10-1: WHAT EXACTLY ARE YOU COUNTING?

I. Essential Features of a counting problem

A. Does order matter?

B. Are repetitions allowed?

2. Vocabulary

A. Set: A collection of elements. Order and repetition don't matter.

Some examples of sets:

$A = \{2, 3, 5, 7, 11, 13, ...\}$ $B = \{red, blue, yellow\}$

$C = \text{All NHHS Students}$

B. String: a list of symbols where order matters

C. Example 2: For symbols A, B, and C:

1. Write all sets of 3 symbols without repetition.

$$\{ABC\}$$

2. Write all sets of 3 symbols with repetition.

$$\{AAA\}, \{AAB\}, \{AAC\}, \{ABB\}, \{ABC\},$$
$$\{ACC\}, \{BBB\}, \{BBC\}, \{BCC\}, \{CCC\}$$

3. write all strings of 3 symbols without repetition

$$ABC, ACB, BAC, BCA, CAB, CBA$$

4. Write all collections of 3 symbols with repetition

$$ABC, ACB, BAC, BCA, CAB, CBA$$
$$AAB, AAC, ABA, ACA, BAA, CAA$$
$$BBA, BBC, BAB, BCB, ABB, CBB$$
$$CCA, CCB, CAC, CBC, ACC, BCC$$
$$AAA, BBB, CCC$$

CW: Lesson Master 10-1 odds

HW: # 3-9, 11-13

10.2: MULTIPLICATION COUNTING PRINCIPLE

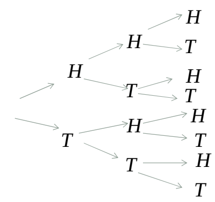
I. Tree Diagrams

A. Example 1: A fair coin is tossed three times.

-State the essential features

Order matters, repetitions are allowed

-Construct a tree diagram.

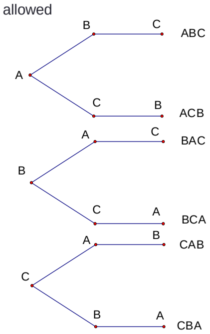


A. Example 2: In how many ways can three students sit in three seats?

-State the essential features

Order matters, repetition is not allowed

-Construct a tree diagram



II. Fundamental Counting Principle

A. Definition: If you can choose one item from a group of *M* items and a second item from a group of *N* items, then the total number of two item choices is M*N

Category 1 (eggs)	Category 2	Category 3 (Meat)
Scrambled Sunny-Side-Up Poached	Pancakes French Toast	Pork Roll Bacon Sausage

B. Example 3: As part of a breakfast meal, a customer can choose one out of each of the categories above.

How many total choices are possible?

$$\underline{3} \cdot \underline{2} \cdot \underline{3} = \boxed{18\ choices}$$

How many choices if you can also choose coffee or tea?

$$18 \cdot 2 = \boxed{36\, choices}$$

C. Example 4: How many ways can you answer a 10 question test in each of the following cases?

1. If the questions are true/false

$$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^{10} = \boxed{1{,}024\, ways}$$

2. If it is multiple choice with five choices

$$5^{10} = \boxed{9{,}765{,}625\, ways}$$

D. Example 5: How many four digit numbers have at least one digit with a 6 (find how many don't have a 6 and subtract)?

Four Digit Numbers:

$$9 \cdot 10 \cdot 10 \cdot 10 = 9{,}000$$

Four Digit Numbers without a 6:

$$8 \cdot 9 \cdot 9 \cdot 9 = 5{,}832$$

Four Digit Numbers with a 6:

$$9{,}000 - 5{,}832 = \boxed{3{,}168\, numbers}$$

Classwork: LM 10-2 #2, 3a, 4, 6

Homework: # 3-6, 9, 11, 13, 15, 17

10-3: Permutations

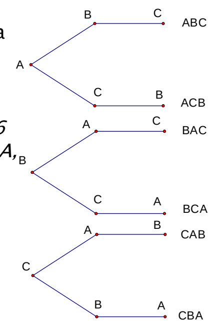
I. Permutations of All Objects

A. Each ordering of a set of n objects (an *n-set*) is called a *permutation.*

B. Permutations: There are $n!$ permutations of an *n-set*.

C. For example, there are $3!=6$ permutations of the letters A, B and C.

$$ABC, ACB, BAC,$$
$$BCA, CAB, CBA$$



D. Example 1: A family with two parents and four kids is going to line up in a row for a picture. The parents must stand next to each other, but the kids can be in any order. How many arrangements are there?

solution: think of the parents as a block, then we have the following five objects to order.

$$P, C_1, C_2, C_3, C_4$$

There are $5! = 120$ ways to do so

The parents can be in $2!=2$ arrangements.

$$(5!)(2!) = 120 \cdot 2 = \boxed{240\, arrangements}$$

II. Permutation Counting Formula (if you are not selecting all objects):

The number of permutations of n objects taken r at a time

is given by:

$$\boxed{{}_nP_r = n(n-1)(n-2)\cdots(n-r+1) \quad \text{or} \quad {}_nP_r = \frac{n!}{(n-r)!}} \quad \text{for } 0 \le r \le n$$

$$\boxed{\text{If } r > n, \text{ then } {}_nP_r = 0}$$

E. Example 4: Evaluate each of the following without a calculator:

1. $${}_8P_4 = \frac{8!}{4!} = \frac{8\cdot7\cdot6\cdot5\cdot4!}{4!} = 8\cdot7\cdot6\cdot5 = \boxed{1680}$$

2. $${}_{11}P_4 = \frac{11!}{7!} = \frac{11\cdot10\cdot9\cdot8\cdot7!}{7!} = 11\cdot10\cdot9\cdot8 = \boxed{7920}$$

3. $${}_nP_2 = n(n-1) = \boxed{n^2 - n}$$

F. Example 5: 100 Senators are being considered to chair the 17 committees in the US senate. How many different ways could the chair people be selected, assuming that no one chairs more than 1 committee?

$${}_{100}P_{17} = \boxed{2.365 \cdot 10^{33}}$$

G. Example 6: using the letters in the word facetiously, how many strings can be made of:

1. 11 letters

$${}_{11}P_{11} = \boxed{39,916,800}$$

2. 7 letters

$${}_{11}P_{7} = \boxed{1,663,200}$$

3. 5 letters

$${}_{11}P_{5} = \boxed{55,440}$$

CW: LM 10-3 #1-5, 6-8 (skills)

HW: #2-8, 10-11, 13, 15, 17

10-4: COMBINATIONS

I. Combinations

A. When the order does not matter for our selections (ABC is the same as ACB and CAB), we are using *combinations*.

B. Formula

The number of combinations of *n* objects taken *r* at a time Is given by:

$$_nC_r = \frac{n!}{r!(n-r)!} \text{ for } 0 \le r \le n$$
$$\text{If } r > n, \text{ then } {}_nC_r = 0$$

C. Example 1: evaluate each of the following with the formula

1. $_{10}C_4 = \frac{10!}{4! \cdot 6!} = \boxed{210}$

2. $_{10}C_6 = \frac{10!}{6! \cdot 4!} = \boxed{210}$

3. $_5C_0 = \frac{5!}{0! \cdot 5!} = \boxed{1}$

D. Example 2: Decide if each of the following scenarios represents a permutation or combination, then solve the problem.

1. There are 10 people at a party, and each pair of people shakes hands exactly once. How many handshakes were there?

Combination $_{10}C_2 = \boxed{45 \textit{ hand shakes}}$

2. A basketball team has 12 players on its roster. How many ways are there to pick a Center, Power Forward, Small Forward, Shooting Guard, and Point Guard?

Permutation $_{12}P_5 = \boxed{95,040 \textit{ ways}}$

3. How many ways are there to select a committee of 6 people from a group of 12?

Combination $_{12}C_6 = \boxed{924 \textit{ ways}}$

4. In a senior class with 200 people, how many ways can a president, vice president, secretary and treasurer be selected.

Permutation $_{200}P_4 = \boxed{1,552,438,800 \textit{ ways}}$

D. Example 3:

The local bank branch has a pool of 8 tellers and 8 customer service reps (CSRs). How many ways can the manager select 4 tellers and 2 CSRs to work on a given day?

$$\text{Tellers: } {}_8C_4 = \frac{8!}{4!4!} = 70$$
$$\text{CSRs: } {}_8C_2 = \frac{8!}{2!6!} = 28$$
$$\text{Choices: } 70 \cdot 28 = \boxed{1960}$$

II. Lottery example

Example 4: in the NJ lottery Pick 6 game, you choose 6 numbers (order doesn't matter) from 01 through 49 (inclusive).

1. Find the probability that you choose the one winning ticket.

Total Number of Possibilities:

$_{49}C_6 = 13,983,816$ *possibilities*

Probability:

$$\frac{1}{13,983,816} \approx \boxed{7.15 \times 10^{-8}}$$

2. Find the probability that you get 3 of the 6 numbers correct

Number of ways to get 3 of 6:

$_6C_3 = 20$ *possibilities*

Probability:

$$\frac{20}{13,983,816} \approx \boxed{1.43 \times 10^{-6}}$$

CW: LM 10-4 #1-4, 8, 11, 12

HW: # 1, 2a, 4-11, 13, 15-18

10-5: The Binomial Theorem

I. Binomial Coefficients

A. Definition: The binomial coefficients that appear in the expansion of $(a+b)^n$ are the values of

$$_nC_r \text{ for } r = 0,1,2,3,\ldots,n$$

Another notation for combinations is the following:

$$\binom{n}{r} = {}_nC_r$$

Both are read as "*n* choose *r*"

The Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k$$

$$= \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \ldots \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$

B. Example 1: Expand the following binomial.

$$(x+y)^4$$

Type the following in your calculator to find the Coefficients

$$4\,{}_nC_r\{0,1,2,3,4\} \quad = \{1,4,6,4,1\}$$

$$\boxed{(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4}$$

C. Example 2: Use Pascal's triangle to expand $(x+y)^7$

$$\boxed{(x+y)^7 = x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7}$$

D. Example 3: Find the coefficient of x^8 in the expansion of $(x-3)^{12}$.

We are only concerned with the following coefficient

$$\binom{12}{8}x^8(-3)^4 = (495)(81)x^8$$

$$\boxed{= 40,095x^8}$$

E. Example 4: Give the 7th term in the expansion of

$$(x+y)^{11}$$

Since we start with the 0th term, we want $k=6$.

$$\binom{n}{k}x^{n-k}y^k = \binom{16}{6}x^{16-6}y^6 = \boxed{8008x^{10}y^6}$$

F. Example 5: Give the 4th term of the expansion of

$$(2b-4a^2)^7$$

$$\binom{n}{k}x^{n-k}y^k = \binom{7}{3}(2b)^4(-4a^2)^3 = \boxed{-35840a^6b^4}$$

HW 1: LM 10-5 Evens

HW 2: #1, 2, 4-10, 13-17

10-6: Counting and the Binomial Theorem

I. Binomial Distributions:

A. Suppose an experiment consists of *n* independent repetitions of an experiment with two outcomes, called "success" and "failure." Let *P(success)=p* and *P(failure)=q*. (note that *q=1-p*).

B. Then the terms in the expansion of *(p+q)^n* give the respective probabilities of exactly *n, n-1, …, 2, 1, 0* successes.

C. In a trial with *r* independent repetitions out of *n*, the probability is below:

$$\boxed{P=\binom{n}{r}p^r q^{n-r}}$$

D. **Example 1: A company owns 400 laptops. Each laptop has an 8% probability of not working. You randomly select 20 laptops for your salespeople.**

$$p=0.08, q=0.92, n=20$$

1. **What is the likelihood that 5 will be broken?**

$$\binom{20}{5}(0.08)^5(0.92)^{15}\approx 0.0145\approx \boxed{1.45\%}$$

2. **What is the likelihood that they will all work?**

$$\binom{20}{0}(0.08)^0(0.92)^{20}\approx 0.187\approx \boxed{18.7\%}$$

3. **What is the likelihood that they will all be broken?**

$$\binom{20}{20}(0.08)^{20}(0.92)^0\approx 1.15\cdot 10^{-22}\approx \boxed{1.15\times 10^{-20}\%}$$

II. More Examples

A. Example 2: I a shipment of 100 batteries, on average, 3% are defective.

$$n = 100 \qquad p = 0.03 \qquad q = 0.97$$

1. Find the probability that there are exactly 3 defective batteries.

$$\binom{100}{3}(0.03)^3(0.97)^{97} = 0.227 = 22.7\%$$

2. The expected value for a binomial distribution is *n*p*. What is the expected number of defective batteries in this shipment?

$$np = (100)(0.03) = 3\ batteries$$

3. Find the probability that at least two batteries are effective.

There are two ways to do this:

P(2)+P(3)+P(4)+…+P(99)+P(100)

Or

1 – P(0) - P(1)

$$1-\binom{100}{0}(0.03)^0(0.97)^{100}-\binom{100}{1}(0.03)^1(0.97)^{99}$$

$$= 80.5\%$$

4. Example 3 (try in small groups): A basketball player makes 70% of her free throw attempts. What is the probability that she makes **at least** 6 of her next 8 attempts.

Note: we want P(at least 6) = P(6)+P(7)+P(8)

$$P(6)=\binom{8}{6}(0.7)^6(0.3)^2 = 29.6\%$$

$$P(7)=\binom{8}{7}(0.7)^7(0.3)^1 = 19.8\%$$

$$P(8)=\binom{8}{8}(0.7)^8(0.3)^0 = 5.8\%$$

P(at least 6) = 55.2%

HW 1: LM 10-6 #3-8

HW 2: #1-5, 7-10, 14-16