

5-5: Theorems About Roots of Polynomial Equations

Algebra 2
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Rational Root Theorem

Let $P(x) = a_nx^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$ be a polynomial with integer coefficients. There are a limited number of possible roots of $P(x) = 0$.

- Integer roots must be factors of a_0 .
- Rational roots must have reduced form $\frac{p}{q}$ where p is an integer factor of a_0 and q is an integer factor of a_n .

Factors of the leading coefficient:
 $\pm 1, \pm 3, \pm 7, \pm 21$

Factors of the constant:
 $\pm 1, \pm 2, \pm 5, \pm 10$

$$21x^2 + 29x + 10 = 0$$

$$\frac{\text{Factors of constant term (10)}}{\text{Factors of leading coefficient (21)}}$$

• Use synthetic division and each combination of factors to find roots •

Ex. 1: What are the rational roots of $x^3 - 2x^2 - 5x + 6 = 0$?

Factors of a_0 (6): $\pm 1, \pm 2, \pm 3, \pm 6$

Factors of a_n (1): ± 1

Factors of a_0 (6): $\pm 1, \pm 2, \pm 3, \pm 6$
 Factors of a_n (1): $\pm 1, \pm 1, \pm 1, \pm 1$

Possible Roots: $\pm 1, \pm 2, \pm 3, \pm 6$

Test possible roots using synthetic division & $x^3 - 2x^2 - 5x + 6 = 0$?

$$\begin{array}{r|rrrr} -1 & 1 & -2 & -5 & 6 \\ & & -1 & 3 & 2 \\ \hline & 1 & -3 & -2 & 8 \end{array}$$

Remainder $\neq 0$; Not a root.

$$\begin{array}{r|rrrr} 1 & 1 & -2 & -5 & 6 \\ & & 1 & -1 & -6 \\ \hline & 1 & -1 & -6 & 0 \end{array}$$

Remainder = 0; Is a root.

Rewrite $x^3 - 2x^2 - 5x + 6 = 0$ as $(x - 1)(x^2 - x - 6) = 0$ and use appropriate method to solve the quadratic (factoring, Quadratic Formula, etc.).

Factor $x^2 - x - 6 \Rightarrow (x - 3)(x + 2)$

Linear Factors:

$(x - 1)(x - 3)(x + 2)$

Zeros: $x = -2, 1, 3$

Rational Root Theorem Summary

1. Find all factors of constant term and lead coefficient.
2. Put them all in the ratio $\frac{\text{Constant Factors}}{\text{Lead Coeff. Factors}}$
3. Use synthetic division to find the roots
 - a) Repeat until you can solve the remaining polynomial or until you have a second-degree polynomial and can solve by factoring or the Quadratic Formula.

Conjugate Root Theorem

If $P(x)$ is a polynomial with rational coefficients, then irrational roots of $P(x) = 0$ that have the form $a + \sqrt{b}$ occur in conjugate pairs. That is, if $a + \sqrt{b}$ is an irrational root with a and b rational, then $a - \sqrt{b}$ is also a root.

If $P(x)$ is a polynomial with real coefficients, then the complex roots of $P(x) = 0$ occur in conjugate pairs. That is, if $a + bi$ is an irrational root with a and b real, then $a - bi$ is also a root.

Ex. 2: A quintic polynomial $P(x)$ has rational coefficients. If $1, \sqrt{3}$ and $2 - 3i$ are roots of $P(x) = 0$, what are the remaining roots?

$-\sqrt{3}$ and $2 + 3i$. 1 is neither irrational or complex so doesn't have a conjugate.

Ex. 3: What polynomial function $P(x)$ with rational coefficients so that $P(x) = 0$ has roots 4 and $3i$? Since $3i$ is a root, so is $-3i$.

Find the linear factors and write the polynomial in factored form.

$$(x-4)(x-3i)(x+3i) = 0$$

Multiply the linear factors and write the polynomial in standard form.

$$(x-4)(x-3i)(x+3i) = 0$$

$$(x-4)(x^2+9) = 0$$

$$x^3 - 4x^2 + 9x - 36 = 0$$