT^2}{4\pi^2} = \frac{I_{\text{cm}}}{h_1} + mh_1 \qquad (1)$$

Because the period of oscillation is the same for point P_2:

$$\frac{I_{\text{cm}}}{h_1} + mh_1 = \frac{I_{\text{cm}}}{h_2} + mh_2$$

Combining like terms yields:

$$\left(\frac{1}{h_1}-\frac{1}{h_2}\right)I_{\text{cm}}=m(h_2-h_1)$$

Provided $h_1 \neq h_2$:

$$I_{\text{cm}}=mh_1h_2$$

Substitute in equation (1) and simplify to obtain:

$$\frac{mgT^2}{4\pi^2}=\frac{mh_1h_2}{h_1}+mh_1 \Rightarrow h_1+h_2=\boxed{\frac{gT^2}{4\pi^2}}$$

74 ••• A physical pendulum consists of a spherical bob of radius r and mass m suspended from a rigid rod of negligible mass as in Figure 14-31. The distance from the center of the sphere to the point of support is L. When r is much less than L, such a pendulum is often treated as a simple pendulum of length L. (*a*) Show that the period for small oscillations is given by $T=T_0\sqrt{1+\frac{2r^2}{5L^2}}$ where $T_0=2\pi\sqrt{L/g}$ is the period of a simple pendulum of length L. (*b*) Show that when r is much smaller than L, the period can be approximated by $T \approx T_0\,(1+r^2/5L^2)$. (*c*) If L = 1.00 m and r = 2.00 cm, find the error in the calculated value when the approximation $T = T_0$ is used for the period. How large must be the radius of the bob for the error to be 1.00 percent?

Picture the Problem (*a*) We can find the period of the physical pendulum in terms of the period of a simple pendulum by starting with $T=2\pi\sqrt{I/mgL}$ and applying the parallel-axis theorem. (*b*) Performing a binomial expansion (with $r << L$) on the radicand of our expression for T will lead to $T \approx T_0\,(1+r^2/5L^2)$.

(*a*) Express the period of the physical pendulum:

$$T=2\pi\sqrt{\frac{I}{mgL}}$$

Using the parallel-axis theorem, relate the moment of inertia of the pendulum about an axis through its center of mass to its moment of inertia about an axis through its point of support:

$$I=I_{\text{cm}}+mL^2$$
$$=\tfrac{2}{5}mr^2+mL^2$$

Substitute for I and simplify to obtain:

$$T=2\pi\sqrt{\frac{\frac{2}{5}mr^2+mL^2}{mgL}}=2\pi\sqrt{\frac{L}{g}}\sqrt{1+\frac{2r^2}{5L^2}}=\boxed{T_0\sqrt{1+\frac{2r^2}{5L^2}}}$$

(*b*) Expanding $\left(1+\frac{2r^2}{5L^2}\right)^{1/2}$ binomially yields:

$$\left(1+\frac{2r^2}{5L^2}\right)^{1/2} = 1+\frac{1}{2}\left(\frac{2r^2}{5L^2}\right)+\frac{1}{8}\left(\frac{2r^2}{5L^2}\right)^2 + \text{higher-order terms}$$

$$\approx 1+\frac{r^2}{5L^2}$$

provided $r << L$

Substitute in our result from Part (*a*) to obtain:

$$T \approx \boxed{T_0\left(1+\frac{r^2}{5L^2}\right)}$$

(*c*) Express the fractional error when the approximation $T = T_0$ is used for this pendulum:

$$\frac{\Delta T}{T} \approx \frac{T-T_0}{T_0} = \frac{T}{T_0}-1$$

$$= 1+\frac{r^2}{5L^2}-1 = \frac{r^2}{5L^2}$$

Substitute numerical values and evaluate $\Delta T/T$:

$$\frac{\Delta T}{T} \approx \frac{(2.00\,\text{cm})^2}{5(100\,\text{cm})^2} = \boxed{0.00800\%}$$

For an error of 1.00%:

$$\frac{r^2}{5L^2} = 0.0100 \Rightarrow r = L\sqrt{0.0500}$$

Substitute the numerical value of L and evaluate r to obtain:

$$r = (100\,\text{cm})\sqrt{0.0500} = \boxed{22.4\,\text{cm}}$$

75 ••• Figure 14-32 shows the pendulum of a clock in your grandmother's house. The uniform rod of length L = 2.00 m has a mass m = 0.800 kg. Attached to the rod is a uniform disk of mass M = 1.20 kg and radius 0.150 m. The clock is constructed to keep perfect time if the period of the pendulum is exactly 3.50 s. (*a*) What should the distance d be so that the period of this pendulum is 2.50 s? (*b*) Suppose that the pendulum clock loses 5.00 min/d. To make sure that your grandmother won't be late for her quilting parties, you decide to adjust the clock back to its proper period. How far and in what direction should you move the disk to ensure that the clock will keep perfect time?

Picture the Problem (*a*) The period of this physical pendulum is given by $T = 2\pi\sqrt{I/MgD}$. We can express its period as a function of the distance d by using the definition of the center of mass of the pendulum to find D in terms of d and the parallel-axis theorem to express I in terms of d. Solving the resulting

quadratic equation yields *d*. (*b*) Because the clock is losing 5 minutes per day, one would reposition the disk so that the clock runs faster; that is, so the pendulum has a shorter period. We can determine the appropriate correction to make in the position of the disk by relating the fractional time loss to the fractional change in its position.

(*a*) Express the period of a physical pendulum:

$$T = 2\pi\sqrt{\frac{I}{m_{\text{tot}}gx_{\text{cm}}}}$$

Solving for $\frac{I}{x_{\text{cm}}}$ yields:

$$\frac{I}{x_{\text{cm}}} = \frac{T^2 g m_{\text{tot}}}{4\pi^2} \qquad (1)$$

Express the moment of inertia of the physical pendulum, about an axis through the pivot point, as a function of *d*:

$$I = I_{\text{cm}} + Md^2 = \tfrac{1}{3}mL^2 + \tfrac{1}{2}Mr^2 + Md^2$$

Substitute numerical values and evaluate *I*:

$$\begin{aligned}\boldsymbol{I} &= \tfrac{1}{3}(0.800\,\text{kg})(2.00\,\text{m})^2 \\ &\quad + \tfrac{1}{2}(1.20\,\text{kg})(0.150\,\text{m})^2 + (1.20\,\text{kg})\boldsymbol{d}^2 \\ &= 1.0802\,\text{kg}\cdot\text{m}^2 + (1.20\,\text{kg})\boldsymbol{d}^2\end{aligned}$$

Locate the center of mass of the physical pendulum relative to the pivot point:

$$\boldsymbol{x}_{\text{cm}} = \frac{(0.800\,\text{kg})(1.00\,\text{m}) + (1.20\,\text{kg})\boldsymbol{d}}{2.00\,\text{kg}}$$

and

$$\boldsymbol{x}_{\text{cm}} = 0.400\,\text{m} + 0.600\boldsymbol{d}$$

Substitute in equation (1) to obtain:

$$\frac{1.0802\,\text{kg}\cdot\text{m}^2 + (1.20\,\text{kg})d^2}{0.400\,\text{m} + 0.600d} = \frac{T^2(9.81\,\text{m/s}^2)(2.00\,\text{kg})}{4\pi^2} = (0.49698\,\text{kg}\cdot\text{m/s}^2)T^2 \qquad (2)$$

Setting $T = 2.50$ s and solving for *d* yields:

$$d = \boxed{1.63574\,\text{m}}$$

where we have kept more than three significant figures for use in Part (*b*).

(*b*) There are 1440 minutes per day. If the clock loses 5.00 min per day, then the period of the clock is related to the perfect period of the clock by:

$$1435T = 1440T_{\text{perfect}} \Rightarrow T = \frac{1440}{1435}T_{\text{perfect}}$$

where $T_{\text{perfect}} = 3.50$ s.

Substitute numerical values and evaluate T:

$$T = \frac{1440}{1435}(3.50\,\text{s}) = 3.51220\,\text{s}$$

Substitute $T = 3.51220$ s in equation (2) and solve for d to obtain:

$$d = 3.40140\,\text{m}$$

Substitute $T = 3.50$ s in equation (2) and solve for d' to obtain:

$$d' = 3.37826\,\text{m}$$

Express the distance the disk needs to be moved upward to correct the period:

$$\Delta d = d - d' = 3.40140\,\text{m} - 3.37826\,\text{m} = \boxed{2.31\,\text{cm}}$$

Damped Oscillations

76 • A 2.00-kg object oscillates with an initial amplitude of 3.00 cm. The force constant of the spring is 400 N/m. Find (*a*) the period, and (*b*) the total initial energy. (*c*) If the energy decreases by 1.00 percent per period, find the linear damping constant *b* and the *Q* factor.

Picture the Problem (*a*) We can find the period of the oscillator from $T = 2\pi\sqrt{m/k}$. (*b*) The total initial energy of the spring-object system is given by $E_0 = \frac{1}{2}kA^2$. (*c*) The *Q* factor can be found from its definition $Q = 2\pi/\left(|\Delta E|/E\right)_{\text{cycle}}$ and the damping constant from $Q = \omega_0 m/b$.

(*a*) The period of the oscillator is given by:

$$T = 2\pi\sqrt{\frac{m}{k}}$$

Substitute numerical values and evaluate T:

$$T = 2\pi\sqrt{\frac{2.00\,\text{kg}}{400\,\text{N/m}}} = \boxed{0.444\,\text{s}}$$

(*b*) Relate the initial energy of the oscillator to its amplitude:

$$E_0 = \tfrac{1}{2}kA^2$$

Substitute numerical values and evaluate E_0:

$$E_0 = \tfrac{1}{2}(400\,\text{N/m})(0.0300\,\text{m})^2 = \boxed{0.180\,\text{J}}$$

(*c*) Relate the fractional rate at which the energy decreases to the *Q* value and evaluate *Q*:

$$Q = \frac{2\pi}{\left(|\Delta E|/E\right)_{\text{cycle}}} = \frac{2\pi}{0.0100} = \boxed{628}$$

Express the Q value in terms of b:

$$Q = \frac{\omega_0 m}{b}$$

Solve for the damping constant b:

$$b = \frac{\omega_0 m}{Q} = \frac{2\pi m}{TQ} = \frac{2\pi m}{2\pi\sqrt{\frac{m}{k}}Q} = \frac{\sqrt{mk}}{Q}$$

Substitute numerical values and evaluate b:

$$b = \frac{\sqrt{(2.00\,\text{kg})(400\,\text{N/m})}}{628} = \boxed{0.0450\,\text{kg/s}}$$

77 •• **[SSM]** Show that the ratio of the amplitudes for two successive oscillations is constant for a linearly damped oscillator.

Picture the Problem The amplitude of the oscillation at time t is $A(t) = A_0 e^{-t/2\tau}$ where $\tau = m/b$ is the decay constant. We can express the amplitudes one period apart and then show that their ratio is constant.

Relate the amplitude of a given oscillation peak to the time at which the peak occurs:

$$A(t) = A_0 e^{-t/2\tau}$$

Express the amplitude of the oscillation peak at $t' = t + T$:

$$A(t+T) = A_0 e^{-(t+T)/2\tau}$$

Express the ratio of these consecutive peaks:

$$\frac{A(t)}{A(t+T)} = \frac{A_0 e^{-t/2\tau}}{A_0 e^{-(t+T)/2\tau}} = e^{-T/2\tau} = \boxed{\text{constant}}$$

78 •• An oscillator has a period of 3.00 s. Its amplitude decreases by 5.00 percent during each cycle. (*a*) By how much does its mechanical energy decrease during each cycle? (*b*) What is the time constant τ? (*c*) What is the Q factor?

Picture the Problem (*a*) We can relate the fractional change in the energy of the oscillator each cycle to the fractional change in its amplitude. (*b*) and (*c*) Both the Q value and the decay constant τ can be found from their definitions.

(*a*) Relate the energy of the oscillator to its amplitude:

$$E = \tfrac{1}{2}kA^2$$

Take the differential of this relationship to obtain:

$$dE = kAdA$$

Divide the second of these equations by the first and simplify to obtain:

$$\frac{dE}{E} = \frac{kAdA}{\frac{1}{2}kA^2} = 2\frac{dA}{A}$$

Approximate dE and dA by ΔE and ΔA and evaluate $\Delta E/E$:

$$\frac{\Delta E}{E} = 2(5.00\%) = \boxed{10.0\%}$$

(*b*) For small damping:

$$\frac{|\Delta E|}{E} = \frac{T}{\tau}$$

and

$$\tau = \frac{T}{|\Delta E|/E} = \frac{3.00\,\text{s}}{0.0100} = \boxed{30.0\,\text{s}}$$

(*c*) The Q factor is given by:

$$Q = \omega_0\tau = \left(\frac{2\pi}{T}\right)\tau$$

Substitute numerical values and evaluate Q:

$$Q = \frac{2\pi}{3.00\,\text{s}}(30.0\,\text{s}) = \boxed{62.8}$$

79 •• A linearly damped oscillator has a Q factor of 20. (*a*) By what fraction does the energy decrease during each cycle? (*b*) Use Equation 14-40 to find the percentage difference between ω' and ω_0. *Hint: Use the approximation* $(1+x)^{1/2} \approx 1 + \frac{1}{2}x$ *for small x.*

Picture the Problem We can use the physical interpretation of Q for small damping $\left(Q = \frac{2\pi}{(|\Delta E|/E)_{\text{cycle}}}\right)$ to find the fractional decrease in the energy of the oscillator each cycle.

(*a*) Express the fractional decrease in energy each cycle as a function of the Q factor and evaluate $|\Delta E|/E$:

$$\frac{|\Delta E|}{E} = \frac{2\pi}{Q} = \frac{2\pi}{20} = 0.314 = \boxed{0.31}$$

(*b*) The percentage difference between ω' and ω_0 is given by:

$$\frac{\omega' - \omega_0}{\omega_0} = \frac{\omega'}{\omega_0} - 1$$

Using the definition of the Q factor, use Equation 14-40 to express the ratio of ω' to ω_0 as a function of Q:

$$\frac{\omega'}{\omega_0}=\left[1-\frac{1}{4}\left(\frac{b^2}{m^2\omega_0^2}\right)\right]^{1/2}=\left[1-\frac{1}{4Q^2}\right]^{1/2}$$

Substituting for $\frac{\omega'}{\omega_0}$ yields:

$$\frac{\omega'-\omega_0}{\omega_0}=\left[1-\frac{1}{4Q^2}\right]^{1/2}-1$$

Use the approximation

$$(1+x)^{1/2}\approx 1+\tfrac{1}{2}x$$

for $x << 1$ to obtain:

$$\left[1-\frac{1}{4Q^2}\right]^{1/2}\approx 1-\frac{1}{8Q^2}$$

Substituting for $\left[1-\frac{1}{4Q^2}\right]^{1/2}$ and simplifying yields:

$$\frac{\omega'-\omega_0}{\omega_0}=1-\frac{1}{8Q^2}-1=-\frac{1}{8Q^2}$$

Substitute the numerical value of Q and evaluate $\frac{\omega'-\omega_0}{\omega_0}$:

$$\frac{\omega'-\omega_0}{\omega_0}=-\frac{1}{8(20)^2}=\boxed{-3.1\times10^{-2}\%}$$

80 •• A linearly damped mass–spring system oscillates at 200 Hz. The time constant of the system is 2.0 s. At $t = 0$ the amplitude of oscillation is 6.0 cm and the energy of the oscillating system is 60 J. (*a*) What are the amplitudes of oscillation at t = 2.0 s and t = 4.0 s? (*b*) How much energy is dissipated in the first 2-s interval and in the second 2-s interval?

Picture the Problem The energy of the spring-and-mass oscillator varies with time according to $E = E_0e^{-t/\tau}$ and its energy is proportional to the square of the amplitude.

(*a*) Using $E = E_0e^{-t/\tau}$ and $E \propto A^2$, solve for the amplitude A as a function of time:

$E = E_0e^{-t/\tau}$ and $E \propto A^2$ imply that $A^2 = A_0^2e^{-t/\tau}$

Hence $A = A_0e^{-t/2\tau}$

Express the amplitude of the oscillations as a function of time:

$$A=(6.0\text{ cm})e^{-t/4\text{ s}}$$

Evaluate the amplitude when t = 2.0 s:

$$A(2.0\text{ s})=(6.0\text{ cm})e^{-2.0\text{ s}/4.0\text{ s}}=\boxed{3.6\text{ cm}}$$

Evaluate the amplitude when $t = 4.0$ s:

$$A(4.0\,\text{s}) = (6.0\,\text{cm})e^{-4.0\,\text{s}/4.0\,\text{s}} = \boxed{2.2\,\text{cm}}$$

(*b*) Express the energy of the system at $t = 0$, $t = 2.0$ s, and $t = 4.0$ s:

$$E(0) = E_0 e^{-0/2.0\,\text{s}} = E_0$$
$$E(2.0\,\text{s}) = E_0 e^{-2.0/2.0\,\text{s}} = E_0 e^{-1}$$
$$E(4.0\,\text{s}) = E_0 e^{-4.0/2.0\,\text{s}} = E_0 e^{-2}$$

The energy dissipated in the first 2.0 s is equal to the negative of the change in mechanical energy:

$$-\Delta E_{0\to 2.0\,\text{s}} = -E_0\left(e^{-2.0\,\text{s}/2.0\,\text{s}} - e^0\right) = (60\,\text{J})\left(1 - e^{-1}\right) = \boxed{38\,\text{J}}$$

The energy dissipated in the second 2.0-s interval is:

$$-\Delta E_{2.0\,\text{s}\to 4.0\,\text{s}} = -E_0\left(e^{-4.0\,\text{s}/2.0\,\text{s}} - e^{-2.0\,\text{s}/2.0\,\text{s}}\right) = (60\,\text{J})e^{-1}\left(1 - e^{-1}\right) = \boxed{14\,\text{J}}$$

81 •• **[SSM]** Seismologists and geophysicists have determined that the vibrating Earth has a resonance period of 54 min and a Q factor of about 400. After a large earthquake, Earth will "ring" (continue to vibrate) for up to 2 months. (*a*) Find the percentage of the energy of vibration lost to damping forces during each cycle. (*b*) Show that after n periods the vibrational energy is given by $E_n = (0.984)^n E_0$, where E_0 is the original energy. (*c*) If the original energy of vibration of an earthquake is E_0, what is the energy after 2.0 d?

Picture the Problem (*a*) We can find the fractional loss of energy per cycle from the physical interpretation of Q for small damping. (*b*) We will also find a general expression for the earth's vibrational energy as a function of the number of cycles it has completed. (*c*) We can then solve this equation for the earth's vibrational energy after any number of days.

(*a*) Express the fractional change in energy as a function of Q:

$$\frac{\Delta E}{E} = \frac{2\pi}{Q} = \frac{2\pi}{400} = \boxed{1.57\%}$$

(*b*) Express the energy of the damped oscillator after one cycle:

$$E_1 = E_0\left(1 - \frac{\Delta E}{E}\right)$$

Express the energy after two cycles:

$$E_2 = E_1\left(1 - \frac{\Delta E}{E}\right) = E_0\left(1 - \frac{\Delta E}{E}\right)^2$$

Generalizing to n cycles:

$$E_n = E_0\left(1-\frac{\Delta E}{E}\right)^n = E_0(1-0.0157)^n$$
$$= \boxed{E_0(0.984)^n}$$

(*c*) Express 2.0 d in terms of the number of cycles; that is, the number of vibrations the earth will have experienced:

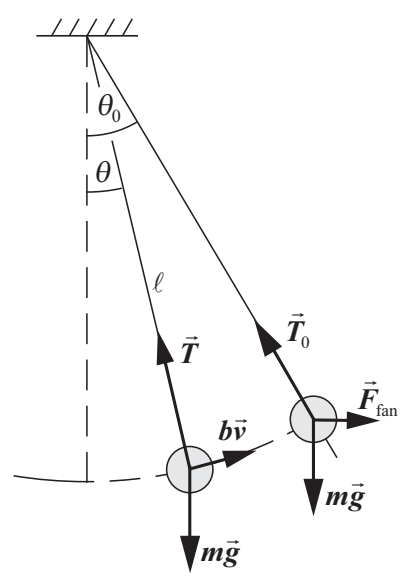
$$2.0\,\text{d} = 2.0\,\text{d}\times\frac{24\,\text{h}}{\text{d}}\times\frac{60\,\text{m}}{\text{h}}$$
$$= 2880\,\text{min}\times\frac{1T}{54\,\text{min}}$$
$$= 53.3T$$

Evaluate $E(2\text{ d})$:

$$E(2\,\text{d}) = E_0(0.9843)^{53.3} = \boxed{0.43E_0}$$

82 ••• A pendulum that is used in your physics laboratory experiment has a length of 75 cm and a compact bob with a mass equal to 15 g. To start the bob oscillating, you place a fan next to it that blows a horizontal stream of air on the bob. With the fan on, the bob is in equilibrium when the pendulum is displaced by an angle of 5.0º from the vertical. The speed of the air from the fan is 7.0 m/s. You turn the fan off, and allow the pendulum to oscillate. (*a*) Assuming that the drag force due to the air is of the form $-bv$, predict the decay time constant τ for this pendulum. (*b*) How long will it take for the pendulum's amplitude to reach 1.0º?

Picture the Problem The diagram shows 1) the pendulum bob displaced through an angle θ_0 and held in equilibrium by the force exerted on it by the air from the fan and 2) the bob accelerating, under the influence of gravity, tension force, and drag force, toward its equilibrium position. We can apply Newton's 2nd law to the bob to obtain the equation of motion of the damped pendulum and then use its solution to find the decay time constant and the time required for the amplitude of oscillation to decay to 1°.



(*a*) Apply $\sum\tau = I\alpha$ to the pendulum to obtain:

$$-mg\ell\sin\theta + \ell F_{\text{d}} = I\frac{d^2\theta}{dt^2}$$

Express the moment of inertia of the pendulum about an axis through its point of support:

$$I = m\ell^2$$

Substitute for I and F_d to obtain:

$$m\ell^2 \frac{d^2\theta}{dt^2} + \ell bv + mg\ell \sin\theta = 0$$

Because $\theta << 1$ and $v = \ell\omega = \ell d\theta/dt$:

$$m\ell^2 \frac{d^2\theta}{dt^2} + \ell^2 b \frac{d\theta}{dt} + mg\ell\theta \approx 0$$

or

$$m\frac{d^2\theta}{dt^2} + b\frac{d\theta}{dt} + \frac{mg}{\ell}\theta \approx 0$$

The solution to this second-order homogeneous differential equation with constant coefficients is:

$$\theta = \theta_0 e^{-t/2\tau}\cos(\omega' t + \delta) \qquad (1)$$

where θ_0 is the maximum amplitude, $\tau = m/b$ is the time constant, and the frequency $\omega' = \omega_0\sqrt{1-(b/2m\omega_0)^2}$.

Apply $\sum \vec{F} = m\vec{a}$ to the bob when it is at its maximum angular displacement to obtain:

$$\sum F_x = F_{\text{fan}} - T\sin\theta_0 = 0$$

and

$$\sum F_y = T\cos\theta_0 - mg = 0$$

Divide the x equation by the y equation to obtain:

$$\frac{F_{\text{fan}}}{mg} = \frac{T\sin\theta_0}{T\cos\theta_0} = \tan\theta_0$$

or

$$F_{\text{fan}} = mg\tan\theta_0$$

When the bob is in equilibrium, the drag force on it equals F_{fan}:

$$bv = mg\tan\theta_0 \Rightarrow \tau = \frac{m}{b} = \frac{v}{g\tan\theta_0}$$

Substitute numerical values and evaluate τ:

$$\tau = \frac{7.0\,\text{m/s}}{(9.81\,\text{m/s}^2)\tan 5.0°} = 8.16\,\text{s} = \boxed{8.2\,\text{s}}$$

(*b*) From equation (1), the angular amplitude of the motion is given by:

$$\theta = \theta_0 e^{-t/2\tau}$$

When the amplitude has decreased to 1.0°:

$$1.0° = 5.0° e^{-t/2\tau} \text{ or } e^{-t/2\tau} = 0.20$$

Take the natural logarithm of both sides of the equation to obtain:

$$-\frac{t}{2\tau} = \ln(0.20) \Rightarrow t = -2\tau\ln(0.20)$$

Substitute the numerical value of τ and evaluate t:

$$t = -2(8.16\,\text{s})\ln(0.20) = \boxed{26\,\text{s}}$$

83 ••• **[SSM]** You are in charge of monitoring the viscosity of oils at a manufacturing plant and you determine the viscosity of an oil by using the following method: The viscosity of a fluid can be measured by determining the decay time of oscillations for an oscillator that has known properties and operates while immersed in the fluid. As long as the speed of the oscillator through the fluid is relatively small, so that turbulence is not a factor, the drag force of the fluid on a sphere is proportional to the sphere's speed relative to the fluid: $F_\text{d} = 6\pi a\eta v$, where η is the viscosity of the fluid and a is the sphere's radius. Thus, the constant b is given by $6\pi a\eta$. Suppose your apparatus consists of a stiff spring that has a force constant equal to 350 N/cm and a gold sphere (radius 6.00 cm) hanging on the spring. (*a*) What is the viscosity of an oil do you measure if the decay time for this system is 2.80 s? (*b*) What is the Q factor for your system?

Picture the Problem (*a*) The decay time for a damped oscillator (with speed-dependent damping) system is defined as the ratio of the mass of the oscillator to the coefficient of v in the damping force expression. (*b*) The Q factor is the product of the resonance frequency and the damping time.

(*a*) From $F_\text{d} = 6\pi a\eta v$ and $F_\text{d} = -bv$, it follows that:

$$b = 6\pi a\eta \Rightarrow \eta = \frac{b}{6\pi a}$$

Because $\tau = m/b$, we can substitute for b to obtain:

$$\eta = \frac{m}{6\pi a\tau}$$

Substituting $m = \rho V$ and simplifying yields:

$$\eta = \frac{\rho V}{6\pi a\tau} = \frac{\frac{4}{3}\pi a^3\rho}{6\pi a\tau} = \frac{2a^2\rho}{9\tau}$$

Substitute numerical values and evaluate η (see Table 13-1 for the density of gold):

$$\eta = \frac{2(0.0600\,\text{m})^2(19.3\times10^3\,\text{kg/m}^3)}{9(2.8\,\text{s})} = \boxed{5.51\,\text{Pa}\cdot\text{s}}$$

(*b*) The Q factor is the product of the resonance frequency and the damping time:

$$Q = \omega_0\tau = \sqrt{\frac{k}{m}}\tau = \sqrt{\frac{k}{\rho V}}\tau = \sqrt{\frac{k}{\frac{4}{3}\pi a^3\rho}}\tau$$

Substitute numerical values and evaluate Q:

$$Q=\sqrt{\frac{3\left(350\,\frac{\text{N}}{\text{cm}}\times\frac{100\text{ cm}}{\text{m}}\right)}{4\pi(0.0600\text{ m})^3\left(19.3\times10^3\text{ kg/m}^3\right)}}(2.80\text{ s})\approx\boxed{125}$$

Driven Oscillations and Resonance

84 • A linearly damped oscillator loses 2.00 percent of its energy during each cycle. (*a*) What is its Q factor? (*b*) If its resonance frequency is 300 Hz, what is the width of the resonance curve $\Delta\omega$ when the oscillator is driven?

Picture the Problem (*a*) We can use the physical interpretation of Q for small damping to find the Q factor for this damped oscillator. (*b*) The width of the resonance curve depends on the Q factor according to $\Delta\omega=\omega_0/Q$.

(*a*) Using the physical interpretation of Q for small damping, relate Q to the fractional loss of energy of the damped oscillator per cycle:

$$Q=\frac{2\pi}{\left(|\Delta E|/E\right)_{\text{cycle}}}$$

Evaluate this expression for $\left(|\Delta E|/E\right)_{\text{cycle}}=2.00\%$:

$$Q=\frac{2\pi}{0.0200}=\boxed{314}$$

(*b*) Relate the width of the resonance curve to the Q value of the oscillatory system:

$$\Delta\omega=\frac{\omega_0}{Q}=\frac{2\pi f_0}{Q}$$

Substitute numerical values and evaluate $\Delta\omega$:

$$\Delta\omega=\frac{2\pi\left(300\text{ s}^{-1}\right)}{314}=\boxed{6.00\text{ rad/s}}$$

85 • Find the resonance frequency for each of the three systems shown in Figure 14-33.

Picture the Problem The resonant frequency of a vibrating system depends on the mass m of the system and on its "stiffness" constant k according to $f_0=\frac{1}{2\pi}\sqrt{\frac{k}{m}}$ or, in the case of a simple pendulum oscillating with small-amplitude vibrations, $f_0=\frac{1}{2\pi}\sqrt{\frac{g}{L}}$.

(*a*) For this spring-and-mass oscillator we have:

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{400.0\,\text{N/m}}{10\,\text{kg}}} = \boxed{1.0\,\text{Hz}}$$

(*b*) For this spring-and-mass oscillator we have:

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{800.0\,\text{N/m}}{5\,\text{kg}}} = \boxed{2\,\text{Hz}}$$

(*c*) For this simple pendulum we have:

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{9.81\,\text{m/s}^2}{2.0\,\text{m}}} = \boxed{0.35\,\text{Hz}}$$

86 •• A damped oscillator loses 3.50 percent of its energy during each cycle. (*a*) How many cycles elapse before half of its original energy is dissipated? (*b*) What is its *Q* factor? (*c*) If the natural frequency is 100 Hz, what is the width of the resonance curve when the oscillator is driven by a sinusoidal force?

Picture the Problem (*a*) We'll find a general expression for the damped oscillator's energy as a function of the number of cycles it has completed. We can then solve this equation for the number of cycles corresponding to the loss of half the oscillator's energy. (*b*) The *Q* factor is related to the fractional energy loss per cycle through $\Delta E/E = 2\pi/Q$. (*c*) The width of the resonance curve is $\Delta\omega = \omega_0/Q$ where ω_0 is the oscillator's natural angular frequency.

(*a*) Express the energy of the damped oscillator after one cycle:

$$E_1 = E_0\left(1 - \frac{\Delta E}{E}\right)$$

Express the energy after two cycles:

$$E_2 = E_1\left(1 - \frac{\Delta E}{E}\right) = E_0\left(1 - \frac{\Delta E}{E}\right)^2$$

Generalizing to *n* cycles:

$$E_n = E_0\left(1 - \frac{\Delta E}{E}\right)^n$$

Substituting numerical values yields:

$$0.50E_0 = E_0(1 - 0.035)^n$$

or

$$0.50 = (0.965)^n$$

Solving for *n* yields:

$$n = \frac{\ln 0.50}{\ln 0.965} = 19.5$$

$$\approx \boxed{20\text{ complete cycles.}}$$

(*b*) Apply the physical interpretation of Q for small damping to obtain:

$$Q = \frac{2\pi}{\Delta E/E} = \frac{2\pi}{0.0350} = \boxed{180}$$

(*c*) The width of the resonance curve is given by:

$$\Delta\omega = \frac{\omega_0}{Q} = \frac{2\pi f_0(\Delta E/E)}{2\pi} = f_0(\Delta E/E)$$

Substitute numerical values and evaluate $\Delta\omega$:

$$\Delta\omega = (100\,\text{Hz})(0.0350) = \boxed{3.50\,\text{rad/s}}$$

87 •• **[SSM]** A 2.00-kg object oscillates on a spring of force constant 400 N/m. The linear damping constant has a value of 2.00 kg/s. The system is driven by a sinusoidal force of maximum value 10.0 N and angular frequency 10.0 rad/s. (*a*) What is the amplitude of the oscillations? (*b*) If the driving frequency is varied, at what frequency will resonance occur? (*c*) What is the amplitude of oscillation at resonance? (*d*) What is the width of the resonance curve $\Delta\omega$?

Picture the Problem (*a*) The amplitude of the damped oscillations is related to the damping constant, mass of the system, the amplitude of the driving force, and the natural and driving frequencies through $A = \dfrac{F_0}{\sqrt{m^2(\omega_0^2-\omega^2)^2 + b^2\omega^2}}$.

(*b*) Resonance occurs when $\omega = \omega_0$. (*c*) At resonance, the amplitude of the oscillations is $A = F_0/\sqrt{b^2\omega^2}$. (*d*) The width of the resonance curve is related to the damping constant and the mass of the system according to $\Delta\omega = b/m$.

(*a*) Express the amplitude of the oscillations as a function of the driving frequency:

$$A = \frac{F_0}{\sqrt{m^2(\omega_0^2-\omega^2)^2 + b^2\omega^2}}$$

Because $\omega_0 = \sqrt{\dfrac{k}{m}}$:

$$A = \frac{F_0}{\sqrt{m^2\left(\dfrac{k}{m}-\omega^2\right)^2 + b^2\omega^2}}$$

Substitute numerical values and evaluate A:

$$A = \frac{10.0\text{ N}}{\sqrt{(2.00\text{ kg})^2\left(\dfrac{400\text{ N/m}}{2.00\text{ kg}} - (10.0\text{ rad/s})^2\right)^2 + (2.00\text{ kg/s})^2(10.0\text{ rad/s})^2}} = \boxed{4.98\text{ cm}}$$

(*b*) Resonance occurs when:

$$\omega = \omega_0 = \sqrt{\frac{k}{m}}$$

Substitute numerical values and evaluate ω:

$$\omega = \sqrt{\frac{400\text{ N/m}}{2.00\text{ kg}}} = 14.14\text{ rad/s}$$
$$= \boxed{14.1\text{rad/s}}$$

(*c*) The amplitude of the motion at resonance is given by:

$$A = \frac{F_0}{\sqrt{b^2\omega_0^2}}$$

Substitute numerical values and evaluate *A*:

$$A = \frac{10.0\text{ N}}{\sqrt{(2.00\text{ kg/s})^2(14.14\text{ rad/s})^2}}$$
$$= \boxed{35.4\text{ cm}}$$

(*d*) The width of the resonance curve is:

$$\Delta\omega = \frac{b}{m} = \frac{2.00\text{ kg/s}}{2.00\text{ kg}} = \boxed{1.00\text{ rad/s}}$$

88 •• Suppose you have the same apparatus described in Problem 74 and the same gold sphere hanging from a weaker spring that has a force constant of only 35.0 N/cm. You have studied the viscosity of ethylene glycol with this device, and found that ethylene glycol has a viscosity value of 19.9 mPa·s. Now you decide to drive this system with an external oscillating force. (*a*) If the magnitude of the driving force for the device is 0.110 N and the device is driven at resonance, how large would be the amplitude of the resulting oscillation? (*b*) If the system were not driven, but were allowed to oscillate, what percentage of its energy would it lose per cycle?

Picture the Problem (*a*) The amplitude of the steady-state oscillations when the system is in resonance is given by $A = F_0/b\omega$. (*b*) We can relate the fractional energy loss to the *Q* value of the oscillator.

(*a*) The amplitude of the steady-state oscillations when the system is in resonance is given by:

$$A = \frac{F_0}{b\omega}$$

Because $b = 6\pi a\eta$, and $\omega = \sqrt{k/m}$:

$$A = \frac{F_0}{6\pi a\eta\omega} = \frac{F_0}{6\pi a\eta}\sqrt{\frac{m}{k}}$$

Substituting $m = \rho V$ and simplifying yields:

$$A = \frac{F_0}{6\pi a\eta}\sqrt{\frac{\rho V}{k}} = \frac{F_0}{6\pi a\eta}\sqrt{\frac{\frac{4}{3}\pi a^3\rho}{k}} = \frac{F_0}{3\pi\eta}\sqrt{\frac{\pi a\rho}{3k}}$$

Substitute numerical values and evaluate A:

$$A = \frac{0.110\text{ N}}{3\pi(19.9\text{ mPa}\cdot\text{s})}\sqrt{\frac{\pi(0.0600\text{ m})(19.3\times10^3\text{ kg/m}^3)}{3(35.0\text{ N/cm})}} = \boxed{34.5\text{ cm}}$$

(*b*) This is a very weakly damped system and so we can relate the fractional energy loss per cycle to the system's Q value:

$$Q = \frac{2\pi}{(\Delta E/E)_{\text{cycle}}} = \omega_0\tau$$

Because $\tau = \frac{m}{b} = \frac{m}{6\pi a\eta}$:

$$\frac{2\pi}{(\Delta E/E)_{\text{cycle}}} = \frac{m\omega_0}{6\pi a\eta}$$

Substituting for m and ω_0 and simplifying yields:

$$\frac{2\pi}{(\Delta E/E)_{\text{cycle}}} = \frac{\rho V\sqrt{\frac{k}{m}}}{6\pi a\eta} = \frac{\frac{4}{3}\pi a^3\rho}{6\pi a\eta}\sqrt{\frac{k}{m}} = \frac{2a^2\rho}{9\eta}\sqrt{\frac{k}{m}}$$

Solve for $(\Delta E/E)_{\text{cycle}}$ to obtain:

$$(\Delta E/E)_{\text{cycle}} = \frac{9\pi\eta}{a^2\rho}\sqrt{\frac{m}{k}}$$

Substitute numerical values and evaluate $(\Delta E/E)_{\text{cycle}}$:

$$(\Delta E/E)_{\text{cycle}} = \frac{9\pi(19.9\text{ mPa}\cdot\text{s})}{(0.0600\text{ m})^2(19.3\times10^3\text{ kg/m}^3)}\sqrt{\frac{17.5\text{ kg}}{35.0\text{ N/cm}}} = \boxed{5.73\times10^{-4}}$$

General Problems

89 • A particle's displacement from equilibrium is given by $x(t) = 0.40\cos(3.0t + \pi/4)$, where x is in meters and t is in seconds. (*a*) Find the frequency and period of its motion. (*b*) Find an expression for the speed of the particle as a function of time. (*c*) What is its maximum speed?

Picture the Problem (*a*) The particle's displacement is of the form $x = A\cos(\omega t + \delta)$. Thus, we have A = 0.40 m, ω = 3.0 rad/s, and $\delta = \pi/4$. We can find the frequency of the motion from its angular frequency and the period from the frequency. (*b*) The particle's velocity is the time derivative of its displacement. (*c*) The particle's maximum speed occurs when $\sin(\omega t + \delta) = -1$.

(*a*) The particle's displacement from equilibrium is of the form

$$x = A\cos(\omega t + \delta).$$

Comparing this to the given equation we see that:

$$\omega = 3.0\text{ rad/s}$$

and so

$$f = \frac{\omega}{2\pi} = \frac{3.0\text{ rad/s}}{2\pi} = 0.477\text{ Hz} = \boxed{0.48\text{ Hz}}$$

The period of the particle's motion is the reciprocal of its frequency:

$$T = \frac{1}{f} = \frac{1}{0.477\text{ s}^{-1}} = 2.09\text{ s} = \boxed{2.1\text{ s}}$$

(*b*) Differentiate $x = A\cos(\omega t + \delta)$ with respect to time to obtain an expression for the particle's velocity:

$$v_x = \frac{dx}{dt} = \frac{d}{dt}[A\cos(\omega t + \delta)] = -\omega A\sin(\omega t + \delta)$$

Substituting for A, ω, and δ yields:

$$v_x = -(3.0\text{ rad/s})(0.40\text{ m})\sin\left[(3.0\text{ rad/s})t + \frac{\pi}{4}\right] = \boxed{-(1.2\text{ m/s})\sin\left[(3.0\text{ rad/s})t + \frac{\pi}{4}\right]}$$

(*c*) The particle's maximum speed occurs when $\sin(\omega t + \delta) = -1$:

$$v_{x\,\max} = -(1.2\text{ m/s})(-1) = \boxed{1.2\text{ m/s}}$$

90 • An astronaut arrives at a new planet, and gets out his simple device to determine the gravitational acceleration there. Prior to his arrival, he noted that the radius of the planet was 7550 km. If his 0.500-m-long pendulum has a period of 1.0 s, what is the mass of the planet?

Picture the Problem We can apply Newton's 2nd law and the law of gravity to an object at the surface of the new planet to obtain an expression for the mass of the planet as a function of the acceleration due to gravity at its surface. We can use the period of the astronaut's pendulum to obtain an expression for the acceleration of gravity a_g at the surface of the new planet.

Apply Newton's 2nd law and the law of gravity to an object of mass m at the surface of the planet:

$$\frac{GM_{\text{planet}}m}{R_{\text{planet}}^2} = ma_g \Rightarrow M_{\text{planet}} = \frac{a_g R_{\text{planet}}^2}{G}$$

The period of the astronaut's simple pendulum is related to the gravitational field a_g at the surface of the new planet:

$$T = 2\pi\sqrt{\frac{L}{a_g}} \Rightarrow a_g = \frac{4\pi^2 L}{T^2}$$

Substituting for a_g and simplifying yields:

$$M_{\text{planet}} = \frac{4\pi^2 R_{\text{planet}}^2 L}{GT^2}$$

Substitute numerical values and evaluate M_{planet}:

$$M_{\text{planet}} = \frac{4\pi^2(7550\text{ km})^2(0.500\text{ m})}{(6.67\times10^{-11}\text{ N}\cdot\text{m}^2/\text{kg}^2)(1.0\text{ s})^2} = \boxed{1.7\times10^{25}\text{ kg}}$$

91 •• A pendulum clock keeps perfect time on Earth's surface. In which case will the error be greater: if the clock is placed in a mine of depth h or if the clock is elevated to a height h? Prove your answer and assume $h << R_E$.

Picture the Problem Assume that the density of Earth ρ is constant and let m represent the mass of the clock. We can decide the question of where the clock is more accurate by applying the law of gravitation to the clock at a depth h below/above the surface of Earth and at Earth's surface and expressing the ratios of the acceleration due to gravity below/above the surface of Earth to its value at the surface of Earth.

Express the gravitational force acting on the clock when it is at a depth h in a mine:

$$mg' = \frac{GM'm}{(R_E - h)^2}$$

where M' is the mass between the location of the clock and the center of Earth.

Express the gravitational force acting on the clock at the surface of Earth:

$$mg = \frac{GM_E m}{R_E^2}$$

Divide the first of these equations by the second and simplify to obtain:

$$\frac{g'}{g} = \frac{\dfrac{GM'}{(R_E - h)^2}}{\dfrac{GM_E}{R_E^2}} = \frac{M'}{M_E}\frac{R_E^2}{(R_E - h)^2}$$

Express M':

$$M' = \rho V' = \tfrac{4}{3}\pi\rho(R_E - h)^3$$

Express M_E:

$$M_E = \rho V = \tfrac{4}{3}\pi\rho R_E^3$$

Substitute for M' and M_E to obtain:

$$\frac{g'}{g}=\frac{\frac{4}{3}\pi\rho(R_E-h)^3}{\frac{4}{3}\pi\rho R_E^3}\frac{R_E^2}{(R_E-h)^2}$$

Simplifying and solving for g' yields:

$$g'=g\left(\frac{R_E-h}{R_E}\right)=g\left(1-\frac{h}{R_E}\right)$$

or

$$g'=g\left(1-\frac{h}{R_E}\right) \qquad (1)$$

Express the gravitational force acting on the clock when it is at an elevation h:

$$mg''=\frac{GM_Em}{(R_E+h)^2}$$

Express the gravitational force acting on the clock at the surface of Earth:

$$mg=\frac{GM_Em}{R_E^2}$$

Divide the first of these equations by the second and simplify to obtain:

$$\frac{g''}{g}=\frac{\frac{GM_E}{(R_E+h)^2}}{\frac{GM_E}{R_E^2}}=\frac{R_E^2}{(R_E-h)^2}$$

Factoring R_E^2 from the denominator yields:

$$\frac{g''}{g}=\frac{1}{\left(1+\frac{h}{R_E}\right)^2}$$

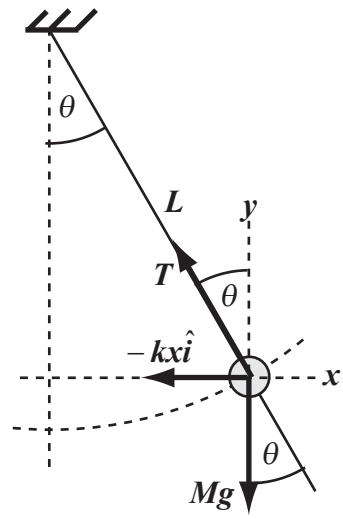
Solve for g'' to obtain:

$$g''=g\left(1+\frac{h}{R_E}\right)^{-2} \qquad (2)$$

Comparing equations (1) and (2), we see that g' is closer to g than is g''. Thus the error is greater if the clock is elevated.

92 •• Figure 14-34 shows a pendulum of length L with a bob of mass M. The bob is attached to a spring that has a force constant k. When the bob is directly below the pendulum support, the spring is unstressed. (*a*) Derive an expression for the period of this oscillating system for small-amplitude vibrations. (*b*) Suppose that M = 1.00 kg and L is such that in the absence of the spring the period is 2.00 s. What is the force constant k if the period of the oscillating system is 1.00 s?

Picture the Problem The figure shows this system when it has an angular displacement θ. The period of the system is related to its angular frequency according to $T = 2\pi/\omega$. We can find the equation of motion of the system by applying Newton's 2nd law. By writing this equation in terms of θ and using a small-angle approximation, we'll find an expression for ω that we can use to express T.



(*a*) The period of the system in terms of its angular frequency is given by:

$$T = \frac{2\pi}{\omega} \qquad (1)$$

Apply $\sum \vec{F} = m\vec{a}$ to the bob:

$$\sum F_x = -kx - T\sin\theta = Ma_x$$

and

$$\sum F_y = T\cos\theta - Mg = 0$$

Eliminate T between the two equations to obtain:

$$-kx - Mg\tan\theta = Ma_x$$

Noting that $x = L\theta$ and $a_x = L\alpha = L\dfrac{d^2\theta}{dt^2}$, eliminate the variable x in favor of θ:

$$ML\frac{d^2\theta}{dt^2} = -kL\theta - Mg\tan\theta$$

For $\theta << 1$, $\tan\theta \approx \theta$:

$$ML\frac{d^2\theta}{dt^2} = -kL\theta - Mg\theta$$

$$= -(kL + Mg)\theta$$

or

$$\frac{d^2\theta}{dt^2} = -\left(\frac{k}{M} + \frac{g}{L}\right)\theta = -\omega^2\theta$$

where $\omega = \sqrt{\dfrac{k}{M} + \dfrac{g}{L}}$

Substitute in equation (1) to obtain:

$$T = \boxed{\frac{2\pi}{\sqrt{\frac{k}{M} + \frac{g}{L}}}}$$

(*b*) When $k = 0$ (no spring), $T = 2.00$ s, and $M = 1.00$ kg we have:

$$2.00\text{ s} = \frac{2\pi}{\sqrt{\frac{g}{L}}} \qquad (2)$$

With the spring present and $T = 1.00$ s we have:

$$1.00\text{ s} = \frac{2\pi}{\sqrt{k\text{ kg}^{-1} + \frac{g}{L}}} \qquad (3)$$

Solving equations (2) and (3) simultaneously yields:

$$k = \boxed{29.6\text{ N/m}}$$

93 •• **[SSM]** A block that has a mass equal to m_1 is supported from below by a frictionless horizontal surface. The block, which is attached to the end of a horizontal spring with a force constant *k,* oscillates with an amplitude *A*. When the spring is at its greatest extension and the block is instantaneously at rest, a second block of mass m_2 is placed on top of it. (*a*) What is the smallest value for the coefficient of static friction μ_s such that the second object does not slip on the first? (*b*) Explain how the total mechanical energy *E*, the amplitude *A*, the angular frequency ω, and the period *T* of the system are affected by the placing of m_2 on m_1, assuming that the coefficient of friction is great enough to prevent slippage.

Picture the Problem Applying Newton's 2nd law to the first object as it is about to slip will allow us to express μ_s in terms of the maximum acceleration of the system which, in turn, depends on the amplitude and angular frequency of the oscillatory motion.

(*a*) Apply $\sum F_x = ma_x$ to the second object as it is about to slip:

$$f_{s,\max} = m_2 a_{\max}$$

Apply $\sum F_y = 0$ to the second object:

$$F_n - m_2 g = 0$$

Use $f_{s,\max} = \mu_s F_n$ to eliminate $f_{s,\max}$ and F_n between the two equations and solve for μ_s:

$$\mu_s m_2 g = m_2 a_{\max} \Rightarrow \mu_s = \frac{a_{\max}}{g}$$

Relate the maximum acceleration of the oscillator to its amplitude and angular frequency and substitute for ω^2:

$$a_{\max} = A\omega^2 = A\frac{k}{m_1 + m_2}$$

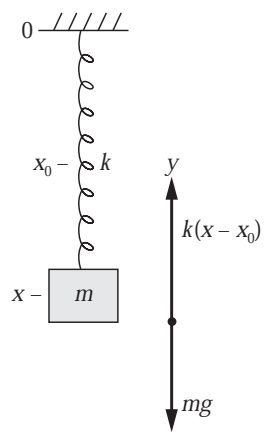
Finally, substitute for $a_{\max}$ to obtain:

$$\mu_s = \boxed{\frac{Ak}{(m_1 + m_2)g}}$$

(*b*) A is unchanged. E is unchanged because $E = \frac{1}{2}kA^2$. ω is reduced and T is increased by increasing the total mass of the system.

94 •• A 100-kg box hangs from the ceiling of a room–suspended from a spring with a force constant of 500 N/m. The unstressed length of the spring is 0.500 m. (*a*) Find the equilibrium position of the box. (*b*) An identical spring is stretched and attached to the ceiling and box and is parallel with the first spring. Find the frequency of the oscillations when the box is released. (*c*) What is the new equilibrium position of the box once it comes to rest?

Picture the Problem The diagram shows the box hanging from the stretched spring and the free-body diagram when the box is in equilibrium. We can apply $\sum F_y = 0$ to the box to derive an expression for x. In (*b*) and (*c*), we can proceed similarly to obtain expressions for the effective force constant, the new equilibrium position of the box, and frequency of oscillations when the box is released.



(*a*) Apply $\sum F_y = 0$ to the box to obtain:

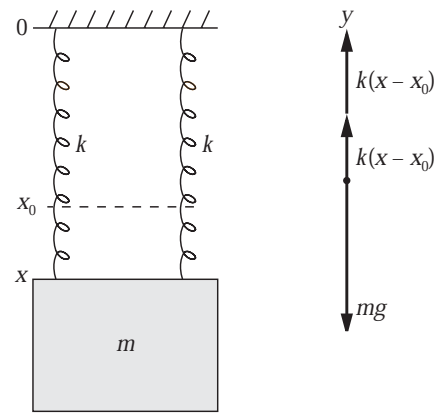
$$k(x - x_0) - mg = 0 \Rightarrow x = \frac{mg}{k} + x_0$$

Substitute numerical values and evaluate x:

$$\boldsymbol{x} = \frac{(100\,\text{kg})(9.81\,\text{m/s}^2)}{500\,\text{N/m}} + 0.500\,\text{m}$$

$$= \boxed{2.46\,\text{m}}$$

(*b*) Draw the free-body diagram for the block with the two springs exerting equal upward forces on it:



Apply $\sum F_y = 0$ to the box to obtain:

$$k(x-x_0)+k(x-x_0)-mg=0$$

or

$$k_{\text{eff}}(x-x_0)-mg=0 \qquad (1)$$

where $k_{\text{eff}} = 2k$

When the box is displaced from this equilibrium position and released, its motion is simple harmonic motion and its frequency is given by:

$$\omega = \sqrt{\frac{k_{\text{eff}}}{m}} = \sqrt{\frac{2k}{m}}$$

Substitute numerical values and evaluate ω:

$$\boldsymbol{\omega} = \sqrt{\frac{2(500\,\text{N/m})}{100\,\text{kg}}} = \boxed{3.16\,\text{rad/s}}$$

(*c*) Solve equation (1) for *x* to obtain:

$$x = \frac{mg}{2k} + x_0$$

Substitute numerical values and evaluate *x*:

$$\boldsymbol{x} = \frac{(100\,\text{kg})(9.81\,\text{m/s}^2)}{2(500\,\text{N/m})} + 0.500\,\text{m}$$

$$= \boxed{1.48\,\text{m}}$$

95 •• The acceleration due to gravity *g* varies with geographical location because of Earth's rotation and because Earth is not exactly spherical. This was first discovered in the seventeenth century, when it was noted that a pendulum clock carefully adjusted to keep correct time in Paris lost about 90 s/d near the equator. (*a*) Show by using the differential approximation that a small change in the acceleration of gravity Δg produces a small change in the period ΔT of a pendulum given by $\Delta T / T \approx -\frac{1}{2}\Delta g / g$. (*b*) How large a change in *g* is needed to account for a 90 s/d change in the period?

Picture the Problem We'll differentiate the expression for the period of simple pendulum $T = 2\pi\sqrt{\frac{L}{g}}$ with respect to g, separate the variables, and use a differential approximation to establish that $\frac{\Delta T}{T} \approx -\frac{1}{2}\frac{\Delta g}{g}$.

(*a*) Express the period of a simple pendulum in terms of its length and the local value of the acceleration due to gravity:

$$T = 2\pi\sqrt{\frac{L}{g}}$$

Differentiate this expression with respect to g to obtain:

$$\frac{dT}{dg} = \frac{d}{dg}\left[2\pi\sqrt{L}g^{-1/2}\right] = -\pi\sqrt{L}g^{-3/2} = -\frac{T}{2g}$$

Separate the variables to obtain:

$$\frac{dT}{T} = -\frac{1}{2}\frac{dg}{g}$$

For $\Delta g << g$ we can approximate dT and dg by ΔT and Δg:

$$\frac{\Delta T}{T} \approx \boxed{-\frac{1}{2}\frac{\Delta g}{g}}$$

(*b*) Solve the equation in Part (*a*) for Δg:

$$\Delta g = -2g\frac{\Delta T}{T}$$

Substitute numerical values and evaluate Δg for a 90 s/d change in the period:

$$\Delta g = -2\left(9.81\,\text{m/s}^2\right)\left(-90\frac{\text{s}}{\text{d}}\times\frac{1\,\text{d}}{24\,\text{h}}\times\frac{1\,\text{h}}{3600\,\text{s}}\right) = \boxed{2.0\,\text{cm/s}^2}$$

96 •• A small block that has a mass equal to m_1 rests on a piston that is vibrating vertically with simple harmonic motion described by the formula $y = A\sin\omega t$. (*a*) Show that the block will leave the piston if $\omega^2 A > g$. (*b*) If $\omega^2 A = 3g$ and $A = 15$ cm, at what time will the block leave the piston?

Picture the Problem If the displacement of the block is $y = A\sin\omega t$, its acceleration is $a = -\omega^2 A\sin\omega t$.

(*a*) At maximum upward extension, the block is momentarily at rest. Its downward acceleration is g. The downward acceleration of the piston is $\omega^2 A$. Therefore, if $\omega^2 A > g$, the block will separate from the piston.

(*b*) Express the acceleration of the small block:

$$a = -A\omega^2 \sin \omega t$$

For $\omega^2 A = 3g$ and A = 15 cm:

$$a = -3g \sin \omega t = -g$$

Solving for t yields:

$$t = \frac{1}{\omega}\sin^{-1}\left(\frac{1}{3}\right) = \sqrt{\frac{A}{3g}}\sin^{-1}\left(\frac{1}{3}\right)$$

Substitute numerical values and evaluate t:

$$t = \sqrt{\frac{0.15\,\text{m}}{3\left(9.81\,\text{m/s}^2\right)}}\sin^{-1}\left(\frac{1}{3}\right) = \boxed{24\,\text{ms}}$$

97 •• **[SSM]** Show that for the situations in Figure 14-35*a* and Figure 14-35*b* the object oscillates with a frequency $f = (1/2\pi)\sqrt{k_{\text{eff}}/m}$, where k_{eff} is given by (*a*) $k_{\text{eff}} = k_1 + k_2$, and (*b*) $1/k_{\text{eff}} = 1/k_1 + 1/k_2$. *Hint: Find the magnitude of the net force F on the object for a small displacement x and write* $F = -k_{\text{eff}}x$. *Note that in Part* (*b*) *the springs stretch by different amounts, the sum of which is x.*

Picture the Problem Choose a coordinate system in which the $+x$ direction is to the right and assume that the object is displaced to the right. In case (*a*), note that the two springs undergo the same displacement whereas in (*b*) they experience the same force.

(*a*) Express the net force acting on the object:

$$F_{\text{net}} = -k_1 x - k_2 x = -(k_1 + k_2)x = -k_{\text{eff}}x$$

where $k_{\text{eff}} = \boxed{k_1 + k_2}$

(*b*) Express the force acting on each spring and solve for x_2:

$$F = -k_1 x_1 = -k_2 x_2 \Rightarrow x_2 = \frac{k_1}{k_2}x_1$$

Express the total extension of the springs:

$$x_1 + x_2 = -\frac{F}{k_{\text{eff}}}$$

Solving for k_{eff} yields:

$$k_{\text{eff}} = -\frac{F}{x_1 + x_2} = -\frac{-k_1 x_1}{x_1 + x_2} = \frac{k_1 x_1}{x_1 + \frac{k_1}{k_2}x_1} = \frac{1}{\frac{1}{k_1} + \frac{1}{k_2}}$$

Take the reciprocal of both sides of the equation to obtain:

$$\frac{1}{k_{\text{eff}}} = \boxed{\frac{1}{k_1} + \frac{1}{k_2}}$$

98 •• During an earthquake, a floor oscillates horizontally in approximately simple harmonic motion. Assume it oscillates at a single frequency with a period of 0.80 s. (*a*) After the earthquake, you are in charge of examining the video of the floor motion and discover that a box on the floor started to slip when the amplitude reached 10 cm. From your data, determine the coefficient of static friction between the box and the floor. (*b*) If the coefficient of friction between the box and floor were 0.40, what would be the maximum amplitude of vibration before the box would slip?

Picture the Problem Applying Newton's 2nd law to the box as it is about to slip will allow us to express μ_s in terms of the maximum acceleration of the platform which, in turn, depends on the amplitude and angular frequency of the oscillatory motion.

(*a*) Apply $\sum F_x = ma_x$ to the box as it is about to slip:

$$f_{s,max} = ma_{max}$$

Apply $\sum F_y = 0$ to the box:

$$F_n - mg = 0$$

Use $f_{s,max} = \mu_s F_n$ to eliminate $f_{s,max}$ and F_n between the two equations:

$$\mu_s mg = ma_{max} \text{ and } \mu_s = \frac{a_{max}}{g}$$

Relate the maximum acceleration of the oscillator to its amplitude and angular frequency:

$$a_{max} = A\omega^2$$

Substitute for a_{max} in the expression for μ_s:

$$\mu_s = \frac{A\omega^2}{g} = \frac{4\pi^2 A}{T^2 g}$$

Substitute numerical values and evaluate μ_s:

$$\mu_s = \frac{4\pi^2(0.10\,\text{m})}{(0.80\,\text{s})^2(9.81\,\text{m/s}^2)} = \boxed{0.63}$$

(*b*) Solve the equation derived above for A_{max}:

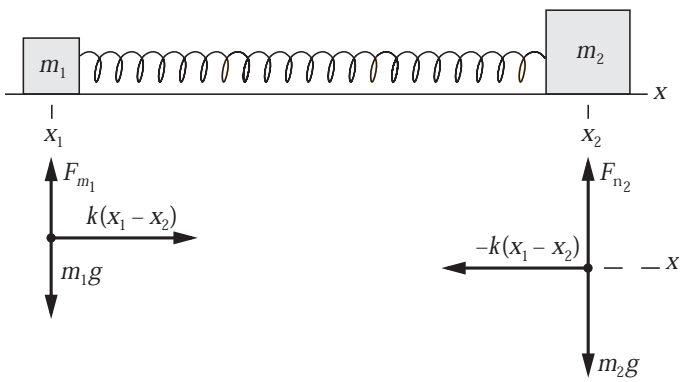
$$A_{max} = \frac{\mu_s g}{\omega^2} = \frac{\mu_s g T^2}{4\pi^2}$$

Substitute numerical values and evaluate A_{max}:

$$A_{max} = \frac{(0.40)(9.81\,\text{m/s}^2)(0.80\,\text{s})^2}{4\pi^2} = \boxed{6.4\,\text{cm}}$$

99 •• If we attach two blocks of masses m_1 and m_2 to either end of a spring of force constant k and set them into oscillation by releasing them from rest with the spring stretched, show that the oscillation frequency is given by $\omega = (k/\mu)^{1/2}$, where $\mu = m_1m_2/(m_1 + m_2)$ is the reduced mass of the system.

Picture the Problem The pictorial representation shows the two blocks connected by the spring and displaced from their equilibrium positions. We can apply Newton's 2nd law to each of these coupled oscillators and solve the resulting equations simultaneously to obtain the equation of motion of the coupled oscillators. We can then compare this equation and its solution to the equation of motion of the simple harmonic oscillator and its solution to show that the oscillation frequency is $\omega = (k/\mu)^{1/2}$ where $\mu = m_1m_2/(m_1 + m_2)$.



Apply $\sum \vec{F} = m\vec{a}$ to the block whose mass is m_1 and solve for its acceleration:

$$k(x_1 - x_2) = m_1a_1 = m_1\frac{d^2x_1}{dt^2}$$

or

$$a_1 = \frac{d^2x_1}{dt^2} = \frac{k}{m_1}(x_1 - x_2)$$

Apply $\sum \vec{F} = m\vec{a}$ to the block whose mass is m_2 and solve for its acceleration:

$$-k(x_1 - x_2) = m_2a_2 = m_1\frac{d^2x_2}{dt^2}$$

or

$$a_2 = \frac{d^2x_2}{dt^2} = \frac{k}{m_2}(x_2 - x_1)$$

Subtract the first equation from the second to obtain:

$$\frac{d^2(x_2 - x_1)}{dt^2} = \frac{d^2x}{dt^2} = -k\left(\frac{1}{m_1} + \frac{1}{m_2}\right)x$$

where $x = x_2 - x_1$

The reduced mass of the system is:

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} \Rightarrow \mu = \frac{m_1m_2}{m_1 + m_2}$$

Substitute to obtain:

$$\frac{d^2x}{dt^2} = -\frac{k}{\mu}x \qquad (1)$$

Compare this equation to the equation of the simple harmonic oscillator:

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

The solution to this equation is:

$$x = x_0\cos(\omega t + \delta)$$

$$\text{where } \omega = \sqrt{\frac{k}{m}}$$

Because of the similarity of the two differential equations, the solution to equation (1) must be:

$$x = x_0\cos(\omega t + \delta)$$

$$\text{where } \omega = \boxed{\sqrt{\frac{k}{\mu}}} \text{ and } \mu = \boxed{\frac{m_1m_2}{m_1 + m_2}}$$

100 •• In one of your chemistry labs you determine that one of the vibrational modes of the HCl molecule has a frequency of 8.969×10^{13} Hz. Using the result of Problem 99, find the effective "spring constant" between the H atom and the Cl atom in the HCl molecule.

Picture the Problem We can use $\omega = (k/\mu)^{1/2}$ and $\mu = m_1m_2/(m_1 + m_2)$ from Problem 99 to find the spring constant for the HCl molecule.

Use the result of Problem 99 to relate the oscillation frequency to the force constant and reduced mass of the HCl molecule:

$$\omega = \sqrt{\frac{k}{\mu}} \Rightarrow k = \mu\omega^2$$

Express the reduced mass of the HCl molecule:

$$\mu = \frac{m_1m_2}{m_1 + m_2}$$

Substitute for μ to obtain:

$$k = \frac{m_1m_2\omega^2}{m_1 + m_2}$$

Express the masses of the hydrogen and Cl atoms:

$m_1 = 1$ amu $= 1.67\times10^{-27}$ kg
and
$m_2 = 35.45$ amu $= 5.92\times10^{-26}$ kg

Substitute numerical values and evaluate k:

$$\boldsymbol{k} = \frac{(1.673\times10^{-27}\text{ kg})(5.92\times10^{-26}\text{ kg})(8.969\times10^{13}\text{ s}^{-1})^2}{1.673\times10^{-27}\text{ kg} + 5.92\times10^{-26}\text{ kg}} = \boxed{13.1\text{ N/m}}$$

101 •• If a hydrogen atom in HCl were replaced by a deuterium atom (forming DCl) in Problem 100, what would be the new vibration frequency of the molecule? Deuterium consists of 1 proton and 1 neutron.

Picture the Problem In Problem 100, we derived an expression for the oscillation frequency of a spring-and-two-block system as a function of the force constant of the spring and the reduced mass of the two blocks. We can solve this problem, assuming that the "spring constant" does not change, by using the result of Problem 101 and the reduced mass of a deuterium atom and a Cl atom in the equation for the oscillation frequency.

Use the result of Problem 100 to relate the oscillation frequency to the force constant and reduced mass of the DCl molecule:

$$\omega = \sqrt{\frac{k}{\mu}}$$

Express the reduced mass of the DCl molecule:

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

The masses of the deuterium and Cl atoms are:

$m_1 = 2$ amu $= 3.34\times10^{-27}$ kg
and
$m_2 = 35.45$ amu $= 5.92\times10^{-26}$ kg

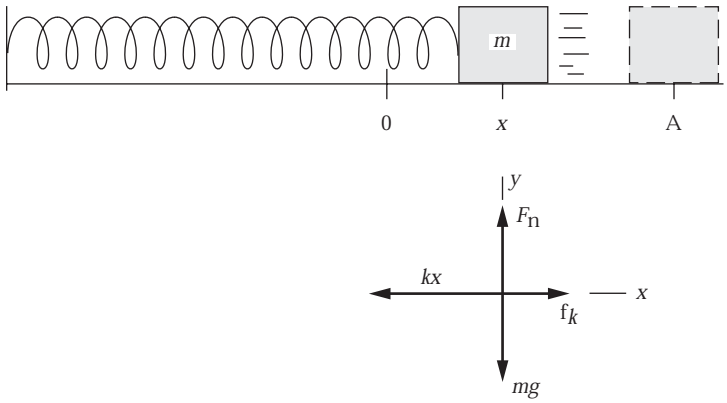
Substitute numerical values and evaluate ω:

$$\omega = \sqrt{\frac{13.1\,\text{N/m}}{\dfrac{(3.34\times10^{-27}\,\text{kg})(5.92\times10^{-26}\,\text{kg})}{3.34\times10^{-27}\,\text{kg}+5.92\times10^{-26}\,\text{kg}}}} = \boxed{6.44\times10^{13}\,\text{rad/s}}$$

102 ••• A block of mass m on a horizontal table is attached to a spring of force constant k, as shown in Figure 14-36. The coefficient of kinetic friction between the block and the table is μ_k. The spring is unstressed if the block is at the origin ($x = 0$), and the $+x$ direction is to the right. The spring is stretched a distance A, where $kA > \mu_k mg$, and the block is released. (*a*) Apply Newton's second law to the block to obtain an equation for its acceleration d^2x/dt^2 for the first half-cycle, during which the block is moving to the left. Show that the resulting equation can be written as $d^2x'/dt^2 = -\omega^2 x'$, where $\omega = \sqrt{k/m}$ and $x' = x - x_0$, with $x_0 = \mu_k mg/k = \mu_k g/\omega^2$. (*b*) Repeat Part (*a*) for the second half-cycle as the block moves to the right, and show that $d^2x''/dt^2 = -\omega^2 x''$, where $x'' = x + x_0$ and x_0 has the same value. (*c*) Use a **spreadsheet program** to graph the first 5 half-cycles for $A = 10x_0$. Describe the motion, if any, after the fifth half-cycle.

Picture the Problem The pictorial representation shows the block moving from right to left with an instantaneous displacement x from its equilibrium position.

The free-body diagram shows the forces acting on the block during the half-cycles that it moves from right to left. When the block is moving from left to right, the directions of the kinetic friction force and the restoring force exerted by the spring are reversed. We can apply Newton's 2nd law to these motions to obtain the equations given in the problem statements and then use their solutions to plot the graph called for in (*c*).



(*a*) Apply $\sum F_x = ma_x$ to the block while it is moving to the left to obtain:

$$f_k - kx = m\frac{d^2x}{dt^2}$$

Using $f_k = \mu_k F_n = \mu_k mg$, eliminate f_k in the equation of motion:

$$m\frac{d^2x}{dt^2} = -kx + \mu_k mg$$

or

$$m\frac{d^2x}{dt^2} = -k\left(x - \frac{\mu_k mg}{k}\right)$$

Let $x_0 = \frac{\mu_k mg}{k}$ to obtain:

$$m\frac{d^2x}{dt^2} = -k(x - x_0)$$

or

$$\boxed{\frac{d^2x'}{dt^2} = -\frac{k}{m}x' = -\omega^2 x'}$$

provided $x' = x - x_0$ and

$$x_0 = \boxed{\frac{\mu_k mg}{k} = \frac{\mu_k g}{\omega^2}}$$

The solution to the equation of motion is:

$$x' = x_0'\cos(\omega t + \delta)$$

and its derivative is

$$v' = -\omega x_0'\sin(\omega t + \delta)$$

The initial conditions are:

$$x'(0) = x - x_0 \text{ and } v'(0) = 0$$

Apply these conditions to obtain:

$$x'(0) = x_0' \cos\delta = x - x_0$$

and

$$v'(0) = -\omega x_0' \sin\delta = 0$$

Solve these equations simultaneously to obtain:

$$\delta = 0 \text{ and } x_0' = x - x_0$$

and

$$x' = (x - x_0)\cos\omega t$$

or

$$x = \boxed{(x - x_0)\cos\omega t + x_0} \qquad (1)$$

(*b*) Apply $\sum \vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ to the block while it is moving to the right to obtain:

$$-f_k - kx = m\frac{d^2x}{dt^2}$$

Using $f_k = \mu_k F_n = \mu_k mg$, eliminate f_k in the equation of motion:

$$m\frac{d^2x}{dt^2} = -kx - \mu_k mg$$

or

$$m\frac{d^2x}{dt^2} = -k\left(x + \frac{\mu_k mg}{k}\right)$$

Let $x_0 = \dfrac{\mu_k mg}{k}$ to obtain:

$$m\frac{d^2x}{dt^2} = -k(x + x_0)$$

or

$$\boxed{\frac{d^2x''}{dt^2} = -\frac{k}{m}x'' = -\omega^2 x''}$$

provided $x'' = x + x_0$ and

$$x_0 = \boxed{\frac{\mu_k mg}{k} = \frac{\mu_k g}{\omega^2}}.$$

The solution to the equation of motion is:

$$x'' = x_0'' \cos(\omega t + \delta)$$

and its derivative is

$$v'' = -\omega x_0'' \sin(\omega t + \delta)$$

The initial conditions are:

$$x''(0) = x + x_0 \text{ and } v''(0) = 0$$

Apply these conditions to obtain:

$$x''(0) = x_0'' \cos\delta = x + x_0$$

and

$$v''(0) = -\omega x_0'' \sin\delta = 0$$

Solve these equations simultaneously to obtain:

$\delta = 0$ and $x_0'' = x + x_0$

and

$x'' = (x + x_0)\cos\omega t$

or

$$\boxed{x = (x + x_0)\cos\omega t - x_0} \qquad (2)$$

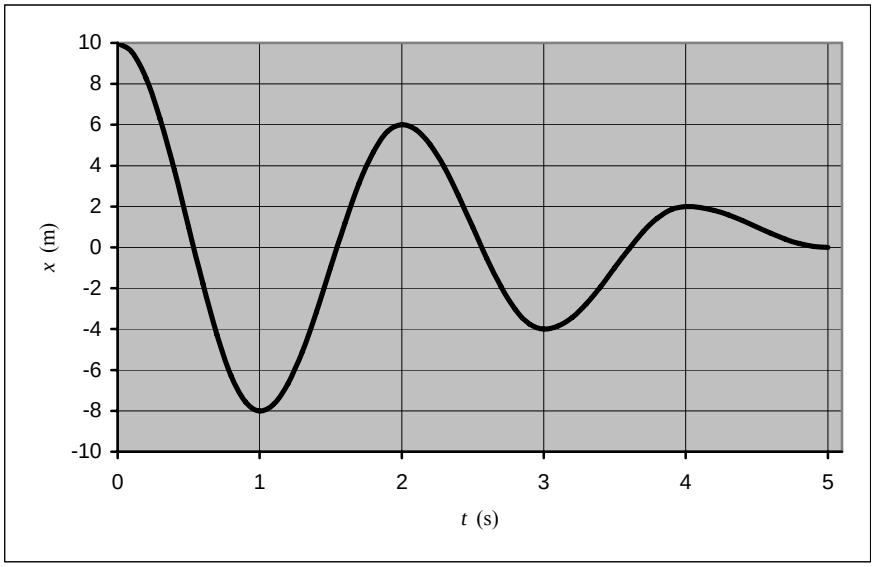
(*c*) A spreadsheet program to calculate the position of the oscillator as a function of time (equations (1) and (2)) is shown below. The constants used in the position functions (x_0 = 1 m and T = 2 s were used for simplicity) and the formulas used to calculate the positions are shown in the table. *After each half-period, one must compute a new amplitude for the oscillation, using the final value of the position from the last half-period.*

Cell	Content/Formula	Algebraic Form
B1	1	x_0
B2	10	A
C7	C6 + 0.1	$t + \Delta t$
D7	($B$2–$B$1)*COS(PI()*C7)+$B$1	$(A - x_0)\cos\pi t + x_0$
D17	(ABS($D$6+$B$1))*COS(PI()*C17)–$B$1	$\lvert x + x_0\rvert\cos\pi t - x_0$
D27	(ABS($D$6–$B$1))*COS(PI()*C27)+$B$1	$\lvert x - x_0\rvert\cos\pi t + x_0$
D37	(ABS($D$36+$B$1))*COS(PI()*C37)–$B$1	$\lvert x + x_0\rvert\cos\pi t - x_0$
D47	($D$46–$B$1)*COS(PI()*C47)+$B$1	$(x - x_0)\cos\pi t + x_0$

	A	B	C	D
1	x_0=	1	m	
2	A=	10		
3				
4			t	x
5			(s)	(m)
6			0.0	10.00
7			0.1	9.56
8			0.2	8.28
9			0.3	6.29
10			0.4	3.78
53			4.7	0.41
54			4.8	0.19
55			4.9	0.05
56			5.0	0.00

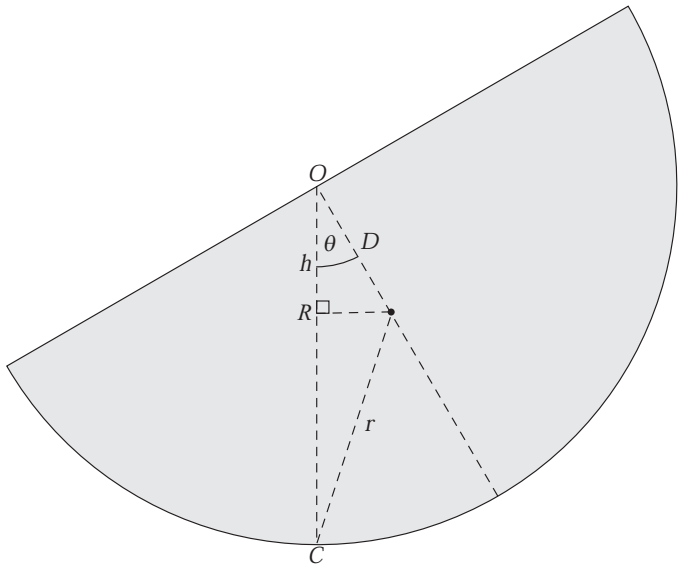
The following graph was plotted using the data from columns C (t) and D (x).

Note that the motion of the block ceases after five half-cycles.



103 ••• Figure 14-37 shows a uniform solid half-cylinder of mass M and radius R resting on a horizontal surface. If one side of this cylinder is pushed down slightly and then released, the half-cylinder will oscillate about its equilibrium position. Determine the period of this oscillation.

Picture the Problem The diagram shows the half-cylinder displaced from its equilibrium position through an angle θ. The frequency of its motion will be found by expressing the mechanical energy E in terms of θ and $d\theta/dt$. For small θ we will obtain an equation of the form $E = \frac{1}{2}\kappa\theta^2 + \frac{1}{2}I\left(\frac{d\theta}{dt}\right)^2$. Differentiating both sides of this equation with respect to time will lead to $0 = \left(\kappa\theta + I\frac{d^2\theta}{dt^2}\right)\frac{d\theta}{dt}$, an equation that must be valid at all times. Because the situation of interest to us requires that $d\theta/dt$ is not always equal to zero, we have $0 = \kappa\theta + I\frac{d^2\theta}{dt^2}$ or $\frac{d^2\theta}{dt^2} + \frac{\kappa}{I}\theta = 0$, the equation of simple harmonic motion with $\omega^2 = \kappa/I$. We'll show that the distance from O to the center of mass D, is given by $D = \frac{4R}{3\pi}$, and let the distance from the contact point C to the center of mass be r. Finally, we'll take the potential energy to be zero where θ is zero and assume that there is no slipping.



Apply conservation of energy to obtain:

$$E = U + K$$
$$= Mg(h - D) + \frac{1}{2} I_C \left(\frac{d\theta}{dt}\right)^2 \quad (1)$$

From Table 9-1, the moment of inertia of a solid cylinder about an axis perpendicular to its face and through its center is given by:

$$I_{0,\text{solid cylinder}} = \frac{1}{2}(2M)R^2 = MR^2$$

where M is the mass of the half-cylinder.

Express the moment of inertia of the half-cylinder about the same axis:

$$I_{0,\text{half cylinder}} = I_0 = \frac{1}{2}\left[MR^2\right] = \frac{1}{2}MR^2$$

Use the parallel-axis theorem to relate I_{cm} to I_0:

$$I_0 = I_{cm} + MD^2$$

Substitute for I_0 and solve for I_{cm}:

$$I_{cm} = I_0 - D^2M = \frac{1}{2}MR^2 - D^2M$$

Apply the parallel-axis theorem a second time to obtain an expression for I_C:

$$I_C = \frac{1}{2}MR^2 - D^2M + Mr^2$$
$$= M\left(\frac{1}{2}R^2 - D^2 + r^2\right) \quad (2)$$

Apply the law of cosines to obtain:

$$r^2 = R^2 + D^2 - 2RD\cos\theta$$

Substitute for r^2 in equation (2) to obtain:

$$I_C = M\left(\frac{1}{2}R^2 - D^2 + R^2 + D^2 - 2RD\cos\theta\right) = MR^2\left(\frac{3}{2} - 2\frac{D}{R}\cos\theta\right)$$

Substitute for h and I_C in equation (1):

$$E = MgD(1-\cos\theta) + \frac{1}{2}MR^2\left(\frac{3}{2} - 2\frac{D}{R}\cos\theta\right)\left(\frac{d\theta}{dt}\right)^2$$

Use the small angle approximation $\cos\theta \approx 1 - \frac{1}{2}\theta^2$ to obtain:

$$E = \frac{1}{2}MgD\theta^2 + \frac{1}{2}MR^2\left(\frac{3}{2} - \frac{D}{R}\left[2-\theta^2\right]\right)\left(\frac{d\theta}{dt}\right)^2$$

Because $\theta^2 << 2$, we can neglect the θ^2 in the square brackets to obtain:

$$E = \frac{1}{2}MgD\theta^2 + \frac{1}{2}MR^2\left(\frac{3}{2} - 2\frac{D}{R}\right)\left(\frac{d\theta}{dt}\right)^2$$

Differentiating both sides with respect to time and simplifying yields:

$$R^2\left(\frac{3}{2} - 2\frac{D}{R}\right)\left(\frac{d^2\theta}{dt^2}\right) + gD\theta = 0,$$

or

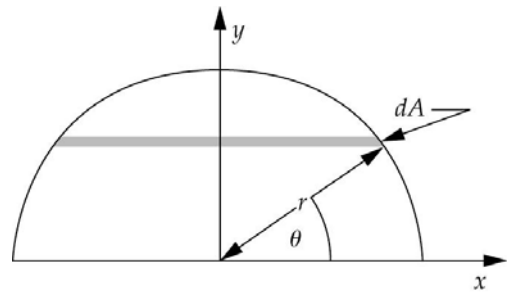
$$\frac{d^2\theta}{dt^2} + \frac{gD}{R^2\left(\frac{3}{2} - 2\frac{D}{R}\right)}\theta = 0,$$

the equation of simple harmonic motion with $\omega^2 = \dfrac{gD}{R^2\left(\frac{3}{2} - 2\frac{D}{R}\right)}$. (3)

D is the y coordinate of the center of mass of the semicircular disk shown. A surface element of area dA is shown in the diagram. Because the disk is a continuous object, we'll use

$$M\vec{r}_{\text{cm}} = \int \vec{r}\,dm$$

to find $y_{\text{cm}} = D$.



Express the coordinates of the center of mass of the semicircular disk:

$x_{\text{cm}} = 0$ by symmetry.

$$y_{\text{cm}} = D = \frac{\int y\sigma\,dA}{M}$$

Express y as a function of r and θ:

$$y = r\sin\theta$$

Express dA in terms of r and θ:

$$dA = r\,d\theta\,dr$$

Substitute and evaluate D:

$$D = \frac{\sigma\int_0^R\int_0^\pi r^2\sin\theta\,d\theta\,dr}{M} = \frac{2\sigma}{M}\int_0^R r^2\,dr$$

$$= \frac{2\sigma}{3M}R^3$$

Express M as a function of r and θ:

$$M = \sigma A_{\text{half disk}} = \tfrac{1}{2}\sigma\pi R^2$$

Substituting for M and simplifying yields:

$$D = \frac{2\sigma}{3\left(\frac{1}{2}\sigma\pi R^2\right)}R^3 = \boxed{\frac{4}{3\pi}R}$$

Substitute for D in equation (3) and simplify to obtain:

$$\omega^2 = \frac{\frac{4}{3\pi}}{\left(\frac{3}{2}-\frac{8}{3\pi}\right)}\frac{g}{R} = \left(\frac{8}{9\pi-16}\right)\frac{g}{R}$$

The period of the motion is given by:

$$T = \frac{2\pi}{\omega}$$

Substituting for ω and simplifying yields:

$$T = 2\pi\sqrt{\left(\frac{9\pi-16}{8}\right)\frac{R}{g}} = \boxed{7.78\sqrt{\frac{R}{g}}}$$

104 ••• A straight tunnel is dug through Earth as shown in Figure 14-38. Assume that the walls of the tunnel are frictionless. (*a*) The gravitational force exerted by Earth on a particle of mass *m* at a distance *r* from the center of Earth when $r < R_E$ is $F_r = -\left(GmM_{\mathrm{E}} / R_{\mathrm{E}}^3\right)r$, where M_{E} is the mass of Earth and R_{E} is its radius. Show that the net force on a particle of mass *m* at a distance *x* from the middle of the tunnel is given by $F_x = -\left(GmM_{\mathrm{E}} / R_{\mathrm{E}}^3\right)x$, and that the motion of the particle is therefore simple harmonic motion. (*b*) Show that the period of the motion is independent of the length of the tunnel and is given by $T = 2\pi\sqrt{R_{\mathrm{E}}/g}$. (*c*) Find its numerical value in minutes.

Picture the Problem The net force acting on the particle as it moves in the tunnel is the *x*-component of the gravitational force acting on it. We can find the period of the particle from the angular frequency of its motion. We can apply Newton's 2nd law to the particle in order to express ω in terms of the radius of Earth and the acceleration due to gravity at the surface of Earth.

(*a*) From the figure we see that:

$$F_x = F_r \sin\theta = -\frac{GmM_{\mathrm{E}}}{R_{\mathrm{E}}^3} r \frac{x}{r} = \boxed{-\frac{GmM_{\mathrm{E}}}{R_{\mathrm{E}}^3} x}$$

Because this force is a linear restoring force, the motion of the particle is simple harmonic motion.

(*b*) Express the period of the particle as a function of its angular frequency:

$$T = \frac{2\pi}{\omega} \qquad (1)$$

Apply $\sum F_x = ma_x$ to the particle:

$$-\frac{GmM_{\mathrm{E}}}{R_{\mathrm{E}}^3} x = ma$$

Solving for *a* yields:

$$a = -\frac{GM_{\mathrm{E}}}{R_{\mathrm{E}}^3} x = -\omega^2 x$$

$$\text{where } \omega = \sqrt{\frac{GM_{\mathrm{E}}}{R_{\mathrm{E}}^3}}$$

Use $GM_{\mathrm{E}} = gR_{\mathrm{E}}^2$ to simplify ω:

$$\omega = \sqrt{\frac{gR_{\mathrm{E}}^2}{R_{\mathrm{E}}^3}} = \sqrt{\frac{g}{R_{\mathrm{E}}}}$$

Substitute in equation (1) to obtain:

$$T = \frac{2\pi}{\sqrt{\frac{g}{R_E}}} = \boxed{2\pi\sqrt{\frac{R_E}{g}}}$$

(*c*) Substitute numerical values and evaluate *T*:

$$T = 2\pi\sqrt{\frac{6.37\times10^6\text{ m}}{9.81\text{ m/s}^2}} = 5.06\times10^3\text{ s}$$

$$= \boxed{84.4\text{ min}}$$

105 ••• **[SSM]** In this problem, derive the expression for the average power delivered by a driving force to a driven oscillator (Figure 14-39).

(*a*) Show that the instantaneous power input of the driving force is given by

$$P = Fv = -A\,\omega F_0 \cos\omega t \sin(\omega t - \delta).$$

(*b*) Use the identity $\sin(\theta_1 - \theta_2) = \sin\theta_1 \cos\theta_2 - \cos\theta_1 \sin\theta_2$ to show that the equation in (*a*) can be written as

$$P = A\omega F_0 \sin\delta \cos^2\omega t - A\omega F_0 \cos\delta \cos\omega t \sin\omega t$$

(*c*) Show that the average value of the second term in your result for (*b*) over one or more periods is zero, and that therefore $P_{av} = \frac{1}{2}A\omega F_0 \sin\delta$.

(*d*) From Equation 14-56 for tan δ, construct a right triangle in which the side opposite the angle δ is $b\omega$ and the side adjacent is $m(\omega_0^2 - \omega^2)$, and use this triangle to show that

$$\sin\delta = \frac{b\omega}{\sqrt{m^2(\omega_0^2-\omega^2)^2 + b^2\omega^2}} = \frac{b\omega A}{F_0}.$$

(*e*) Use your result for Part (*d*) to eliminate ωA from your result for Part (*c*) so that the average power input can be written as

$$P_{av} = \frac{1}{2}\frac{F_0^2}{b}\sin^2\delta = \frac{1}{2}\left[\frac{b\omega^2F_0^2}{m^2(\omega_0^2-\omega^2)^2 + b^2\omega^2}\right].$$

Picture the Problem We can follow the step-by-step instructions provided in the problem statement to obtain the desired results.

(*a*) Express the average power delivered by a driving force to a driven oscillator:

$$P = \vec{F}\cdot\vec{v} = Fv\cos\theta$$

or, because θ is 0°,

$$P = Fv$$

Express *F* as a function of time:

$$F = F_0\cos\omega t$$

Express the position of the driven oscillator as a function of time:

$$x = A\cos(\omega t - \delta)$$

Differentiate this expression with respect to time to express the velocity of the oscillator as a function of time:

$$v = -A\omega\sin(\omega t - \delta)$$

Substitute to express the average power delivered to the driven oscillator:

$$P = (F_0\cos\omega t)[-A\omega\sin(\omega t - \delta)] = \boxed{-A\omega F_0\cos\omega t\sin(\omega t - \delta)}$$

(*b*) Expand $\sin(\omega t - \delta)$ to obtain:

$$\sin(\omega t - \delta) = \sin\omega t\cos\delta - \cos\omega t\sin\delta$$

Substitute in your result from (*a*) and simplify to obtain:

$$P = -A\omega F_0\cos\omega t(\sin\omega t\cos\delta - \cos\omega t\sin\delta) = \boxed{A\omega F_0\sin\delta\cos^2\omega t - A\omega F_0\cos\delta\cos\omega t\sin\omega t}$$

(*c*) Integrate $\sin\theta\cos\theta$ over one period to determine $\langle\sin\theta\cos\theta\rangle$:

$$\langle\sin\theta\cos\theta\rangle = \frac{1}{2\pi}\left[\int_0^{2\pi}\sin\theta\cos\theta d\theta\right] = \frac{1}{2\pi}\left[\frac{1}{2}\sin^2\theta\Big|_0^{2\pi}\right] = 0$$

Integrate $\cos^2\theta$ over one period to determine $\langle\cos^2\theta\rangle$:

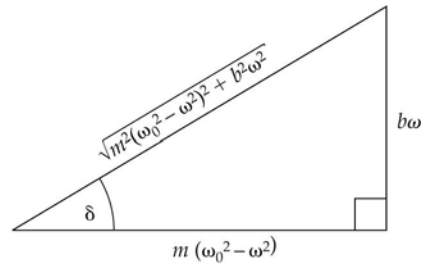
$$\langle\cos^2\theta\rangle = \frac{1}{2\pi}\int_0^{2\pi}\cos^2\theta d\theta = \frac{1}{2\pi}\left[\frac{1}{2}\int_0^{2\pi}(1+\cos 2\theta)d\theta\right] = \frac{1}{2\pi}\left[\tfrac{1}{2}\int_0^{2\pi}d\theta + \tfrac{1}{2}\int_0^{2\pi}\cos 2\theta d\theta\right] = \frac{1}{2\pi}(\pi + 0) = \frac{1}{2}$$

Substitute and simplify to express P_{av}:

$$P_{av} = A\omega F_0\sin\delta\langle\cos^2\omega t\rangle - A\omega F_0\cos\delta\langle\cos\omega t\sin\omega t\rangle = \tfrac{1}{2}A\omega F_0\sin\delta - A\omega F_0\cos\delta(0) = \boxed{\tfrac{1}{2}A\omega F_0\sin\delta}$$

(*d*) Construct a triangle that is consistent with

$$\tan\delta = \frac{b\omega}{m\left(\omega_0^2 - \omega^2\right)}:$$



Using the triangle, express $\sin\delta$:

$$\sin\delta = \boxed{\frac{b\omega}{\sqrt{m^2\left(\omega_0^2-\omega^2\right)^2 + b^2\omega^2}}}$$

Using Equation 14-56, reduce this expression to the simpler form:

$$\sin\delta = \boxed{\frac{b\omega A}{F_0}}$$

(*e*) Solve $\sin\delta = \dfrac{b\omega A}{F_0}$ for ω:

$$\omega = \frac{F_0}{bA}\sin\delta$$

Substitute in the expression for P_{av} to eliminate ω:

$$P_{av} = \boxed{\frac{F_0^2}{2b}\sin^2\delta}$$

Substitute for $\sin\delta$ from (*d*) to obtain:

$$P_{av} = \boxed{\frac{1}{2}\left[\frac{b\omega^2F_0^2}{m^2\left(\omega_0^2-\omega^2\right)^2 + b^2\omega^2}\right]}$$

106 ••• In this problem, you are to use the result of Problem 105 to derive Equation 14-51. At resonance, the denominator of the fraction in brackets in Problem 105(*e*) is $b^2\omega_0^2$ and P_{av} has its maximum value. For a sharp resonance, the variation in ω in the numerator in this equation can be neglected. Then the power input will be half its maximum value at the values of ω, for which the denominator is $2b^2\omega_0^2$.

(*a*) Show that ω then satisfies $m^2(\omega-\omega_0)^2(\omega+\omega_0)^2 \approx b^2\omega_0^2$.

(*b*) Using the approximation $\omega+\omega_2 \approx 2\omega_0$, show that $\omega-\omega_0 \approx \pm b/2m$.

(*c*) Express *b* in terms of *Q*.

(*d*) Combine the results of (*b*) and (*c*) to show that there are two values of ω for which the power input is half that at resonance and that they are given by

$$\omega_1 = \omega_0 - \frac{\omega_0}{2Q} \text{ and } \omega_2 = \omega_0 - \frac{\omega_0}{2Q}$$

Therefore, $\omega_2 - \omega_1 = \Delta\omega = \omega_0/Q$, which is equivalent to Equation 14-51.

Picture the Problem We can follow the step-by-step instructions given in the problem statement to derive the given results.

(*a*) Express the condition on the denominator of Equation 14-56 when the power input is half its maximum value:

$$m^2\left(\omega_0^2-\omega^2\right)^2+b^2\omega^2=2b^2\omega_0^2$$

and, for a sharp resonance,

$$m^2\left(\omega_0^2-\omega^2\right)^2\approx b^2\omega_0^2$$

Factor the difference of two squares to obtain:

$$m^2\left[(\omega_0-\omega)(\omega_0+\omega)\right]^2\approx b^2\omega_0^2$$

or

$$\boxed{m^2(\omega_0-\omega)^2(\omega_0+\omega)^2\approx b^2\omega_0^2}$$

(*b*) Use the approximation $\omega+\omega_0\approx 2\omega_0$ to obtain:

$$m^2(\omega_0-\omega)^2(2\omega_0)^2\approx b^2\omega_0^2$$

Solving for $\omega_0-\omega$ yields:

$$\omega_0-\omega=\boxed{\pm\frac{b}{2m}} \qquad (1)$$

(*c*) Using its definition, express Q:

$$Q=\frac{\omega_0 m}{b}\Rightarrow b=\boxed{\frac{\omega_0 m}{Q}}$$

(*d*) Substitute for b in equation (1) to obtain:

$$\omega_0-\omega=\pm\frac{\omega_0}{2Q}\Rightarrow\omega=\omega_0\pm\frac{\omega_0}{2Q}$$

Express the two values of ω:

$$\omega_+=\boxed{\omega_0+\frac{\omega_0}{2Q}}\text{ and }\omega_-=\boxed{\omega_0-\frac{\omega_0}{2Q}}$$

Remarks: Note that the width of the resonance at half-power is $\Delta\omega=\omega_+-\omega_-=\omega_0/Q$, in agreement with Equation 14-51.

107 ••• The Morse potential, which is often used to model interatomic forces, can be written in the form $U(r)=D\left(1-e^{-\beta(r-r_0)}\right)^2$, where r is the distance between the two atomic nuclei. (*a*) Using a **spreadsheet program** or **graphing calculator**, make a graph of the Morse potential using $D=5.00$ eV, $\beta=0.20$ nm^{-1}, and $r_0=0.750$ nm. (*b*) Determine the equilibrium separation and "spring constant" for small displacements from equilibrium for the Morse potential. (*c*) Determine an expression for the oscillation frequency for a *homonuclear* diatomic molecule (that is, two of the same atoms), where the atoms each have mass m.

Picture the Problem We can find the equilibrium separation for the Morse potential by setting $dU/dr=0$ and solving for r. The second derivative of U will give the "spring constant" for small displacements from equilibrium. In (*c*), we

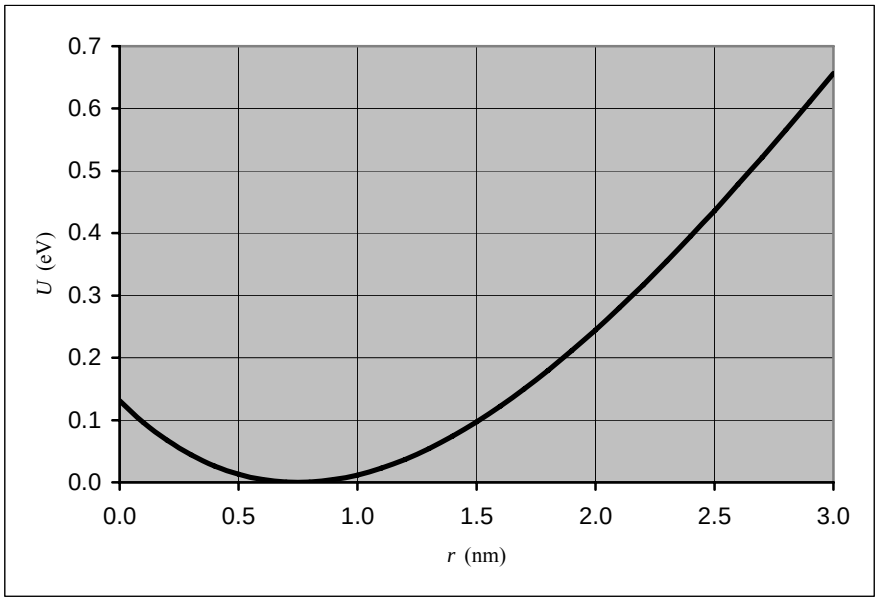
can use $\omega = \sqrt{k/\mu}$, where k is our result from (b) and μ is the reduced mass of a homonuclear diatomic molecule, to find the oscillation frequency of the molecule.

(a) A spreadsheet program to calculate the Morse potential as a function of r is shown below. The constants and cell formulas used to calculate the potential are shown in the table.

Cell	Content/Formula	Algebraic Form
B1	5	D
B2	0.2	β
C9	C8 + 0.1	$r + \Delta r$
D8	$B$1*(1−EXP(−$B$2*(C8−$B$3)))^2	$D\left[1-e^{-\beta(r-r_0)}\right]^2$

	A	B	C	D
1	D=	5	eV	
2	β=	0.2	nm^{-1}	
3	r_0=	0.75	nm	
4				
5				
6			r	$U(r)$
7			(nm)	(eV)
8			0.0	0.13095
9			0.1	0.09637
10			0.2	0.06760
11			0.3	0.04434
12			0.4	0.02629
235			22.7	4.87676
236			22.8	4.87919
237			22.9	4.88156
238			23.0	4.88390
239			23.1	4.88618

The graph shown below was plotted using the data from columns C (r) and D ($U(r)$).



(b) Differentiate the Morse potential with respect to r to obtain:

$$\frac{dU}{dr} = \frac{d}{dr}\left\{D\left[1-e^{-\beta(r-r_0)}\right]^2\right\} = -2\beta D\left[1-e^{-\beta(r-r_0)}\right]$$

This derivative is equal to zero for extrema:

$$-2\beta D\left[1-e^{-\beta(r-r_0)}\right] = 0 \Rightarrow r = \boxed{r_0}$$

Evaluate the second derivative of $U(r)$ to obtain:

$$\frac{d^2U}{dr^2} = \frac{d}{dr}\left\{-2\beta D\left[1-e^{-\beta(r-r_0)}\right]\right\} = 2\beta^2 D e^{-\beta(r-r_0)}$$

Evaluate this derivative at $r = r_0$:

$$\left.\frac{d^2U}{dr^2}\right|_{r=r_0} = 2\beta^2 D \qquad (1)$$

Recall that the potential function for a simple harmonic oscillator is:

$$U = \tfrac{1}{2}kx^2$$

Differentiate this expression twice to obtain:

$$\frac{d^2U}{dx^2} = k$$

By comparison with equation (1) we have:

$$k = \boxed{2\beta^2 D}$$

(*c*) Express the oscillation frequency of the diatomic molecule:

$$\omega = \sqrt{\frac{k}{\mu}}$$

where μ is the reduced mass of the molecule.

Express the reduced mass of the homonuclear diatomic molecule:

$$\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{m^2}{2m} = \frac{m}{2}$$

Substitute for ω and simplify to obtain:

$$\omega = \sqrt{\frac{2\beta^2 D}{\frac{m}{2}}} = \boxed{2\beta\sqrt{\frac{D}{m}}}$$

Remarks: An alternative approach in (*b*) is to expand the Morse potential in a Taylor series

$$U(r) = U(r_0) + (r - r_0)U'(r_0) + \frac{1}{2!}(r - r_0)^2 U''(r_0) + \text{higher order terms}$$

to obtain $U(r) \approx \beta^2 D(r - r_0)^2$. Comparing this expression to the energy of a spring-and-mass oscillator we see that, as was obtained above, $k = 2\beta^2 D$.