5.1 and 5.2: Rational Expressions

I. Definition

A rational expression has the form

$$r(x) = \frac{f(x)}{g(x)}$$

Where *p(x)* and *q(x)* are polynomials ($q(x) \neq 0$)

II. Rewriting in Lowest Terms

A. Example 1:

$$\frac{x^5 - x^3}{x^2 + 3x + 2}$$
$$\frac{x^3(x^2 - 1)}{(x+2)(x+1)}$$
$$\frac{x^3(x-1)(x+1)}{(x+2)(x+1)}$$
$$\boxed{\frac{x^3(x-1)}{(x+2)}}$$
$$x \neq -2, -1$$

II. Rewriting in Lowest Terms

B. Example 2:

$$\frac{\frac{1}{x+1} - 1}{\frac{1}{x+1} + 1}$$
$$\frac{\frac{1}{x+1} - 1}{\frac{1}{x+1} + 1}\left(\frac{x+1}{x+1}\right)$$
$$\frac{1 - (x+1)}{1 + (x+1)}$$
$$\boxed{\frac{2 - x}{2 + x}}$$
$$x \neq -1, -2$$

III. Arithmetic Operations

A. Example 3:

$$\frac{\frac{3x}{2y^3}}{\frac{9x^2}{14y^5}}$$

$$\frac{3x}{2y^3} \cdot \frac{14y^5}{9x^2}$$

$$\frac{42xy^5}{18x^2y^3} = \boxed{\frac{7y^2}{3x}}$$

$$x, y \neq 0$$

III. Arithmetic Operations

B. Example 4:

$$\frac{a^2 + 6a + 8}{a^2 - 5z - 14} \cdot \frac{a - 7}{a^2 - 16}$$

$$\frac{(a + 2)(a + 4)}{(a - 7)(a + 2)} \cdot \frac{a - 7}{(a - 4)(a + 4)}$$

$$\frac{(a + 2)(a + 4)(a - 7)}{(a - 7)(a + 2)(a - 4)(a + 4)}$$

$$\boxed{\frac{1}{(a - 4)}}$$

$$a \neq -4, -2, 4, 7$$

III. Arithmetic Operations

C. Example 5:

$$\frac{4x}{x - 1} - \frac{1}{x^2 + 2x - 3}$$

$$\frac{4x}{x - 1} - \frac{1}{(x - 1)(x + 3)}$$

$$\frac{4x}{x - 1}\left(\frac{x + 3}{x + 3}\right) - \frac{1}{(x - 1)(x + 3)}$$

$$\frac{4x^2 + 12x}{(x - 1)(x + 3)} - \frac{1}{(x - 1)(x + 3)}$$

$$\frac{4x^2 + 12x - 1}{(x - 1)(x + 3)}$$

$$\boxed{\frac{4x^2 + 12x - 1}{x^2 + 2x - 3}}$$

$$x \neq 1, -3$$

Homework

p. 293-294 #2-11, 14-18

5.3: ONE-SIDED LIMITS

I. Methods

Try these in order

1. Direct substitution
2. Simplify
3. One-Sided Limits

I. Methods

A. Direct Substitution

Example 1: Evaluate $\lim\limits_{x\to 1}\frac{x^2+x+2}{x+1}$

$$\lim_{x\to 1}\frac{x^2+x+2}{x+1}=\frac{(1)^2+(1)+2}{(1)+1}=2$$

I. Methods

B. Simplifying

Example 2: evaluate $\lim\limits_{x\to 1}\frac{x^3-1}{x-1}$

Try direct substitution.

$$\lim_{x\to 1}\frac{x^3-1}{x-1}=\frac{(1)^3-1}{(1)-1}=\frac{0}{0}$$

Simplify:

$$\lim_{x\to 1}\frac{x^3-1}{x-1}=\lim_{x\to 1}\frac{(x-1)(x^2+x+1)}{x-1}$$

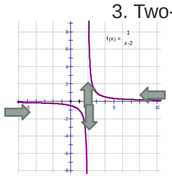
$$=\lim_{x\to 1}(x^2+x+1)=(1)^2+(1)+1=3$$

I. Methods

C. One Sided Limits

Example 3: Evaluate $\lim\limits_{x\to 2}\frac{1}{x-2}$

1. Direct substitution: $\lim\limits_{x\to 2}\frac{1}{x-2}=\frac{1}{2-2}=\frac{1}{0}$
2. You can't simplify
3. Two-sided limits (look at the graph)



$$\lim_{x\to 2^+}\frac{1}{x+2}=\infty$$

$$\lim_{x\to 2^-}\frac{1}{x+2}=-\infty$$

$\lim\limits_{x\to 2}\frac{1}{x-2}$ does not exist

II. Two-Sided Limits

A function *f(x)* has a two-sided limit iff the left and right hand limits equal the same number

$$\lim_{x\to c}f(x)=L\Leftrightarrow\lim_{x\to c^+}f(x)=\lim_{x\to c^-}f(x)=L$$

III. Examples

A. Example 4: Find $\lim_{x\to 3}\frac{x+3}{x^2-9}$.

1. Substitution:

$$\lim_{x\to 3}\frac{x+3}{x^2-9}=\frac{(3)+3}{(3)^2-9}=\frac{6}{0}$$

2. Simplify:

$$\lim_{x\to 3}\frac{x+3}{x^2-9}=\lim_{x\to 3}\frac{x+3}{(x+3)(x-3)}=\lim_{x\to 3}\frac{1}{(x-3)}=\frac{1}{(3)-3}=\frac{1}{0}$$

3. One-Sided Limits:

$$\lim_{x\to 3^-}\frac{1}{x-3}=\frac{1}{0^-}=-\infty$$

$$\lim_{x\to 3^+}\frac{1}{x-3}=\frac{1}{0^+}=+\infty$$

The limit Does Not Exist

III. Examples

B. Example 5: Examine $f(x)=\frac{2}{x-4}$

1. $\lim_{x\to\infty} f(x)=0$
2. $\lim_{x\to -\infty} f(x)=0$

$\Big\}$ *Horizontal Asymptote* $y=0$

3. $\lim_{x\to 4^+} f(x)=\infty$
4. $\lim_{x\to 4^-} f(x)=-\infty$

$\Big\}$ *Vertical Asymptote* $x=4$

IV. Asymptotes and Limits

A. Vertical Asymptotes:

$x=a$ is a vertical asymptote if either

$\lim_{x\to a^+} f(x)=\pm\infty$ or $\lim_{x\to a^-} f(x)=\pm\infty$

B. Horizontal Asymptotes:

$y=b$ is a horizontal asymptote if either

$\lim_{x\to\infty} f(x)=b$ or $\lim_{x\to -\infty} f(x)=b$

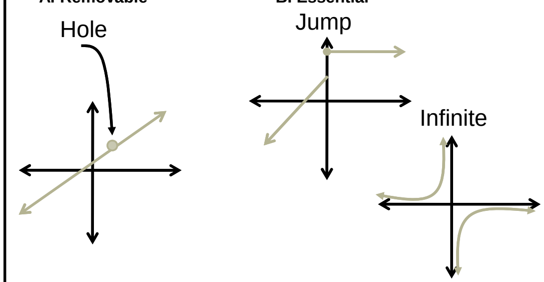
Homework

- P. 298-299 #4-8, 11-15

5.4: RATIONAL FUNCTIONS

I. DISCONTINUITIES

A. Removable **B. Essential**



II. REMOVABLE DISCONTINUITIES

A. Ex 1: Graph $f(x) = \frac{x^2-16}{x+4}$

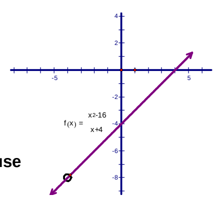
1. This is a line with a hole at *x=-4*.

2. It is discontinuous at *x=-4.*

3. -4 is a point of discontinuity.

4. This discontinuity is *removable*, because it can be 'filled' with one point (-4,-8)

5. This is the continuous extension of *f(x)*.

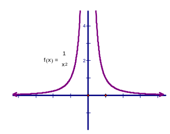


$$g(x) = \begin{cases} \frac{x^2-16}{x+4}; x \neq -4 \\ -8; x = -4 \end{cases}$$

III. ESSENTIAL DISCONTINUITIES

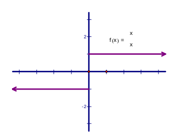
A. Ex 2: graph $f(x) = \frac{1}{x^2}$

Infinite Discontinuity



B. Ex 3: graph $f(x) = \frac{|x|}{x}$

Jump Discontinuity



III. ESSENTIAL DISCONTINUITIES

Ex 4: Is $f(x) = \frac{x^2+3x+2}{x^2+4x+3}$ continuous at *x=-1*.

1. Direct substitution yields $\frac{0}{0}$.

2. Factor: $f(x) = \frac{x^2+3x+2}{x^2+4x+3} = \frac{(x+2)(x+1)}{(x+1)(x+3)} = \frac{x+2}{x+3}$

Now: $f(-1) = \frac{1}{2}$ not continuous

3. $\lim_{x\to -1^-} f(x) = \lim_{x \to -1^+} f(x) = \frac{1}{2} \neq f(-1)$

4. Fill hole:

$$g(x) = \begin{cases} \frac{x^2+3x+2}{x^2+4x+3} & x \neq -1 \\ \frac{1}{2} & x = -1 \end{cases}$$

HOMEWORK

p. 305-307 #6-14, 16, 19, 20

5.5: End Behavior of Rational Functions

I. End Behavior

A. Ex. 1: find the end behavior of f(x) $= \frac{x^2+3}{2x^2-1}$

1. Method 1: Table of values

x	2	10	100	1000
f(x) $= \frac{x^2+3}{2x^2-1}$	1	0.51759	0.50018	0.50000

2. $\lim_{x\to\infty} f(x) = \frac{1}{2}$

I. End Behavior

2. Method 2: divide by highest power.

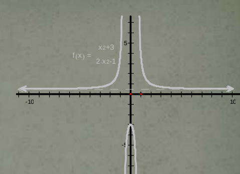
$$\lim_{x\to\infty} \frac{x^2+3}{2x^2-1} = \lim_{x\to\infty} \frac{\frac{x^2}{x^2}+\frac{3}{x^2}}{\frac{2x^2}{x^2}-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{1+\frac{3}{x^2}}{2-\frac{1}{x^2}} = \frac{1+0}{2-0} = \frac{1}{2}$$

3. Method 3: Long Division

$$\lim_{x\to\infty} \frac{x^2+3}{2x^2-1} = \lim_{x\to\infty} \frac{1}{2} + \frac{1/2}{2x^2-1} = \frac{1}{2} + 0 = \frac{1}{2}$$

I. End Behavior

B. Graph, what do you notice?



There is a horizontal asymptote of $y = \frac{1}{2}$

C. Definition: The line *y=b* is a *horizontal asymptote* iff either

$$\lim_{x\to\infty} f(x) = b \text{ or } \lim_{x\to-\infty} f(x) = b$$

II. Asymptotes

A. Ex 2: Investigate all asymptotes using limits for

$$g(x) = \frac{x-5}{x-4}$$

1. Horizontal Asymptotes:

$$\lim_{x\to\infty} \frac{x-5}{x-4} = \lim_{x\to\infty} \frac{\frac{x}{x}-\frac{5}{x}}{\frac{x}{x}-\frac{4}{x}} = \frac{1-0}{1-0} = 1$$

$$y = 1$$

II. Asymptotes

2. Vertical Asymptotes:

$$\lim_{x\to4^+} \frac{x-5}{x-4} = \frac{4^+-5}{4^+-4} = \frac{-1^+}{0^+} = -\infty$$

$$\lim_{x\to4^-} \frac{x-5}{x-4} = \frac{4^--5}{4^--4} = \frac{-1^-}{0^-} = -\infty$$

$\lim_{x\to4} \frac{x-5}{x-4}$ Does Not Exist

3. Asymptotes:

$$y = 1, x = 4$$

II. Asymptotes

B. Ex. 3 Find the end behavior of $f(x) = \frac{3x^2+5}{x+3}$

$$\lim_{x\to\infty} \frac{3x^2+5}{x+3} = \lim_{x\to\infty} \frac{\frac{3x^2}{x^2}+\frac{5}{x^2}5}{\frac{x}{x^2}+\frac{3}{x^2}} = \frac{3^+}{0^+} = \infty$$

$$\lim_{x\to-\infty} \frac{3x^2+5}{x+3} = \lim_{x\to-\infty} \frac{\frac{3x^2}{x^2}+\frac{5}{x^2}5}{\frac{x}{x^2}+\frac{3}{x^2}} = \frac{3^-}{0^-} = -\infty$$

C. Find all asymptotes

Vertical: $x = -3$

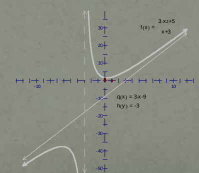
Horizontal: None

III. Slant Asymptotes

A. Ex. 4: Find the slant asymptote for $f(x) = \frac{3x^2+5}{x+3}$.

$$f(x) = \frac{3x^2+5}{x+3} = 3x - 9 + \frac{32}{x-3}$$

Slant Asymptote: $q(x) = 3x - 9$



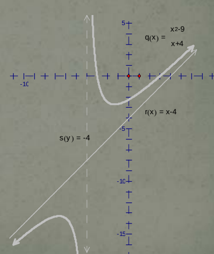
III. Slant Asymptotes

B. Ex. 5: Identify all asymptotes of $f(x) = \frac{x^2-9}{x+4}$

Vertical Asymptote: $x = -4$

Horizontal Asymptote: none

Slant Asymptote: $y = x - 4$



Homework

p. 313-314 # 1-7, 10, 12-14

5.6: IRRATIONAL NUMBERS

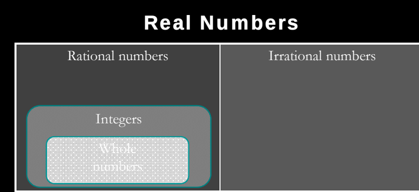
I. RATIONAL VS. IRRATIONAL

A. A *rational* number can be written as a fraction made up of integers.

As a fraction, they will terminate or repeat.

B. An *irrational* number cannot.

As a fraction, they will not terminate and not repeat.



II. RATIONALIZING THE DENOMINATOR

A. Rationalizing the Denominator: no radicals in the denominator.

1. Example 1: Simplify $\frac{1}{\sqrt{2}}$

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

2. Example 2: Simplify $\frac{6}{3\sqrt{x}}$

$$\frac{6}{3\sqrt{x}} = \frac{2}{\sqrt{x}} = \frac{2}{\sqrt{x}} \cdot \frac{\sqrt{x}}{\sqrt{x}} = \frac{2\sqrt{x}}{x}$$

3. Ex 3: Simplify $\frac{\sqrt[3]{54}}{\sqrt[3]{x^2}}$

$$\frac{\sqrt[3]{54}}{\sqrt[3]{x^2}} \cdot \frac{\sqrt[3]{x}}{\sqrt[3]{x}} = \frac{\sqrt[3]{54x}}{\sqrt[3]{x}} = \frac{3\sqrt[3]{2x}}{\sqrt[3]{x}}$$

II. RATIONALIZING THE DENOMINATOR

D. **To rationalize a denominator of $a + \sqrt{b}$,** multiply top and bottom by:

$a - \sqrt{b}$ or $-a + \sqrt{b}$

E. Example 4: Simplify $\frac{4}{2+\sqrt{5}}$

$$\frac{4}{2+\sqrt{5}} \cdot \frac{2-\sqrt{5}}{2-\sqrt{5}} = \frac{8-4\sqrt{5}}{4-2\sqrt{5}+2\sqrt{5}-25} = \frac{8-4\sqrt{5}}{-21}$$

F. Example 5: Simplify $\frac{2+3\sqrt{5}}{2-3\sqrt{5}}$

$$\frac{2+3\sqrt{5}}{2-3\sqrt{5}} \cdot \frac{2+3\sqrt{5}}{2+3\sqrt{5}} = \frac{4+6\sqrt{5}+6\sqrt{5}+45}{4+6\sqrt{5}-6\sqrt{5}-45} = \frac{49+12\sqrt{5}}{-41}$$

III. DECIMALS FOR RATIONAL NUMBERS

A. Example 6. Rewrite the decimal $x = 0.1\overline{25}$ as a fraction.

$$\begin{cases} 100x = 12.5\overline{25} \\ x = 0.1\overline{25} \end{cases}$$

By subtracting these equations

$$99x = 12.4$$

$$x = \frac{12.4}{99} = \frac{62}{495}$$

B. Example 7: Rewrite the decimal $x = 0.\overline{9}$

$$\begin{cases} 10x = 9.\overline{9} \\ x = 0.\overline{9} \end{cases}$$

By subtracting these equations

$$9x = 9$$

$$x = 1$$

HOMEWORK

p. 320-321 #6, 7, 9, 10, 13, 14, 17, 18, 21, 22

5.8 DAY 1: RATIONAL EQUATIONS

I. RATIONAL WORD PROBLEM

- Example 1: It takes Willy 6 hours to do the same job that it takes Nilly 10 hours to do. How long would it take if they worked together?

Willy does 1/6 each hour and Nilly does 1/10 each hour.

- $\frac{t}{6}+\frac{t}{10}=1$
- $60\left(\frac{t}{6}+\frac{t}{10}\right)=(1)60$
- $10t+6t=60$
- $16t=60$
- $t=\frac{60}{16}=3\frac{3}{4}$
- $t=3\ hours, 45\ minutes$

II. RATIONAL EQUATIONS

A. Ex 2: $\frac{1}{x}+\frac{1}{2x}=\frac{1}{6}$ $\quad x\neq 0$

- $(6x)\left(\frac{1}{x}+\frac{1}{2x}\right)=(6x)\left(\frac{1}{6}\right)$
- $6+3=x$
- $9=x$

- B. Ex 3: $\frac{n-5}{n-4}=\frac{3n}{3n+1}$ $\quad x\neq 4,-\frac{1}{3}$
  - $(3n+1)(n-4)\left(\frac{n-5}{n-4}\right)=(3n+1)(n-4)\left(\frac{3n}{3n+1}\right)$
  - $(3n+1)(n-5)=(n-4)(3n)$
  - $3n^2-14n-5=3n^2-12n$
  - $-5=2n$
  - $-\frac{5}{2}=n$

II. RATIONAL EQUATIONS

- C. $\frac{2}{y+2}-\frac{y}{2-y}=\frac{y^2+4}{y^2-4}$ $\quad y\neq\pm 2$
  - $\frac{2}{y+2}-\frac{y}{-(-2+y)}=\frac{y^2+4}{(y-2)(y+2)}$
  - $\frac{2}{y+2}+\frac{y}{y-2}=\frac{y^2+4}{(y-2)(y+2)}$
  - $(y-2)(y+2)\left(\frac{2}{y+2}+\frac{y}{y-2}\right)=(y-2)(y+2)\frac{y^2+4}{(y-2)(y+2)}$
  - $2(y-2)+y(y+2)=y^2+4$
  - $2y-4+y^2+2y=y^2+4$
  - $4y=8$
  - $y=2$
  - No solution

II. RATIONAL EQUATIONS

- D. $\frac{24}{10+c}+\frac{24}{10-c}=5$ $y\neq\pm10$
  - $(10+c)(10-c)\left(\frac{24}{10+c}+\frac{24}{10-c}\right)=(10+c)(10-c)5$
    - $24(10-c)+24(10+c)=5(100-c^2)$
      - $240-24c+240+24c=500-5c^2$
        - $-20=-5c^2$
          - $4=c^2$
          - $\pm2\approx c$

II. RATIONAL EQUATIONS

- E. $\frac{6}{x-1}=\frac{12}{x^2-1}-2$ $x\neq\pm1$
  - $\frac{6}{x-1}=\frac{12}{x^2-1}-2$
  - $(x-1)(x+1)\left(\frac{6}{x-1}\right)=(x-1)(x+1)\left(\frac{12}{(x-1)(x+1)}-2\right)$
  - $6(x+1)=12-2(x-1)(x+1)$
  - $6x+6=12-2(x^2-1)$
  - $6x+6=12-2x^2+2$
  - $2x^2+6x-8=0$
  - $x^2+3x-4=0$
  - $(x+4)(x-1)=0$
  - $x=-4,1$
  - $x=-4$

II. RATIONAL EQUATIONS

- F. $\frac{r}{r-1}-\frac{2}{1-r^2}=\frac{8}{r+1}$ $r\neq\pm1$
  - $\frac{r}{r-1}+\frac{2}{r^2-1}=\frac{8}{r+1}$
  - $\frac{r}{r-1}+\frac{2}{(r-1)(r+1)}=\frac{8}{r+1}$
  - $(r-1)(r+1)\left(\frac{r}{r-1}+\frac{2}{(r-1)(r+1)}\right)=(r-1)(r+1)\left(\frac{8}{r+1}\right)$
  - $r(r+1)+2=8(r-1)$
  - $r^2+r+2=8r-8$
  - $r^2-7r+10=0$
  - $(r-2)(r-5)=0$
  - $r=2,5$

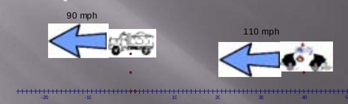
HOMEWORK

- Worksheet #1

5.8 DAY 2: MOTION PROBLEMS

I. Motion Problems

A. Example 1: A truck enters I-78 and drives 90 km/hr. At the same time a State police officer, 40 km east of the truck starts west at 110 km/h. How long will it take the policeman to catch the truck?

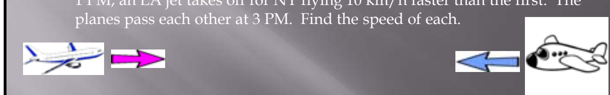


	rate	time	Distance
Truck	90	t	90t
Police Car	110	t	110t

$$90t + 40 = 110t$$
$$40 = 20t$$
$$2 = t$$
2 hours

I. Motion Problems

A. B. Example 2: At noon a jet leaves NY bound for LA, 4000 miles away. At 1 PM, an LA jet takes off for NY flying 10 km/h faster than the first. The planes pass each other at 3 PM. Find the speed of each.



	rate	time	Distance
Plane 1	$r\ km/h$	3 h	$3r$
Plane 2	$r + 10\ km/h$	2 h	$2(r + 30)$

$$3r + 2(r + 10) = 4000$$
$$5r = 3980$$
$$r = 796\ km/h$$
Plane 1 is 796 km/h, Plane 2 is 806 km/h

Homework

Motion Problem Worksheet

5.8 Part III: Work Problems

II. Work Problems

1. Example 1: Working alone a painter can paint a small apartment in 10 hours. Her helper can paint the same apartment in 15 hours. How long will it take if they work together?

	Rate	Time	Part of Job
Painter	1/10	t	t/10
Helper	1/15	t	t/15

$$\frac{t}{10}+\frac{t}{15}=1$$
$$30\left(\frac{t}{10}+\frac{t}{15}\right)=(30)1$$
$$3t+2t=30$$
$$5t=30$$
$$t=6hours$$

II. Work Problems

2. Example 2: Pete can mow a lawn in 50 minutes while his brother Jim can do it in half the time. How long will it take them if they work together?

	Rate	Time	Part of Job
Pete	1/50	t	t/50
Jim	1/25	T	t/25

$$\frac{t}{50}+\frac{t}{25}=1$$
$$50\left(\frac{t}{50}+\frac{t}{25}\right)=50(1)$$
$$t+2t=50$$
$$3t=50$$
$$t=16.\bar{6}\ minutes$$

About 16.6 minutes

Homework:

Work Problem Worksheet