

# Chapter 11: Inference for Distributions of Categorical Data

Section 11.1

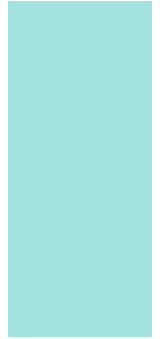
Chi-Square Goodness-of-Fit Tests

The Practice of Statistics, 4<sup>th</sup> edition – For AP\*  
STARNES, YATES, MOORE



## Section 11.1

# Chi-Square Goodness-of-Fit Tests



### Learning Objectives

After this section, you should be able to...

- ✓ COMPUTE expected counts, conditional distributions, and contributions to the chi-square statistic
- ✓ CHECK the Random, Large sample size, and Independent conditions before performing a chi-square test
- ✓ PERFORM a chi-square goodness-of-fit test to determine whether sample data are consistent with a specified distribution of a categorical variable
- ✓ EXAMINE individual components of the chi-square statistic as part of a follow-up analysis

## ■ Introduction

In the previous chapter, we discussed inference procedures for comparing the proportion of successes for two populations or treatments. Sometimes we want to examine the distribution of a single categorical variable in a population. The **chi-square goodness-of-fit test** allows us to determine whether a hypothesized distribution seems valid.

We can decide whether the distribution of a categorical variable differs for two or more populations or treatments using a **chi-square test for homogeneity**. In doing so, we will often organize our data in a two-way table.

It is also possible to use the information in a two-way table to study the relationship between two categorical variables. The **chi-square test for association/independence** allows us to determine if there is convincing evidence of an association between the variables in the population at large.

## ■ **Activity: The Candy Man Can**

- Mars, Incorporated makes milk chocolate candies. Here's what the company's Consumer Affairs Department says about the color distribution of its M&M'S Milk Chocolate Candies:
- *On average, the new mix of colors of M&M'S Milk Chocolate Candies will contain*
  - *13 percent of each of browns and reds,*
  - *14 percent yellows,*
  - *16 percent greens,*
  - *20 percent oranges and*
  - *24 percent blues.*

## ■ Chi-Square Goodness-of-Fit Tests

The **one-way table** below summarizes the data from a **sample bag** of M&M'S Milk Chocolate Candies. In general, one-way tables display the distribution of a categorical variable for the individuals in a sample.

Color	Blue	Orange	Green	Yellow	Red	Brown	Total
Count	9	8	12	15	10	6	60



The sample proportion of blue M&M's is  $\hat{p} = \frac{9}{60} = 0.15$ .

Since the company claims that 24% of all M&M'S Milk Chocolate Candies are blue, we might believe that something fishy is going on. We could use the one-sample z test for a proportion from Chapter 9 to test the hypotheses

$$H_0: p = 0.24$$

$$H_a: p \neq 0.24$$

where  $p$  is the true population proportion of blue M&M'S. We could then perform additional significance tests for each of the remaining colors.

However, performing a one-sample z test for each proportion would be pretty inefficient and would lead to the problem of multiple comparisons.

## ■ Comparing Observed and Expected Counts

More important, performing one-sample z tests for each color wouldn't tell us how likely it is to get a random sample of 60 candies with a color distribution that differs as much from the one claimed by the company as this bag does (taking *all* the colors into consideration at one time).

For that, we need a new kind of significance test, called a **chi-square goodness-of-fit test**.

The null hypothesis in a chi-square goodness-of-fit test should state a claim about the distribution of a single categorical variable in the population of interest. In our example, the appropriate null hypothesis is

$H_0$ : The company's stated color distribution for M&M'S Milk Chocolate Candies is correct.

The alternative hypothesis in a chi-square goodness-of-fit test is that the categorical variable does *not* have the specified distribution. In our example, the alternative hypothesis is

$H_a$ : The company's stated color distribution for M&M'S Milk Chocolate Candies is not correct.

## ■ Comparing Observed and Expected Counts

We can also write the hypotheses in symbols as

$$H_0: p_{blue} = 0.24, p_{orange} = 0.20, p_{green} = 0.16, \\ p_{yellow} = 0.14, p_{red} = 0.13, p_{brown} = 0.13, \\ H_a: \text{At least one of the } p_i\text{'s is incorrect}$$

where  $p_{color}$  = the true population proportion of M&M'S Milk Chocolate Candies of that color.

The idea of the chi-square goodness-of-fit test is this: we compare the **observed counts** from our sample with the counts that would be expected if  $H_0$  is true. The more the observed counts differ from the **expected counts**, the more evidence we have against the null hypothesis.

In general, the expected counts can be obtained by multiplying the proportion of the population distribution in each category by the sample size.

## ■ Example: Computing Expected Counts

A sample bag of M&M's milk Chocolate Candies contained 60 candies. Calculate the expected counts for each color.

Assuming that the color distribution stated by Mars, Inc., is true, 24% of all M&M's milk Chocolate Candies produced are blue.

For random samples of 60 candies, the average number of blue M&M's should be  $(0.24)(60) = 14.40$ . This is our expected count of blue M&M's.

Using this same method, we can find the expected counts for the other color categories:

**Orange:  $(0.20)(60) = 12.00$**

**Green:  $(0.16)(60) = 9.60$**

**Yellow:  $(0.14)(60) = 8.40$**

**Red:  $(0.13)(60) = 7.80$**

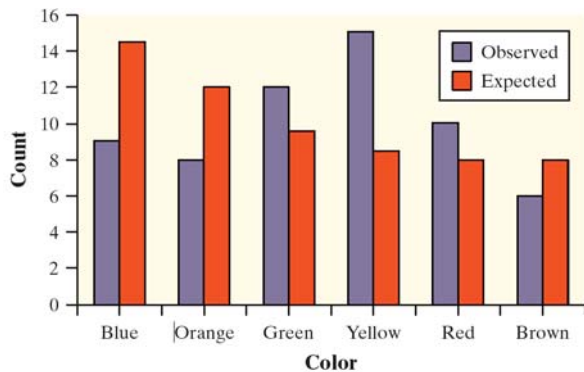
**Brown:  $(0.13)(60) = 7.80$**

Color	Observed	Expected
Blue	9	14.40
Orange	8	12.00
Green	12	9.60
Yellow	15	8.40
Red	10	7.80
Brown	6	7.80



## ■ The Chi-Square Statistic

To see if the data give convincing evidence against the null hypothesis, we compare the observed counts from our sample with the expected counts assuming  $H_0$  is true. If the observed counts are far from the expected counts, that's the evidence we were seeking.



We see some fairly large differences between the observed and expected counts in several color categories. How likely is it that differences this large or larger would occur just by chance in random samples of size 60 from the population distribution claimed by Mars, Inc.?

To answer this question, we calculate a statistic that measures how far apart the observed and expected counts are. The statistic we use to make the comparison is the **chi-square statistic**.

### Definition:

The **chi-square statistic** is a measure of how far the observed counts are from the expected counts. The formula for the statistic is

$$\chi^2 = \sum \frac{(\text{Observed} - \text{Expected})^2}{\text{Expected}}$$

where the sum is over all possible values of the categorical variable.

## ■ Example: Return of the M&M's

The table shows the observed and expected counts for our sample of 60 M&M's Milk Chocolate Candies. Calculate the chi-square statistic.

$$\chi^2 = \sum \frac{(\text{Observed} - \text{Expected})^2}{\text{Expected}}$$

Color	Observed	Expected
Blue	9	14.40
Orange	8	12.00
Green	12	9.60
Yellow	15	8.40
Red	10	7.80
Brown	6	7.80

$$\chi^2 = \frac{(9 - 14.40)^2}{14.40} + \frac{(8 - 12.00)^2}{12.00} + \frac{(12 - 9.60)^2}{9.60} + \frac{(15 - 8.40)^2}{8.40} + \frac{(10 - 7.80)^2}{7.80} + \frac{(6 - 7.80)^2}{7.80}$$

$$\chi^2 = 2.025 + 1.333 + 0.600 + 5.186 + 0.621 + 0.415 = 10.180$$

Think of  $\chi^2$  as a measure of the distance of the observed counts from the expected counts. Large values of  $\chi^2$  are stronger evidence against  $H_0$  because they say that the observed counts are far from what we would expect if  $H_0$  were true. Small values of  $\chi^2$  suggest that the data are consistent with the null hypothesis.

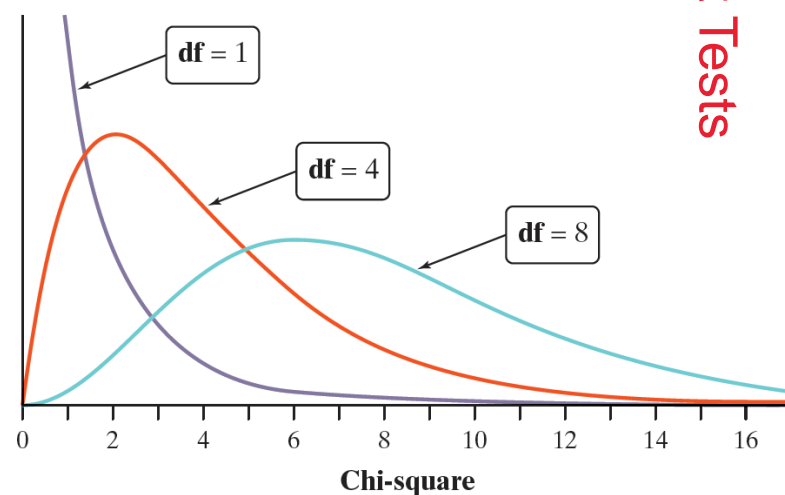
## ■ The Chi-Square Distributions and *P*-Values

The sampling distribution of the chi - square statistic is not a Normal distribution. It is a right - skewed distribution that allows only positive values because  $\chi^2$  can never be negative.

When the expected counts are all at least 5, the sampling distribution of the  $\chi^2$  statistic is close to a **chi - square distribution** with degrees of freedom (df) equal to the number of categories minus 1.

### The Chi-Square Distributions

The chi-square distributions are a family of distributions that take only positive values and are skewed to the right. A particular chi-square distribution is specified by giving its degrees of freedom. The chi-square goodness-of-fit test uses the chi-square distribution with degrees of freedom = the number of categories - 1.



## ■ Example: Return of the M&M's

We computed the chi - square statistic for our sample of 60 M&M's to be  $\chi^2 = 10.180$ . Because all of the expected counts are at least 5, the  $\chi^2$  statistic will follow a chi - square distribution with  $df = 6 - 1 = 5$  reasonably well when  $H_0$  is true.

To find the  $P$  - value, use Table C and look in the  $df = 5$  row.

df	$P$		
	.15	.10	.05
4	6.74	7.78	9.49
5	8.12	9.24	11.07
6	9.45	10.64	12.59

The value  $\chi^2 = 10.180$  falls between the critical values 9.24 and 11.07. The corresponding areas in the right tail of the chi - square distribution with  $df = 5$  are 0.10 and 0.05.

So, the  $P$  - value for a test based on our sample data is between 0.05 and 0.10.

## ■ Example: Return of the M&M's

We computed the chi - square statistic for our sample of 60 M&M's to be  $\chi^2 = 10.180$ . Because all of the expected counts are at least 5, the  $\chi^2$  statistic will follow a chi - square distribution with  $df = 6 - 1 = 5$  reasonably well when  $H_0$  is true.

To find the  $P$  - value, use Table C and look in the  $df = 5$  row.

df	$P$		
	.15	.10	.05
4	6.74	7.78	9.49
5	8.12	9.24	11.07
6	9.45	10.64	12.59

Since our  $P$ -value is between 0.05 and 0.10, it is greater than  $\alpha = 0.05$ . Therefore, we fail to reject  $H_0$ . We don't have sufficient evidence to conclude that the company's claimed color distribution is incorrect.

## ■ Carrying Out a Test

The chi-square goodness-of-fit test uses some approximations that become more accurate as we take more observations. Our rule of thumb is that all expected counts must be at least 5. This **Large Sample Size condition** takes the place of the Normal condition for  $z$  and  $t$  procedures. To use the chi-square goodness-of-fit test, we must also check that the Random and Independent conditions are met.

*Conditions:* Use the chi-square goodness-of-fit test when

- ✓ **Random** The data come from a random sample or a randomized experiment.
- ✓ **Large Sample Size** All expected counts are at least 5.
- ✓ **Independent** Individual observations are independent. When sampling without replacement, check that the population is at least 10 times as large as the sample (the 10% condition).

## ■ Carrying Out a Test

### The Chi-Square Goodness-of-Fit Test

Suppose the Random, Large Sample Size, and Independent conditions are met. To determine whether a categorical variable has a specified distribution, expressed as the proportion of individuals falling into each possible category, perform a test of

$H_0$ : The specified distribution of the categorical variable is correct.

$H_a$ : The specified distribution of the categorical variable is not correct.

We can also write these hypotheses symbolically using  $p_i$  to represent the proportion of individuals that fall in category  $i$ :

$H_0$ :  $p_1 = \text{---}, p_2 = \text{---}, \dots, p_k = \text{---}$ .

$H_a$ : At least one of the  $p_i$ 's is incorrect.

Start by finding the expected count for each category assuming that  $H_0$  is true. Then calculate the chi-square statistic

$$\chi^2 = \sum \frac{(\text{Observed} - \text{Expected})^2}{\text{Expected}}$$

where the sum is over the  $k$  different categories. The  $P$  - value is the area to the right of  $\chi^2$  under the density curve of the chi - square distribution with  $k - 1$  degrees of freedom.

## ■ Carrying Out a Test

### The Chi-Square Goodness-of-Fit Test

Suppose the Random, Large Sample, and Independent conditions are met. To determine whether the observed distribution is significantly different from the specified distribution, we calculate the chi-square test statistic for each possible outcome.

Before we start using the chi-square goodness-of-fit test, we have two important cautions to offer.

1. The chi-square test statistic compares observed and expected *counts*. Don't try to perform calculations with the observed and expected *proportions* in each category.
2. When checking the Large Sample Size condition, be sure to examine the *expected counts*, not the observed counts.

where the sum is the area to the right of  $\chi^2$  under a chi-square distribution with  $k-1$  degrees of freedom.



+ **End of Day 1...**

**Homework...**

Chapter 11

#'s, 5, 6, 7

**Only do the “State and Do” parts of the significance test.  
Complete the “Do” part by hand.**



## ■ Example: When Were You Born?

Are births evenly distributed across the days of the week? The one-way table below shows the distribution of births across the days of the week in a random sample of 140 births from local records in a large city. Do these data give significant evidence that local births are not equally likely on all days of the week?

Day	Sun	Mon	Tue	Wed	Thu	Fri	Sat
Births	13	23	24	20	27	18	15

**State:** We want to perform a test of

$H_0$ : Birth days in this local area are evenly distributed across the days of the week.

$H_a$ : Birth days in this local area are not evenly distributed across the days of the week.

The null hypothesis says that the proportions of births are the same on all days. In that case, all 7 proportions must be  $1/7$ . So we could also write the hypotheses as

$$H_0: p_{Sun} = p_{Mon} = p_{Tues} = \dots = p_{Sat} = 1/7.$$

$H_a$ : At least one of the proportions is not  $1/7$ .

We will use  $\alpha = 0.05$ .

**Plan:** If the conditions are met, we should conduct a chi-square goodness-of-fit test.

- *Random* The data came from a random sample of local births.
- *Large Sample Size* Assuming  $H_0$  is true, we would expect one-seventh of the births to occur on each day of the week. For the sample of 140 births, the expected count for all 7 days would be  $1/7(140) = 20$  births. Since  $20 \geq 5$ , this condition is met.
- *Independent* Individual births in the random sample should occur independently (assuming no twins). Because we are sampling without replacement, there need to be at least  $10(140) = 1400$  births in the local area. This should be the case in a large city.



## Example: When Were You Born?

**Do:** Since the conditions are satisfied, we can perform a chi-square goodness-of-fit test. We begin by calculating the test statistic.

**Test statistic:**

$$\chi^2 = \sum \frac{(\text{Observed} - \text{Expected})^2}{\text{Expected}}$$

$$= \frac{(13-20)^2}{20} + \frac{(23-20)^2}{20} + \frac{(24-20)^2}{20} + \frac{(20-20)^2}{20}$$

$$+ \frac{(27-20)^2}{20} + \frac{(18-20)^2}{20} + \frac{(15-20)^2}{20}$$

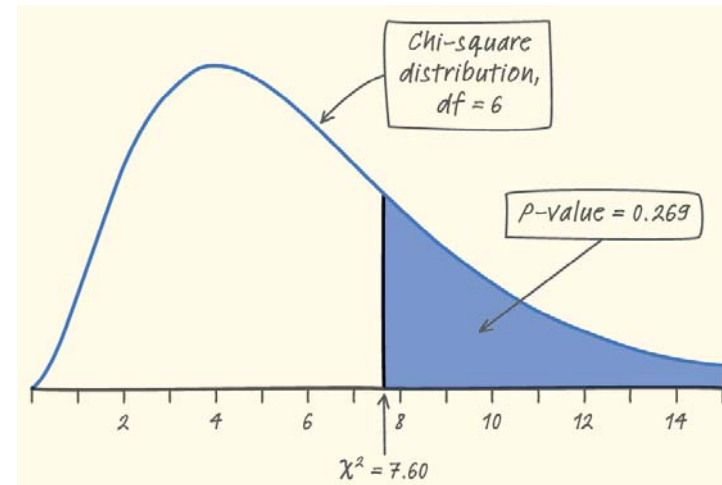
$$= 2.45 + 0.45 + 0.80 + 0.00 + 2.45 + 0.20 + 1.25$$

$$= 7.60$$

**P-Value:**

*Using Table C:*  $\chi^2 = 7.60$  is less than the smallest entry in the  $df = 6$  row, which corresponds to tail area 0.25. The  $P$ -value is therefore greater than 0.25.

*Using technology:* We can find the exact  $P$ -value with a calculator:  $\chi^2\text{cdf}(7.60, 1000, 6) = 0.269$ .



Chi-Square Goodness-of-Fit Tests

**Conclude:** Because the  $P$ -value, 0.269, is greater than  $\alpha = 0.05$ , we fail to reject  $H_0$ . These 140 births don't provide enough evidence to say that all local births in this area are not evenly distributed across the days of the week.



## ■ Example: Inherited Traits

Biologists wish to cross pairs of tobacco plants having genetic makeup  $Gg$ , indicating that each plant has one dominant gene ( $G$ ) and one recessive gene ( $g$ ) for color. Each offspring plant will receive one gene for color from each parent.

		Parent 2 passes on:	
		$G$	$g$
Parent 1 passes on:	$G$	$GG$	$Gg$
	$g$	$Gg$	$gg$

The Punnett square suggests that the expected ratio of green ( $GG$ ) to yellow-green ( $Gg$ ) to albino ( $gg$ ) tobacco plants should be 1:2:1. In other words, the biologists predict that 25% of the offspring will be green, 50% will be yellow-green, and 25% will be albino.

To test their hypothesis about the distribution of offspring, the biologists mate 84 randomly selected pairs of yellow-green parent plants.

Of 84 offspring, 23 plants were green, 50 were yellow-green, and 11 were albino.

Do these data differ significantly from what the biologists have predicted? Carry out an appropriate test at the  $\alpha = 0.05$  level to help answer this question.



## ■ Example: Inherited Traits

**State:** We want to perform a test of

$H_0$ : The biologists' predicted color distribution for tobacco plant offspring is correct.

That is,  $p_{green} = 0.25$ ,  $p_{yellow-green} = 0.5$ ,  $p_{albino} = 0.25$

$H_a$ : The biologists' predicted color distribution isn't correct. That is, at least one of the stated proportions is incorrect.

We will use  $\alpha = 0.05$ .

**Plan:** If the conditions are met, we should conduct a chi-square goodness-of-fit test.

- *Random* The data came from a random sample of local births.
- *Large Sample Size* We check that all expected counts are at least 5. Assuming  $H_0$  is true, the expected counts for the different colors of offspring are green:  $(0.25)(84) = 21$ ; yellow-green:  $(0.50)(84) = 42$ ; albino:  $(0.25)(84) = 21$   
The complete table of observed and expected counts is shown below.

- *Independent* Individual offspring inherit their traits independently from one another. Since we are sampling without replacement, there would need to be at least  $10(84) = 840$  tobacco plants in the population. This seems reasonable to believe.

Offspring color	Observed	Expected
Green	23	21
Yellow-green	50	42
Albino	11	21

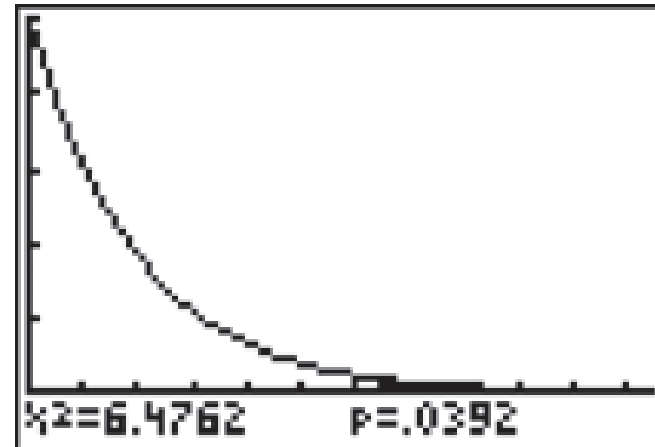


## ■ Example: Inherited Traits

**Do:** Since the conditions are satisfied, we can perform a chi-square goodness-of-fit test. We begin by calculating the test statistic.

**Test statistic :**

$$\begin{aligned}\chi^2 &= \sum \frac{(\text{Observed} - \text{Expected})^2}{\text{Expected}} \\ &= \frac{(23 - 21)^2}{21} + \frac{(50 - 42)^2}{50} + \frac{(11 - 21)^2}{21} \\ &= 6.476\end{aligned}$$



Chi-Square Goodness-of-Fit Tests

**P-Value:**

Note that  $df = \text{number of categories} - 1 = 3 - 1 = 2$ . Using  $df = 2$ , the  $P$ -value from the calculator is 0.0392

**Conclude:** Because the  $P$ -value, 0.0392, is less than  $\alpha = 0.05$ , we will reject  $H_0$ . We have convincing evidence that the biologists' hypothesized distribution for the color of tobacco plant offspring is incorrect.

## ■ Follow-up Analysis

In the chi-square goodness-of-fit test, we test the null hypothesis that a categorical variable has a specified distribution. If the sample data lead to a statistically significant result, we can conclude that our variable has a distribution different from the specified one.

When this happens, start by examining which categories of the variable show large deviations between the observed and expected counts.

Then look at the individual terms that are added together to produce the test statistic  $\chi^2$ . These **components** show which terms contribute most to the chi-square statistic.

In the tobacco plant example, we can see that the component for the albino offspring made the largest contribution to the chi - square statistic.

$$\chi^2 = \frac{(23-21)^2}{21} + \frac{(50-42)^2}{50} + \frac{(11-21)^2}{21}$$

$$= 0.190 + 1.524 + 4.762 = 6.476$$

Offspring color	Observed	Expected
Green	23	21
Yellow-green	50	42
Albino	11	21



## Section 11.1

# Chi-Square Goodness-of-Fit Tests

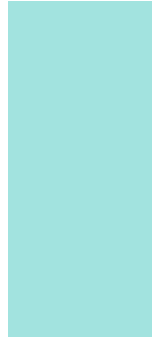
### Summary

In this section, we learned that...

- ✓ A **one-way table** is often used to display the distribution of a categorical variable for a sample of individuals.
- ✓ The **chi-square goodness-of-fit test** tests the null hypothesis that a categorical variable has a specified distribution.
- ✓ This test compares the **observed count** in each category with the counts that would be expected if  $H_0$  were true. The **expected count** for any category is found by multiplying the specified proportion of the population distribution in that category by the sample size.
- ✓ The **chi-square statistic** is

$$\chi^2 = \sum \frac{(\text{Observed} - \text{Expected})^2}{\text{Expected}}$$

where the sum is over all possible values of the categorical variable.





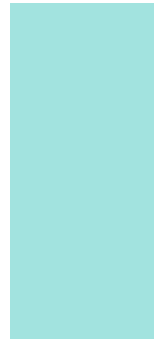


## Section 11.1

# Chi-Square Goodness-of-Fit Tests

### Summary

- ✓ The test compares the value of the statistic  $\chi^2$  with critical values from the **chi-square distribution with** degrees of freedom  $df = \text{number of categories} - 1$ . Large values of  $\chi^2$  are evidence against  $H_0$ , so the  $P$ -value is the area under the chi-square density curve to the right of  $\chi^2$ .
- ✓ The chi-square distribution is an approximation to the sampling distribution of the statistic  $\chi^2$ . *You can safely use this approximation when all expected cell counts are at least 5 (Large Sample Size condition).*
- ✓ Be sure to check that the Random, Large Sample Size, and Independent conditions are met before performing a chi-square goodness-of-fit test.
- ✓ If the test finds a statistically significant result, do a follow-up analysis that compares the observed and expected counts and that looks for the largest **components** of the chi-square statistic.





## End of Day 2...

### Homework...

Chapter 11

#'s, 15, 16, 17 (Complete the 4 part significance test)

The AP readers are not as concerned about you using the formula to do all of the work by hand for the “DO” part as they are for other tests. You can just write down  $\chi^2 = \sum (\text{observed} - \text{expected})^2 / \text{expected}$ .

Observed = { , , , , , }    Expected = { , , , , , }

Then you can use your calculator to do all of the work to find your test statistic and p-value.