1-3: Continuity, End Behavior and Limits

ADV PRECALCULUS

MR. GALLO

Analytical Look at Limits

Given the following function, what happens to f(x) as x gets closer to 3?

$$f(x)=\frac{x^2-5x+6}{x-3}$$

as $x \to 3$, $f(x) \to 1$

Given the following function, what happens to g(x) as x gets closer to 3?

$$g(x)=x-2$$

as $x \to 3$, $g(x) \to 1$

Given the following function, what happens to h(x) as x gets closer to 3?

$$h(x)=\begin{cases}\frac{x^2-5x+6}{x-3}, & x\neq 3\\ 7, & x=3\end{cases}$$

as $x \to 3$, $h(x) \to 1$

In general:

$\lim_{x\to a} f(x)$ means "as x gets closer to the value a, the values of f are approaching what y value?

Intro to Limits

Methods of finding limits

Try these in this order when finding limits of rational functions:

1. Direct substitution
2. Factor and Simplify (then use direct substitution)
3. One-Sided Limits – examine graph

Example 1: Method – Direct Substitution

Evaluate $\lim_{x \to 1} \frac{x^2 + x + 2}{x+1} = \frac{1^2 + 1 + 2}{1+1}$

$= \frac{4}{2} = 2$

Therefore, $\lim_{x \to 1} \frac{x^2 + x + 2}{x+1} = 2$

Try these:

$\lim_{x \to -5} \frac{x+1}{x^2+3}$ — $-\frac{1}{7}$

$\lim_{x \to 4} \left(x^3 - 3x^2 - 5x + 7\right)$ — 3

$\lim_{x \to -8} \sqrt{x+6}$ — Not possible, $f(-8) = \sqrt{-2}$

Example 2: Method – Simplify

Evaluate $\lim_{x \to 1} \frac{x^3 - 1}{x - 1}$

Direct Substitution: $\frac{1^3 - 1}{1 - 1} = \frac{0}{0}$ No

Factor & Simplify: $\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = \frac{(x-1)(x^2 + x + 1)}{x - 1}$

$\lim_{x \to 1} (x^2 + x + 1) = 1^2 + 1 + 1 = 3$

Try these:

$\lim_{x \to -2} \frac{x^3 - 3x^2 - 4x + 12}{x + 2}$ 20

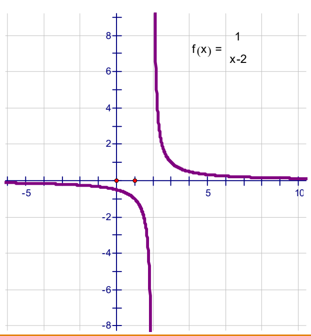
$\lim_{x \to 6} \frac{x^2 - 7x + 6}{3x^2 - 11x - 42}$ $\frac{1}{5}$

Example 3: Method – One-Sided Limits

Evaluate $\lim_{x \to 2} \frac{1}{x - 2} =$

1. Direct substitution? $\frac{1}{2 - 2} = \frac{1}{0}$ No

2. Simplify? Can't Simplify Further



3. Two-sided limits (look at the graph)

$\lim_{x \to 2^+} \frac{1}{x - 2} = \infty$ $\lim_{x \to 2^-} \frac{1}{x - 2} = -\infty$

Since limits ≠, the limit DNE

Two-Sided Limits

A function *f(x)* has a two-sided limit if and only if the left and right hand limits equal the same number (**L** is a real number, not ±∞)

$$\lim_{x \to c} f(x) = L \Leftrightarrow \lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = L$$

If $\lim_{x \to c^+} f(x) \neq \lim_{x \to c^-} f(x)$, then $\lim_{x \to c} f(x) = DNE$

More Examples

Find $\lim_{x \to 3} \frac{x+3}{x^2-9}$

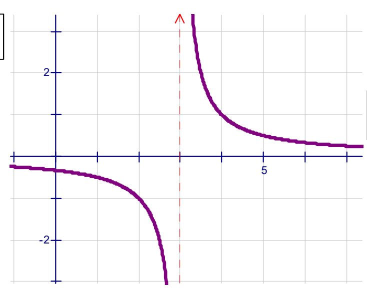
Direct Substitution $\frac{3+3}{3^2-9} = \frac{6}{0}$ No

Factor & Simplify $\frac{x+3}{(x+3)(x-3)} = \frac{1}{x-3} = \frac{1}{3-3} = \frac{1}{0}$ No

One Sided Limits

$\lim_{x \to 3^+} \frac{1}{x-3} = \infty$

$\lim_{x \to 3^-} \frac{1}{x-3} = -\infty$



The limit DNE

Asymptotes and Limits

A. Vertical Asymptotes:

$x = a$ is a vertical asymptote if either

$$\lim_{x \to a^+} f(x) = \pm\infty \text{ or } \lim_{x \to a^-} f(x) = \pm\infty$$

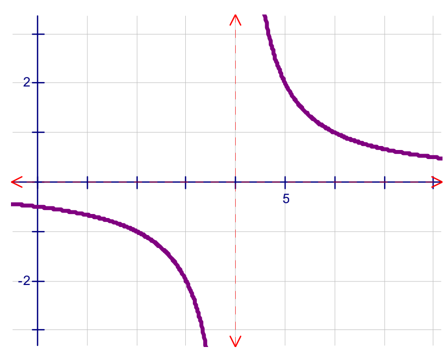
B. Horizontal Asymptotes:

$y = b$ is a horizontal asymptote if either

$$\lim_{x \to \infty} f(x) = b \text{ or } \lim_{x \to -\infty} f(x) = b$$

More Examples – Identify Asymptotes

Examine $f(x) = \dfrac{2}{x-4}$



1. $\lim_{x \to \infty} f(x) = 0$
2. $\lim_{x \to -\infty} f(x) = 0$

Horizontal Asymptote $y = 0$

3. $\lim_{x \to 4^+} f(x) = \infty$
4. $\lim_{x \to 4^-} f(x) = -\infty$

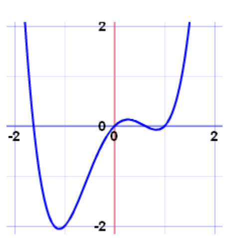
Vertical Asymptote $x = 4$

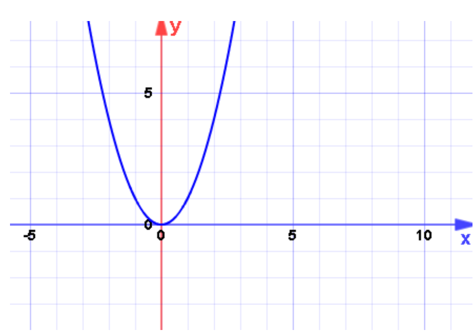
Homework: p.754 #13, 15-20 (Use one of the three methods to evaluate the limits), 23, 25, 27, 30, 32, 35-45 odd

What is a Continuity?

A function is <u>continuous</u> if you can draw or trace the function without lifting your pencil from the paper.

For example:

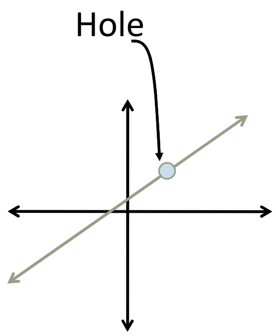




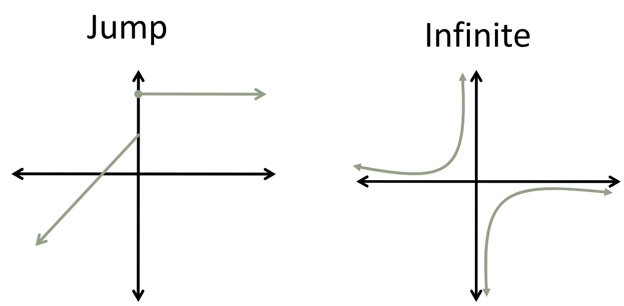
Two Important Discontinuities

A function is **discontinuous** if when drawing or tracing it, you must lift your pencil off the paper to continue drawing or tracing.

A. Removable



B. Non-Removable



Example 1

Is this function, $f(x)=\frac{x^2-16}{x+4}$, continuous?

What is the value of the function when x = -4?

$$f(-4)=\frac{x^2-16}{x+4}=\frac{0}{0}$$

- What are the restrictions for this function?

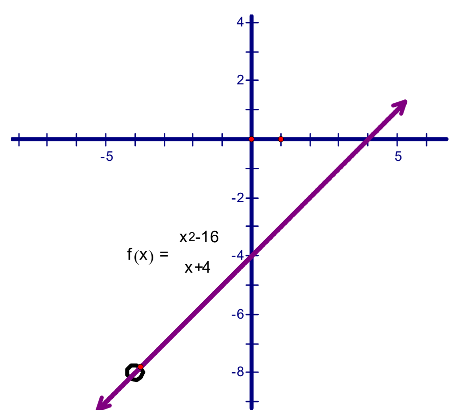
$$(-\infty,-4)\cup(-4,\infty) \quad \{x : x \neq -4, x \in \mathbb{R}\}$$

- Factor to simplify the function – what results?

$$\frac{x^2-16}{x+4}=\frac{(x+4)(x-4)}{x+4}=x-4$$

- What is the value of this simplified function at x = -4?

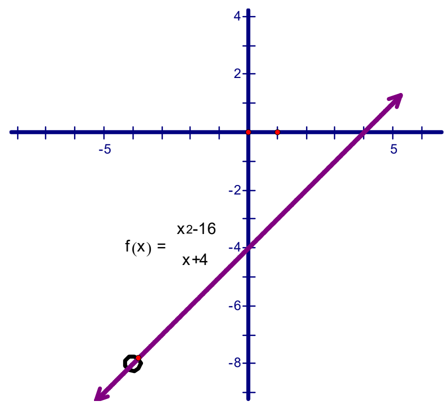
$$-4-4=-8$$



Removable Discontinuities

In Summary: For the graph $f(x) = \frac{x^2-16}{x+4}$

1. This is a line with a hole at *x=-4*.
2. It is discontinuous at *x=-4.*
3. *-4* is a point of discontinuity.
4. This discontinuity is ***removable***, because it can be 'filled' with one point (-4,-8)
5. This is the continuous extension of *f(x)*.



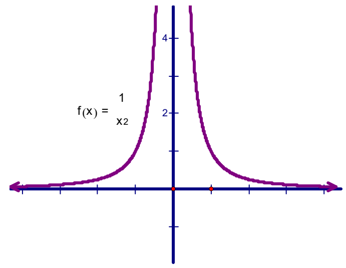
$$g(x)=\begin{cases}\frac{x^2-16}{x+4}; x\neq -4\\ -8; x=-4\end{cases}$$

Infinite Discontinuities

A discontinuity that cannot be removed by insertion of a single point is called an ***infinite discontinuity*** (most of the time these are vertical asymptotes).

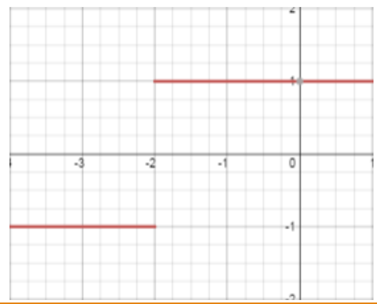
Discontinuities

Example 2: graph $\mathbf{f(x)} = \frac{\mathbf{1}}{\boldsymbol{x^2}}$



Infinite Discontinuity

Example 3: graph $\boldsymbol{f(x)} = \frac{\boldsymbol{x+2}}{\boldsymbol{|x+2|}}$



Jump Discontinuity

Test for Continuity

A function $f(x)$ is continuous at $x = c$ if it satisfies the following conditions:

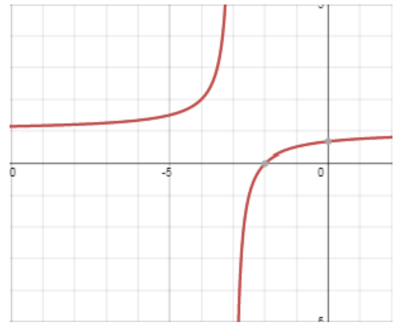
$f(x)$ is defined at c. That is, $f(c)$ exists.

$f(x)$ approaches the same value from either side, That is, $\lim_{x \to c} f(x)$ exists.

The value that $f(x)$ approaches from each side of c is $f(c)$. That is, $\lim_{x \to c} f(x) = f(c)$

Test for Continuity

Is $f(x) = \frac{x^2+3x+2}{x^2+4x+3}$ continuous at $x=-1$.



1. Direct substitution yields $\frac{0}{0}$.

2. Factor: $f(x) = \frac{x^2+3x+2}{x^2+4x+3} = \frac{(x+2)(x+1)}{(x+3)(x+1)}$

   - Now find $f(-1) = \frac{-1+2}{-1+3} = \frac{1}{2}$

3. $\lim_{x \to -1^-} f(x) = \frac{1}{2}$ $\lim_{x \to -1^+} f(x) = \frac{1}{2}$

4. Fill hole: at $x=-1$: $\frac{x+2}{x+3} = \frac{-1+2}{-1+3} = \frac{1}{2}$ $g(x) = \begin{cases} \frac{x^2+3x+2}{x^2+4x+3} & \text{for } x \neq -1 \\ \frac{1}{2} & \text{for } x = -1 \end{cases}$

Homework: p.30 #1-12, 58, 59

End Behavior and Limits

End Behavior

- how a function behaves as x approaches infinity ∞ (on the right) or negative infinity -∞ (on the left).
- Can be represented by ***limit notation***:

$$\lim_{x\to\infty} f(x)$$

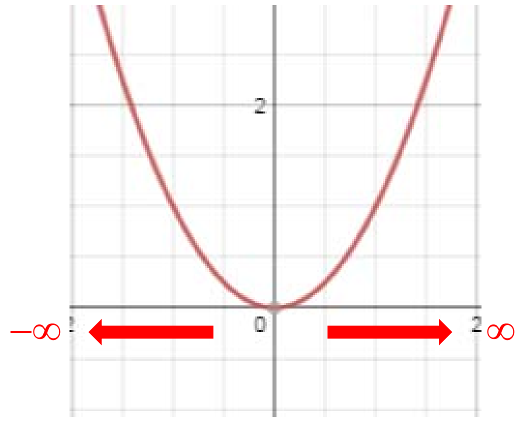
- Read "the limit as $f(x)$ goes to infinity"
- x values are growing very large.
- Look at limits at both ends:

Left-End Behavior $-\infty$

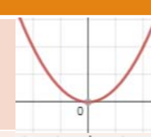
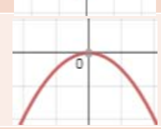
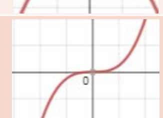
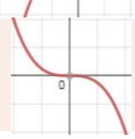
$$\lim_{x\to-\infty} f(x)$$

Right-End Behavior ∞

$$\lim_{x\to\infty} f(x)$$



End Behavior and Limits

Sign of Lead Coefficient	Functions	End Behavior	Graph
Positive **+**	$x^2\ x^4\ x^6$ Even	Up and Up	
Negative **-**	$x^2\ x^4\ x^6$ Even	Down and Down	
Positive **+**	$x\ x^3\ x^5$ Odd	Down and Up	
Negative **-**	$x\ x^3\ x^5$ Odd	Up and Down	

Describe End Behavior

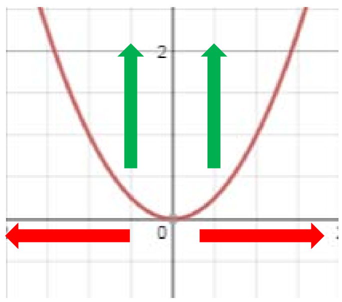
Describe the end behavior of: $y = x^2$

$\lim_{x \to \infty} x^2 = \infty$

What are the y-values doing as the x-values grow larger?

$\lim_{x \to -\infty} x^2 = \infty$

What are the y-values doing as the x-values grow smaller?



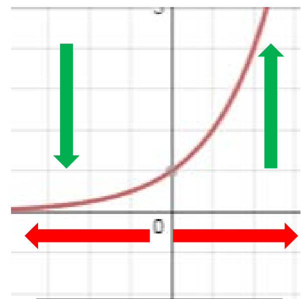
Describe the end behavior of: $y = 2^x$

$\lim_{x \to \infty} 2^x = \infty$

What are the y-values doing as the x-values grow larger?

$\lim_{x \to -\infty} 2^x = 0$

What are the y-values doing as the x-values grow smaller?



Asymptote at $y = 0$

If the limit approaches a constant as x goes to + or – infinity, we say it has a horizontal asymptote at y = that constant.

Describe End Behavior

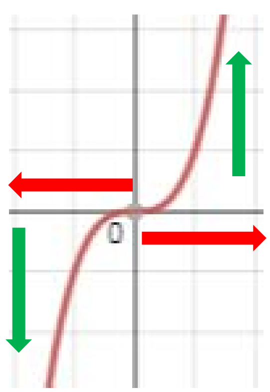
Describe the end behavior of: $y = x^3$

$\lim_{x \to \infty} x^3 = \infty$

What are the y-values doing as the x-values grow larger?

$\lim_{x \to -\infty} x^3 = -\infty$

What are the y-values doing as the x-values grow smaller?



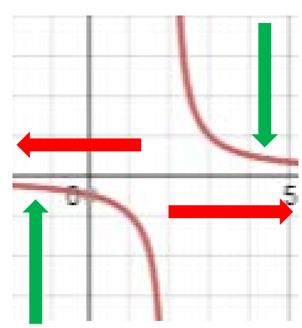
Describe the end behavior of: $f(x) = \frac{1}{x-2}$

$\lim_{x \to \infty} f(x) = 0$

What are the y-values doing as the x-values grow larger?

$\lim_{x \to -\infty} f(x) = 0$

What are the y-values doing as the x-values grow smaller?



Asymptote at $y = 0$

Describe End Behavior

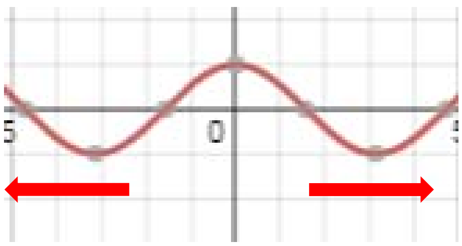
Describe the end behavior of:

$$f(x) = \cos x$$

$\lim_{x\to\infty} f(x) =$ *DNE* $\lim_{x\to-\infty} f(x) =$ *DNE*

What are the y-values doing as the x-values grow larger?

What are the y-values doing as the x-values grow smaller?



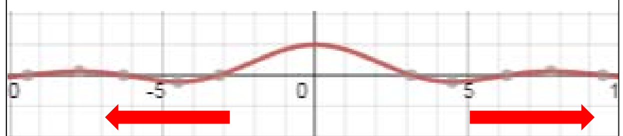
Describe the end behavior of:

$$f(x) = \frac{\sin x}{x}$$

$\lim_{x\to\infty} f(x) = 0$ $\lim_{x\to-\infty} f(x) = 0$

What are the y-values doing as the x-values grow larger?

What are the y-values doing as the x-values grow smaller?



End Behavior

Can find end behavior by: $\dfrac{\text{Term w/ Greatest Power (N)}}{\text{Term w/ Greatest Power (D)}} = \text{End Behavior}$

$f(x) = -x^4 + 8x^3 + 3x^2 + 6x - 80$

End behavior will mimic the greatest power $-x^4$ so:

$\lim_{x\to\infty} f(x) = -\infty$

$\lim_{x\to-\infty} f(x) = -\infty$

$f(x) = \dfrac{x}{x^2 - 2x + 8}$

When x is in the denominator, the limit will always go to 0. $\dfrac{\text{Greatest Power (N)}}{\text{Greatest Power (D)}} = \dfrac{x}{x^2} = \dfrac{1}{x}$

$\lim_{x\to\infty} f(x) = 0$

$\lim_{x\to-\infty} f(x) = 0$

End Behavior

$$f(x)=\frac{3x-2}{x+1} \qquad \frac{\textit{Greatest Power (N)}}{\textit{Greatest Power (D)}}=\frac{3x}{x}=3$$

$$\lim_{x\to\infty} f(x)= \quad 3$$

$$\lim_{x\to-\infty} f(x)= \quad 3$$

Try these:

$$f(x)=5^x$$

$$\lim_{x\to\infty} f(x)=\infty$$

$$\lim_{x\to-\infty} f(x)=0$$

$$f(x)=x^3+3x+1$$

$$\lim_{x\to\infty} f(x)=\infty$$

$$\lim_{x\to-\infty} f(x)=-\infty$$

$$f(x)=\frac{2x^2+3x}{x^2+4}+3$$

$$\lim_{x\to\infty} f(x)=5$$

$$\lim_{x\to-\infty} f(x)=5$$

Possible End Behaviors

There are three possible end behaviors:

1. The values of *f(x)* can increase or decrease without bound (to ∞ or $-\infty$)
2. The values of *f(x)* can approach some number *L*
3. The values of *f(x)* can follow neither of these patterns (such as oscillating between two values).

Homework: p.30 #22-29, 33-40, 62-65

Piecewise Functions

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Piecewise Functions

Piecewise function

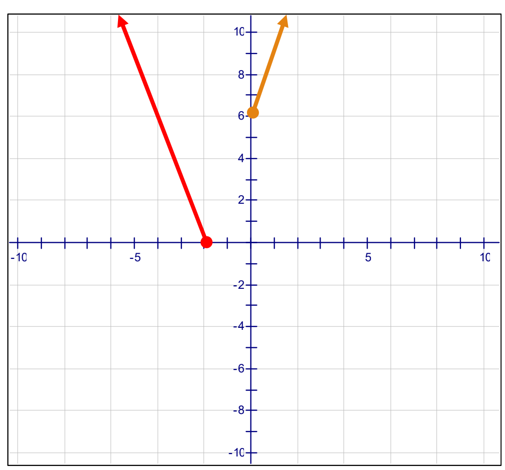
- function whose definition changes depending on the value of the independent variable.

Graph the equation $f(x) = 3x + 6$ for all $x \geq 0$

On the same coordinate system graph

$f(x) = -3x - 6$

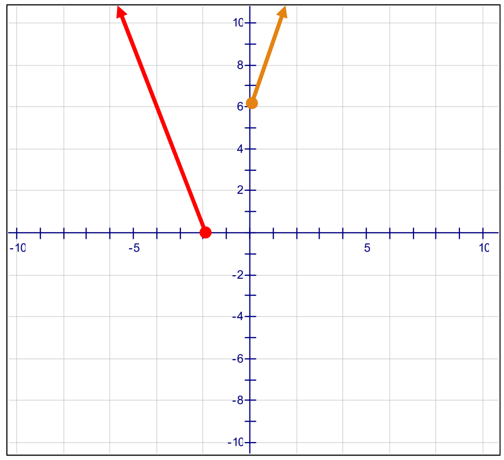
for all $x \leq -2$



This is a piecewise function because there are split domains.

Write the two functions as a single piecewise function:

$$\begin{cases} f(x) = 3x + 6 & x \geq 0 \\ f(x) = -3x - 6 & x \leq -2 \end{cases}$$



Given the function, find each value using *f(x)*:

a) $f(1) =$ 1

b) $f(10) =$ 12

c) $f(5) =$ 7

d) $f(0) =$ 0

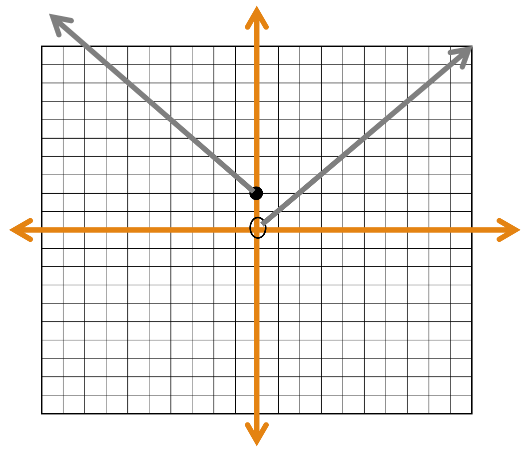
e) $f(-1) =$ –1

f) $f(-10) =$ 10

$$f(x) = \begin{cases} x + 2 & x > 3 \\ x & -1 \leq x \leq 3 \\ -x & x < -1 \end{cases}$$

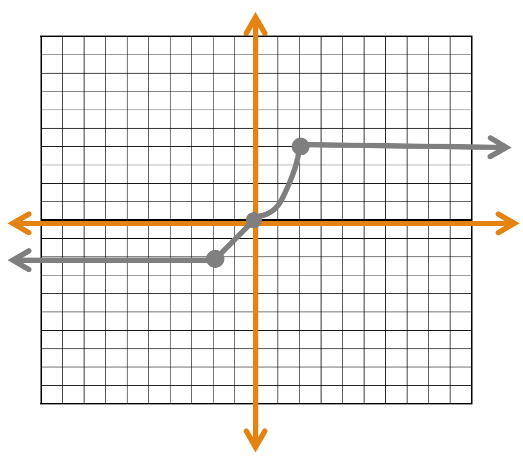
Graph the following functions:

$$f(x) = \begin{cases} -x+2 & x \le 0 \\ x & x > 0 \end{cases}$$



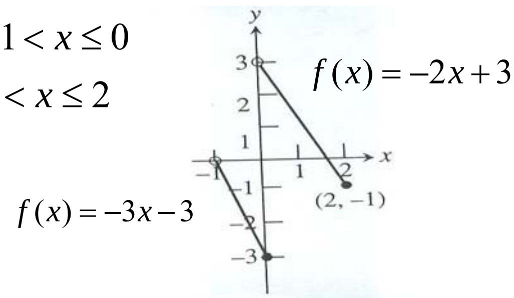
Graph the following function

$$g(x) = \begin{cases} -2 & x<-2 \\ x & -2 \le x \le 0 \\ x^2 & 0 < x < 2 \\ 4 & x \ge 2 \end{cases}$$



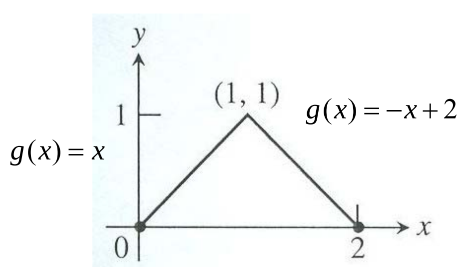
Write a function for the graph

$$f(x)=\begin{cases}-3x-3, & -1<x\le 0\\ -2x+3, & 0<x\le 2\end{cases}$$



Write a function for the graph

$$f(x)=\begin{cases}x, & 0\le x\le 1\\ -x+2, & 1<x\le 2\end{cases}$$



Homework: Piecewise Function WS