Chapter 6: Probability: What are the Chances?

Section 6.3
Conditional Probability and Independence

The Practice of Statistics, 4th edition – For AP*
STARNES, YATES, MOORE

Section 6.3
Conditional Probability and Independence

Learning Objectives

After this section, you should be able to…

- ✓ DETERMINE probabilities from two-way tables
- ✓ CONSTRUCT Venn diagrams and DETERMINE probabilities
- ✓ DEFINE conditional probability
- ✓ COMPUTE conditional probabilities
- ✓ DESCRIBE chance behavior with a tree diagram
- ✓ DEFINE independent events
- ✓ DETERMINE whether two events are independent
- ✓ APPLY the general multiplication rule to solve probability questions

Two-Way Tables and Probability

When finding probabilities involving two events, a two-way table can display the sample space in a way that makes probability calculations easier.

Consider the example on page 303. Suppose we choose a student at random. Find the probability that the student

	Pierced Ears?		
Gender	**Yes**	**No**	**Total**
Male	19	71	**90**
Female	84	4	**88**
Total	**103**	**75**	**178**

(a) has pierced ears.

(b) is a male with pierced ears.

(c) is a male or has pierced ears.

Define events *A*: is male and *B*: has pierced ears.

(a) Each student is equally likely to be chosen. 103 students have pierced ears. So, *P*(pierced ears) = *P*(*B*) = 103/178.

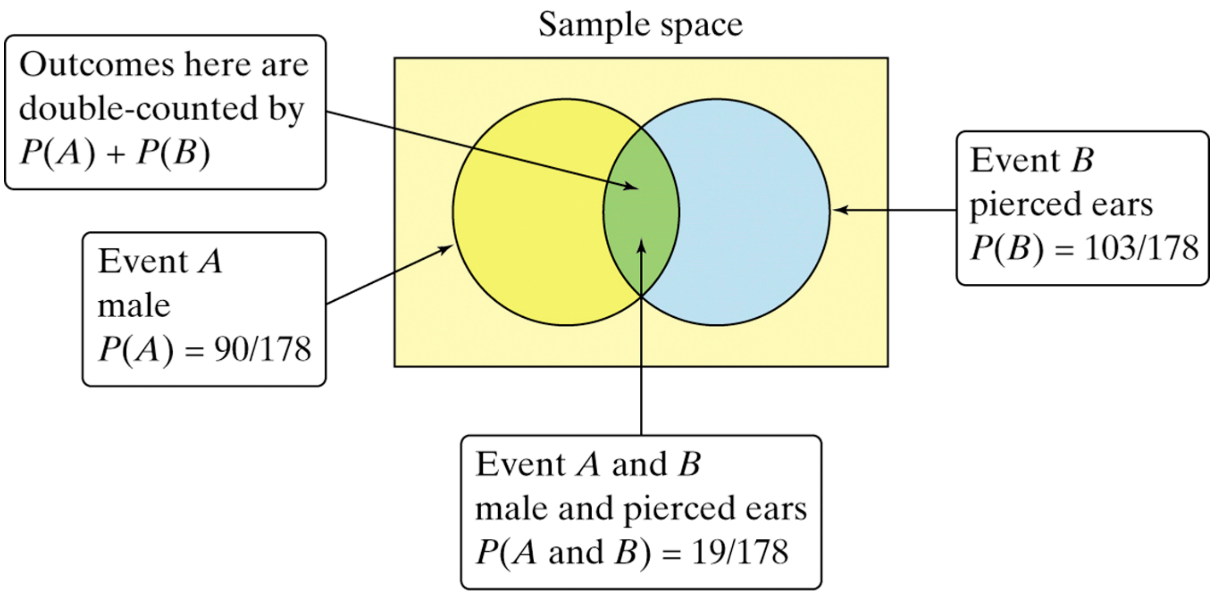
(b) We want to find *P*(male and pierced ears), that is, *P*(*A* and *B*). Look at the intersection of the "Male" row and "Yes" column. There are 19 males with pierced ears. So, *P*(*A* and *B*) = 19/178.

(c) We want to find *P*(male or pierced ears), that is, *P*(*A* or *B*). There are 90 males in the class and 103 individuals with pierced ears. However, 19 males have pierced ears – don't count them twice! *P*(*A* or *B*) = (19 + 71 + 84)/178. So, *P*(*A* or B) = 174/178

Two-Way Tables and Probability

Note, the previous example illustrates the fact that we can't use the addition rule for mutually exclusive events unless the events have no outcomes in common.

The **Venn diagram** below illustrates why.



General Addition Rule for Two Events

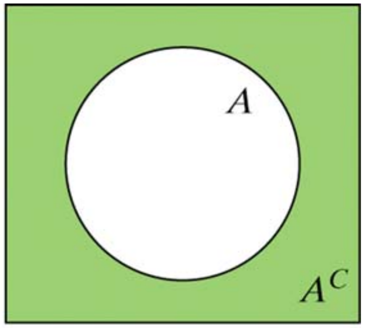
If A and B are any two events resulting from some chance process, then

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

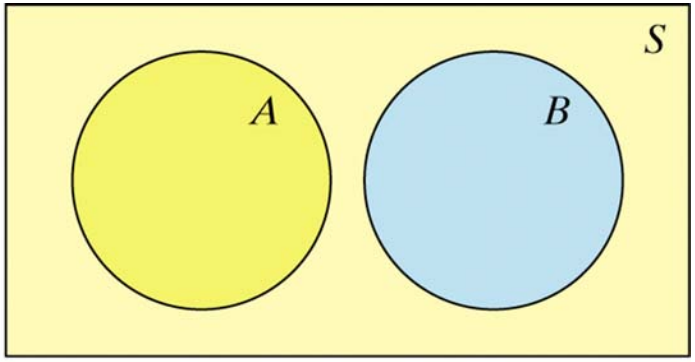
Venn Diagrams and Probability

Because Venn diagrams have uses in other branches of mathematics, some standard vocabulary and notation have been developed.

The complement A^C contains exactly the outcomes that are not in *A*.

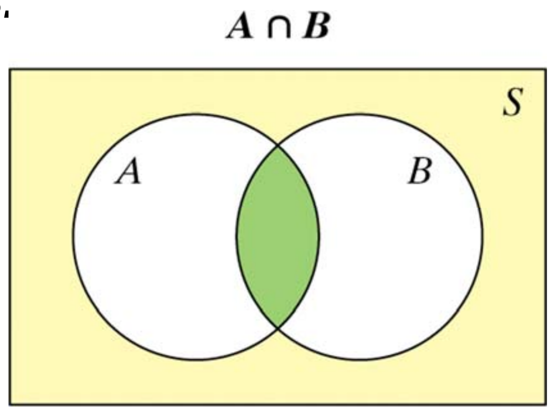


The events *A* and *B* are mutually exclusive (disjoint) because they do not overlap. That is, they have no outcomes in common.

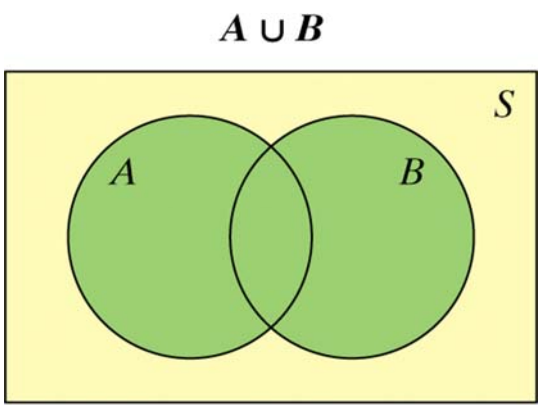


Venn Diagrams and Probability

The intersection of events *A* and *B* (*A* ∩ *B*) is the set of all outcomes in both events *A and B*.



The union of events *A* and *B* (*A* ∪ *B*) is the set of all outcomes in either event *A or B*.

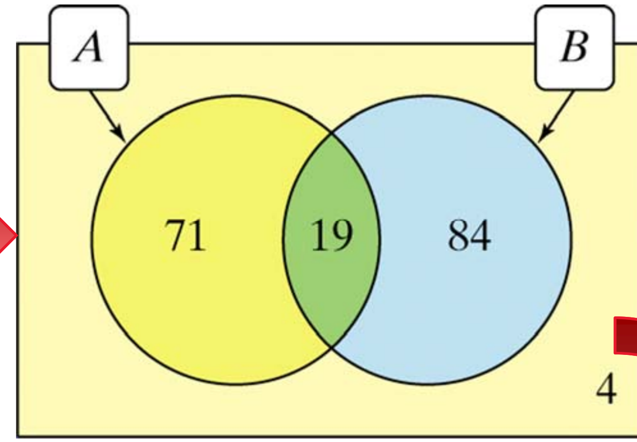


Hint: To keep the symbols straight, remember ∪ for union and ∩ for intersection.

Venn Diagrams and Probability

Recall the example on gender and pierced ears. We can use a Venn diagram to display the information and determine probabilities.

	Pierced Ears?		
Gender	**Yes**	**No**	**Total**
Male	19	71	**90**
Female	84	4	**88**
Total	**103**	**75**	**178**



Define events *A*: is male and *B*: has pierced ears.

Region in Venn diagram	In words	In symbols	Count
In the intersection of two circles	Male and pierced ears	$A \cap B$	19
Inside circle *A*, outside circle *B*	Male and no pierced ears	$A \cap B^C$	71
Inside circle *B*, outside circle *A*	Female and pierced ears	$A^C \cap B$	84
Outside both circles	Female and no pierced ears	$A^C \cap B^C$	4

What is Conditional Probability?

The probability we assign to an event can change if we know that some other event has occurred. This idea is the key to many applications of probability.

When we are trying to find the probability that one event will happen under the *condition* that some other event is already known to have occurred, we are trying to determine a **conditional probability.**

Definition:

The probability that one event happens given that another event is already known to have happened is called a **conditional probability.** Suppose we know that event *A* has happened. Then the probability that event *B* happens *given* that event *A* has happened is denoted by $P(B \mid A)$.

Read | as "given that" or "under the condition that"

Example: Grade Distributions

Consider the two-way table on page 314. Define events

E: the grade comes from an EPS course, and

L: the grade is lower than a B.

	Grade Level			
School	A	B	Below B	Total
Liberal Arts	2,142	1,890	2,268	6300
Engineering and Physical Sciences	368	432	800	1600
Health and Human Services	882	630	588	2100
Total	3392	2952	3656	*10000*

Find *P*(*L*)

$$P(L) = 3656 / 10000 = 0.3656$$

Find *P*(*E* | *L*)

$$P(E \mid L) = 800 / 3656 = 0.2188$$

Find *P*(*L* | *E*)

$$P(L \mid E) = 800 / 1600 = 0.5000$$

Conditional Probability and Independence

When knowledge that one event has happened does not change the likelihood that another event will happen, we say the two events are **independent.**

Definition:

Two events *A* and *B* are **independent** if the occurrence of one event has no effect on the chance that the other event will happen. In other words, events *A* and *B* are independent if

$$P(A \mid B) = P(A) \text{ and } P(B \mid A) = P(B).$$

Example:

	Dominant Hand		
Gender	Right	Left	Total
Male	20	3	**23**
Female	23	4	**27**
Total	**43**	**7**	**50**

Are the events "male" and "left-handed" independent? Justify your answer.

***P*(left-handed | male) = 3/23 = 0.13**

***P*(left-handed) = 7/50 = 0.14**

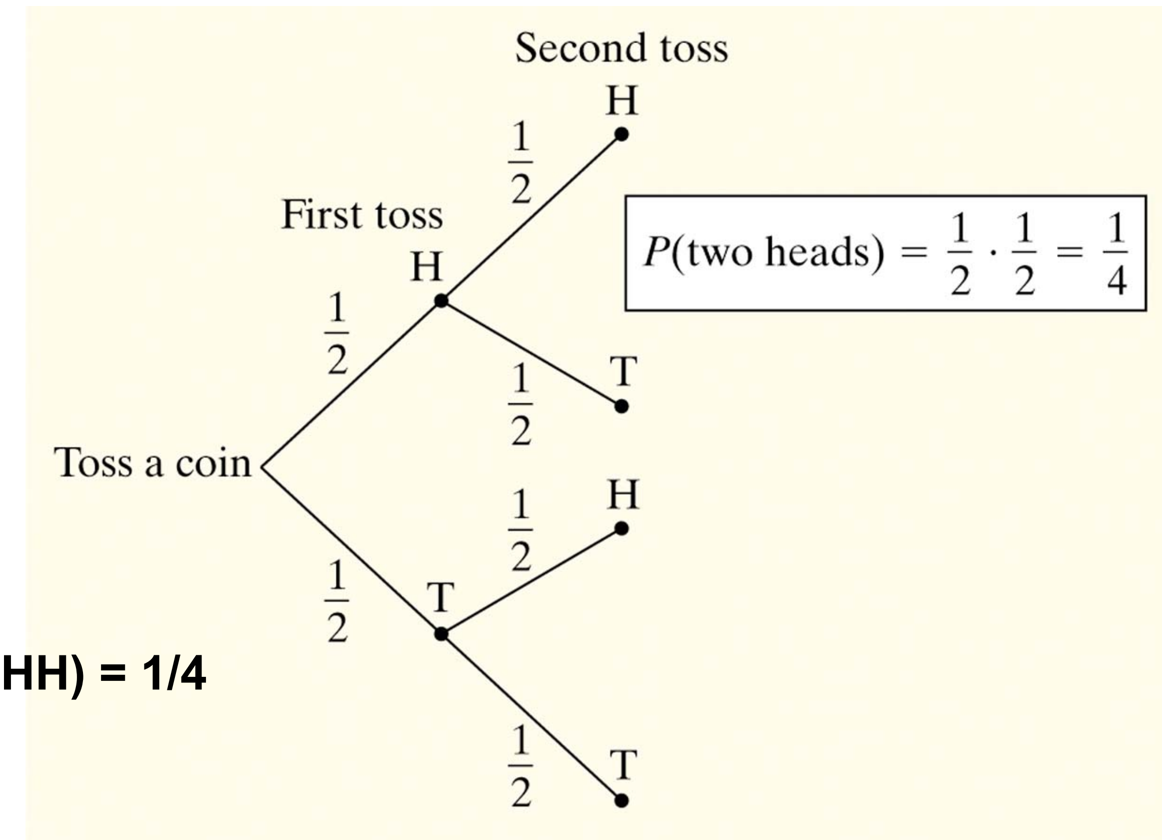
These probabilities are not equal, therefore the events "male" and "left-handed" are not independent.

Tree Diagrams

We learned how to describe the sample space *S* of a chance process in Section 5.2. Another way to model chance behavior that involves a sequence of outcomes is to construct a **tree diagram**.

Consider flipping a coin twice.

What is the probability of getting two heads?



<u>Sample Space:</u>
HH HT TH TT

So, *P*(two heads) = *P*(HH) = 1/4

General Multiplication Rule

The idea of multiplying along the branches in a tree diagram leads to a general method for finding the probability $P(A \cap B)$ that two events happen together.

General Multiplication Rule

The probability that events A and B both occur can be found using the **general multiplication rule**

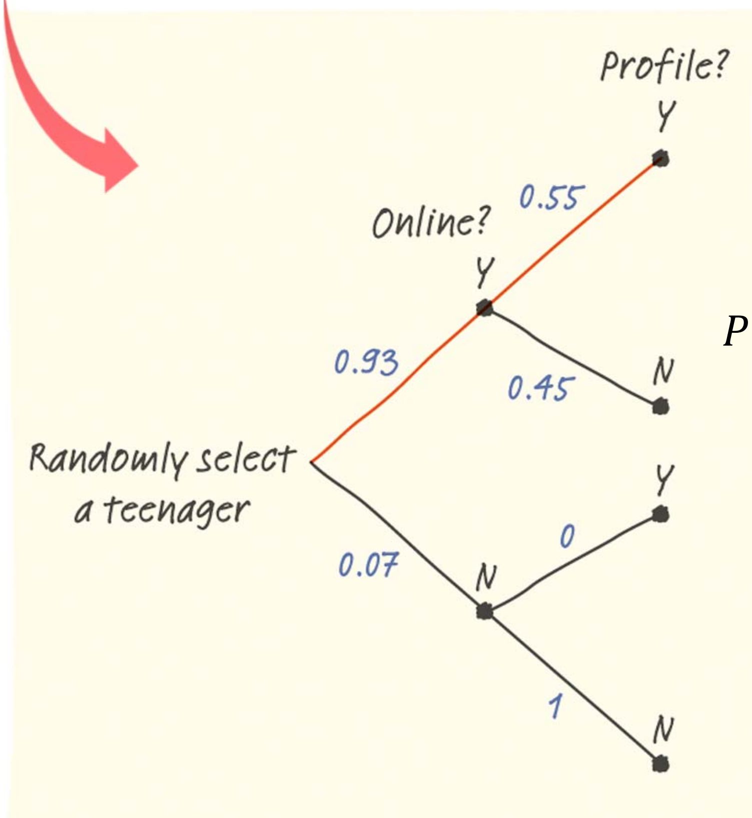
$$P(A \cap B) = P(A) \cdot P(B \mid A)$$

where $P(B \mid A)$ is the conditional probability that event B occurs given that event A has already occurred.

Example: Teens with Online Profiles

The Pew Internet and American Life Project finds that 93% of teenagers (ages 12 to 17) use the Internet, and that 55% of online teens have posted a profile on a social-networking site.

What percent of teens are online *and* have posted a profile?



$$P(\text{online}) = 0.93$$

$$P(\text{profile} \mid \text{online}) = 0.55$$

$$P(\text{online and have profile}) = P(\text{online}) \cdot P(\text{profile} \mid \text{online})$$

$$= (0.93)(0.55)$$

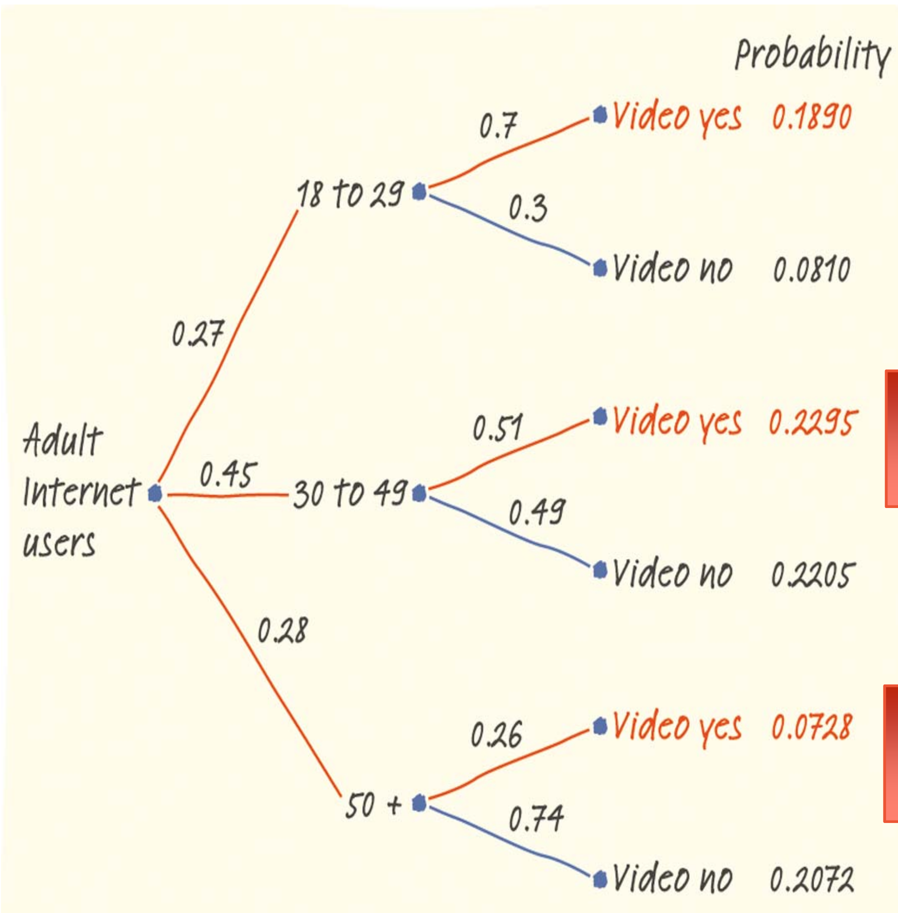
$$= 0.5115$$

51.15% of teens are online *and* have posted a profile.

Example: Who Visits YouTube?

See the example on page 320 regarding adult Internet users.

What percent of all adult Internet users visit video-sharing sites?



P(video yes ∩ 18 to 29) = 0.27 • 0.7 =0.1890

P(video yes ∩ 30 to 49) = 0.45 • 0.51 =0.2295

P(video yes ∩ 50 +) = 0.28 • 0.26 =0.0728

P(video yes) = 0.1890 + 0.2295 + 0.0728 = 0.4913

Independence: A Special Multiplication Rule

When events *A* and *B* are independent, we can simplify the general multiplication rule since $P(B|A) = P(B)$.

Definition:

Multiplication rule for independent events

If *A* and *B* are independent events, then the probability that *A* and *B* both occur is

$$P(A \cap B) = P(A) \cdot P(B)$$

Example:

Following the Space Shuttle *Challenger* disaster, it was determined that the failure of O-ring joints in the shuttle's booster rockets was to blame. Under cold conditions, it was estimated that the probability that an individual O-ring joint would function properly was 0.977. Assuming O-ring joints succeed or fail independently, what is the probability all six would function properly?

***P*(joint1 OK and joint 2 OK and joint 3 OK and joint 4 OK and joint 5 OK and joint 6 OK)**

=P(joint 1 OK) • P(joint 2 OK) • … • P(joint 6 OK)

=(0.977)(0.977)(0.977)(0.977)(0.977)(0.977) = 0.87

Calculating Conditional Probabilities

If we rearrange the terms in the general multiplication rule, we can get a formula for the conditional probability $P(B \mid A)$.

General Multiplication Rule

$$P(A \cap B) = P(A) \cdot P(B \mid A)$$

Conditional Probability Formula

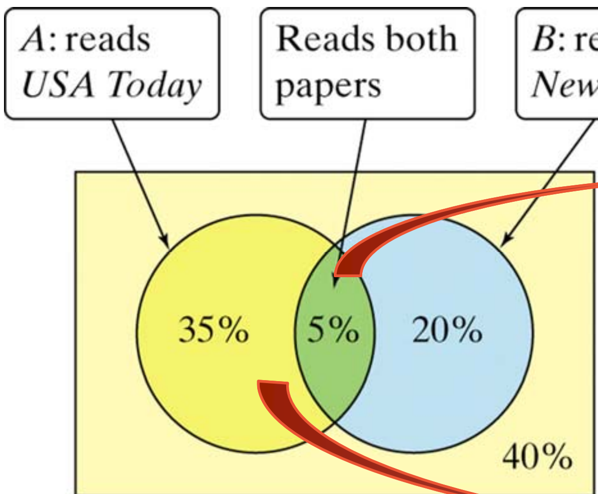
To find the conditional probability P(B | A), use the formula

= ________________

Example: Who Reads the Newspaper?

In Section 5.2, we noted that residents of a large apartment complex can be classified based on the events *A*: reads *USA Today* and *B*: reads the *New York Times*. The Venn Diagram below describes the residents.

What is the probability that a randomly selected resident who reads *USA Today* also reads the *New York Times*?



$$P(B\,|\,A)=\frac{P(A\cap B)}{P(A)}$$

$$P(A\cap B)=0.05$$

$$P(A)=0.40$$

$$P(B\,|\,A)=\frac{0.05}{0.40}=0.125$$

There is a 12.5% chance that a randomly selected resident who reads *USA Today* also reads the *New York Times*.

Section 6.3
Conditional Probability and Independence

Summary

In this section, we learned that…

- ✓ A **two-way table** or a **Venn diagram** can be used to display the sample space for a chance process.
- ✓ The **intersection ($A \cap B$)** of events *A* and *B* consists of outcomes in both *A* and *B*.
- ✓ The **union ($A \cup B$)** of events *A* and *B* consists of all outcomes in event *A*, event *B*, or both.
- ✓ The **general addition rule** can be used to find *P*(*A* or *B*):

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

Section 6.3
Conditional Probability and Independence

Summary

In this section, we learned that…

- ✓ If one event has happened, the chance that another event will happen is a **conditional probability**. $P(B|A)$ represents the probability that event B occurs given that event A has occurred.
- ✓ Events A and B are **independent** if the chance that event B occurs is not affected by whether event A occurs. If two events are mutually exclusive (disjoint), they cannot be independent.
- ✓ When chance behavior involves a sequence of outcomes, a **tree diagram** can be used to describe the sample space.
- ✓ The **general multiplication rule** states that the probability of events A and B occurring together is $P(A \cap B)=P(A) \bullet P(B|A)$
- ✓ In the special case of *independent* events, $P(A \cap B)=P(A) \bullet P(B)$
- ✓ The **conditional probability formula** states $P(B|A) = P(A \cap B) / P(A)$

Homework…

Chapter 6, #'s: 43, 44, 46, 47, 49, 50, 53, 54, 55, 60, 61