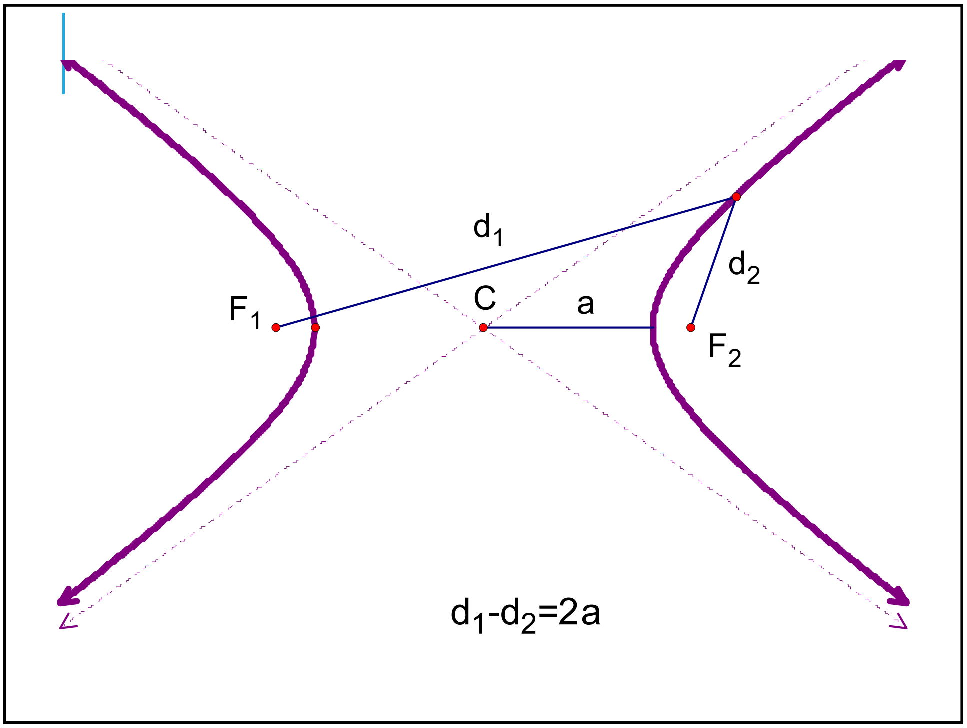
8.3: HYPERBOLAS

I. Geometry of a Hyperbola

A. A hyperbola is the set of all points in a plane whose distances from two fixed points in the plane have a constant *difference*.

B. Vocabulary:

1. The fixed points are the *foci* of the hyperbola
2. The line through the foci is the *focal axis.*
3. The point on the focal axis midway between the foci is the *center*.
4. The distance between the center and the foci, this is c.
5. The points where the hyperbola intersects its focal axis are the *vertices* of the hyperbola.



C. HYPERBOLAS - Center at (0,0)

St. fm..	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$
Focal axis	$x-axis$	$y-axis$
Foci	$(\pm c,0)$	$(0,\pm c)$
Vertices	$(\pm a,0)$	$(0,\pm a)$
Semi-Trans.	a	a
Semi-Conj.	b	b
Pyth. Rel.	$c^2=a^2+b^2$	$c^2=a^2+b^2$
Asymptotes	$y=\pm\frac{b}{a}x$	$y=\pm\frac{a}{b}x$

D. **HYPERBOLAS -** Center at (h, k)

St. fm..	$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$	$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$
Focal axis	$y=k$	$x=h$
Foci	$(h\pm c,k)$	$(h,k\pm c)$
Vertices	$(h\pm a,k)$	$(h,k\pm a)$
Semi-Trans.	a	a
Semi-Conj.	b	b
Pyth. Rel.	$c^2=a^2+b^2$	$c^2=a^2+b^2$
Asymptotes	$y=\pm\frac{b}{a}(x-h)+k$	$y=\pm\frac{a}{b}(x-h)+k$

E. Example 1: Graph the hyperbola below:

$$\frac{(x+2)^2}{4}-\frac{(y-2)^2}{16}=1$$

$(h,k)=(-2,2)$

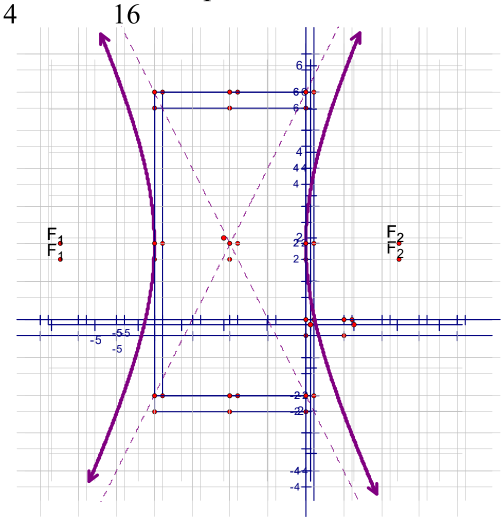
$a=2, b=4$

$c=\sqrt{a^2+b^2}$

$c=\sqrt{4+16}$

$c=2\sqrt{5}$

$foci:\left(-2\pm 2\sqrt{5},2\right)$



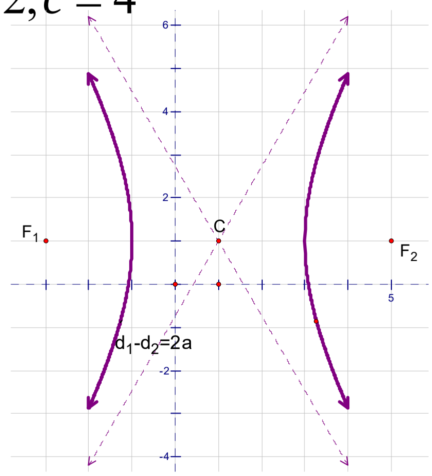
asymp: $y=\pm 2(x+2)+2$

F. Example 2: Determine the equation of a hyperbola with foci at *(-3,1)* and *(5,1)*, and transverse axis of length 4.

$$(h,k)=(1,1)$$
$$a=2, c=4$$

$$b^2=c^2-a^2$$
$$b^2=4^2-2^2$$
$$b^2=12$$
$$b=2\sqrt{3}$$

$$\frac{(x-1)^2}{4}-\frac{(y-1)^2}{12}=1$$



G. Example 3: Sketch the graph of the following equation

$$x^2-2x-y^2+6y-7=0$$
$$(x^2-2x)-(y^2-6y)=7$$
$$(x^2-2x+1)-(y^2-6y+9)=7+1-9$$
$$(x-1)^2-(y-3)^2=-1$$
$$-(x-1)^2+(y-3)^2=1$$
$$\frac{(y-3)^2}{1^2}-\frac{(x-1)^2}{1^2}=1$$

b

II. The eccentricity of a Hyperbola

A. The *eccentricity* of a hyperbola is

$$e = \frac{c}{a} = \frac{\sqrt{a^2 + b^2}}{a}$$

Where *a* is the semitransverse axis, *b* is the semiconjugate axis, and c is the distance from the center to either focus.

B. Eccentricity of all conics:

Circle	$e = 0$
Ellipse	$0 \le e < 1$
Parabola	$e = 1$
Hyperbola	$e > 1$

C. Astronomy Notes:

- The orbit of a comet or an asteroid is always a conic section.
- The eccentricity is proportional to the total energy of the comet.
- As a comet increases speed, it can move from an elliptical orbit to a parabolic, and finally to a hyperbolic orbit

III. General Equations of the Second Degree

A. Using the following general form, we can determine what kind of conic we have

- note that we leave out the *xy* term, as it is outside the purview of our study.
- Also note that if both A and B are zero, we have a degenerate conic

$$Ax^2 + By^2 + Cx + Dy + E = 0$$

Parabola	Either A or B is zero
Ellipse	A and B nonzero and of the same sign
Circle	An ellipse, in which $A = B$
Hyperbola	A and B nonzero and of opposite signs

B. Example 4: Determine which type of conic is the graph of each equation

1. $5x^2 + 3y^2 - 7x + 3y = 1$ *Ellipse*
2. $-5x^2 + 3y^2 + 12x - 4 = 0$ *Hyperbola*
3. $10x^2 + 10y^2 + 3x - 2y - 100 = 0$ *Circle*
4. $x + 7y^2 + 3y + 5 = 0$ *Parabola*