8.2: CIRCLES AND ELLIPSES

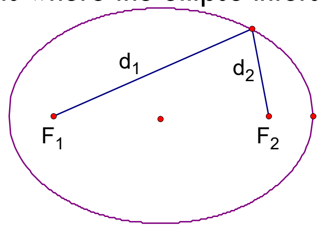
GEOMETRY OF AN ELLIPSE

Geometry of an Ellipse

Definition: An *ellipse* is the set of all points in a plane whose distance from two fixed points in the plane have a constant sum.

Vocabulary

- The fixed points are the *foci* of the ellipse.
- The line through the foci is the *focal axis*.
- The point on the focal axis midway between the foci is the *center*.
- The points where the ellipse intersects its axis are the *vertices*.



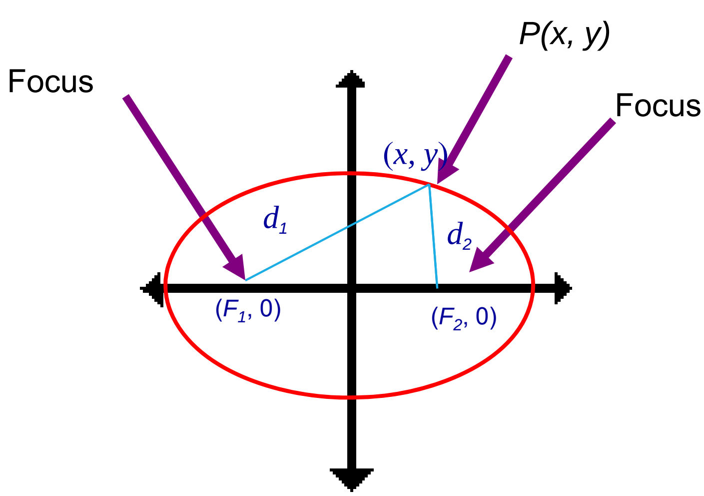
$d_1 + d_2$ is constant

GEOMETRY OF AN ELLIPSE

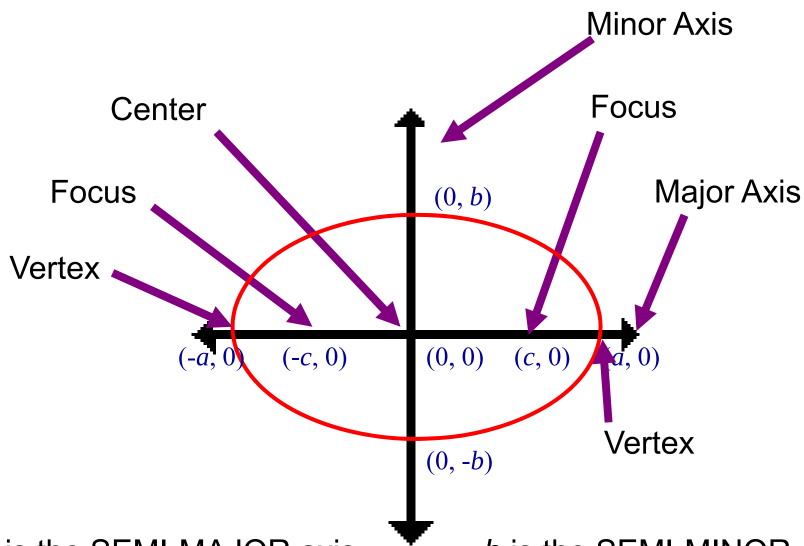
More vocabulary

- A line segment with endpoints on an ellipse is a ***chord***.
- The chord lying on the focal axis is the ***major axis*** (the length is $2a$).
- The chord through the center perpendicular to the focal axis is the ***minor axis*** (the length is $2b$).
- The number a is the **semimajor axis**, and the number b is the **semiminor axis**.

PICTURES – BY DEFINITION



PICTURES -EXPANDED



a is the SEMI-MAJOR axis

b is the SEMI-MINOR axis

ELLIPSES - CENTER AT (0,0)

St. Fm.	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	$\frac{y^2}{a^2}+\frac{x^2}{b^2}=1$
Focal axis	$x-axis$	$y-axis$
Foci	$(\pm c,0)$	$(0,\pm c)$
Semi-Major	a	a
Semi-Minor	b	b
Pyth. Rel.	$a^2=b^2+c^2$	$a^2=b^2+c^2$

ELLIPSES - CENTER AT (*H*, *K*)

St. fm.	$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$	$\frac{(y-k)^2}{a^2}+\frac{(x-h)^2}{b^2}=1$
Focal axis	$y=k$	$x=h$
Foci	$(h\pm c,k)$	$(h,k\pm c)$
Semi-Major	a	a
Semi-Minor	b	b
Pyth. Rel.	$a^2=b^2+c^2$	$a^2=b^2+c^2$

Example 1: Graph the following ellipse $\frac{(x+2)^2}{4}+\frac{(y-2)^2}{16}=1$

$(h,k)=(-2,2)$

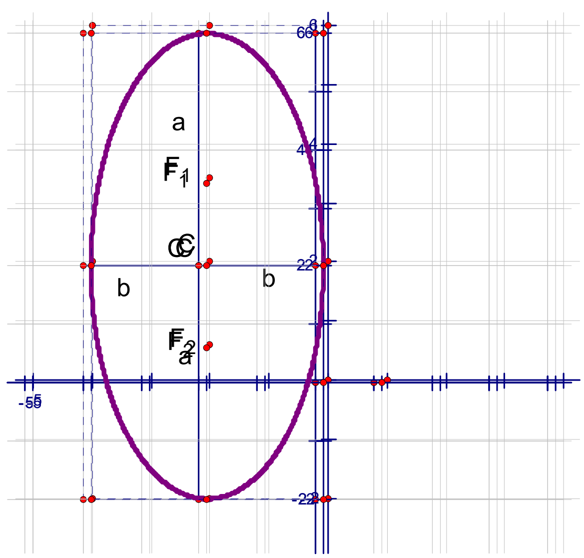
$a=4, b=2$

$c=\sqrt{a^2-b^2}$

$c=\sqrt{4-2}$

$c=\sqrt{2}$

$foci:\left(-2,2\pm\sqrt{2}\right)$



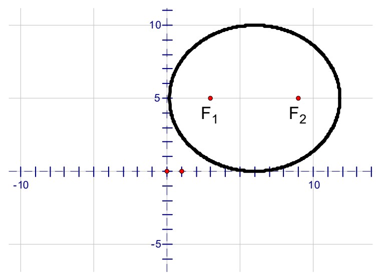
Example 2: Determine the equation of the ellipse with foci at *(3,5)* and *(9,5)* and minor axis of length 10.

c is half the distance between foci, so c=3

The center is the midpoint between the foci, so *(6,5)*

$$2b = 10$$
$$b = 5$$
$$a^2 = b^2 + c^2$$
$$a^2 = 5^2 + 3^2$$
$$a^2 = 25 + 9$$
$$a^2 = 34$$
$$a = \sqrt{34}$$

$$\boxed{\frac{(x-6)^2}{34} + \frac{(y-5)^2}{25} = 1}$$



Example 3: Sketch the graph of the ellipse with the equation below, then determine the foci, vertices, semimajor axis, and semiminor axis.

$$4x^2 + 25y^2 = 100$$

$$\frac{4x^2}{100} + \frac{25y^2}{100} = 1$$
$$\frac{x^2}{25} + \frac{y^2}{4} = 1$$
$$\frac{x^2}{(5)^2} + \frac{y^2}{(2)^2} = 1$$

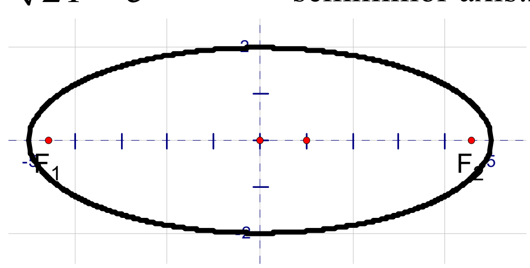
$$a^2 = b^2 + c^2$$
$$25 = 4 + c^2$$
$$\sqrt{21} = c$$

foci: $\left(\pm\sqrt{21}, 0\right)$

vertices: $(\pm 5, 0)$

semimajor axis: 5

semiminor axis: 2



Example 4: Sketch the graph of the ellipse with the equation below, then find the foci, vertices, semimajor axis, and seminor axis.

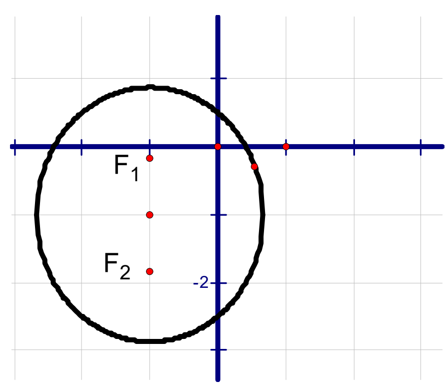
$$5x^2+10x+4y^2+8y-5=0$$
$$5\left(x^2+2x\right)+4\left(y^2+2y\right)=5$$
$$5\left(x^2+2x+1\right)+4\left(y^2+2y+1\right)=5+5+4$$
$$5\left(x+1\right)^2+4\left(y+1\right)^2=14$$
$$\frac{5\left(x+1\right)^2}{14}+\frac{4\left(y+1\right)^2}{14}=1$$

$$\frac{\left(x+1\right)^2}{14/5}+\frac{\left(y+1\right)^2}{7/2}=1 \qquad \frac{\left(x+1\right)^2}{\left(\sqrt{14/5}\right)^2}+\frac{\left(y+1\right)^2}{\left(\sqrt{7/2}\right)^2}=1$$

$$\text{Semimajor axis} = \sqrt{\frac{7}{2}}=\boxed{\frac{\sqrt{14}}{2}}\text{, vertical}$$

$$\text{Seminor axis} = \sqrt{\frac{14}{5}}=\boxed{\frac{\sqrt{70}}{5}}\text{, horizontal}$$

$$c^2=a^2-b^2$$
$$\to\ c^2=\frac{7}{2}-\frac{14}{5}$$
$$\to\ \boxed{c=\frac{\sqrt{70}}{10}}$$
$$\text{foci: }\left(-1,-1\pm\frac{\sqrt{70}}{10}\right)$$



ECCENTRICITY AND ORBITS

Definition: suppose that an ellipse has semimajor axis *a* and focal length c. Then the *eccentricity* e of the ellipse is defined as

$$\boxed{e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a}}$$

Note that since the focal length for an ellipse must be less than the semimajor axis, the eccentricity must be between 0 and 1. (or more specifically $e \in [0,1)$)

Example 5: Determine the eccentricity of the ellipse in example 3

$$a = \frac{\sqrt{14}}{2}, c = \frac{\sqrt{70}}{10} \qquad e = \frac{c}{a} = \frac{\sqrt{70}/10}{\sqrt{14}/2} = \boxed{\frac{\sqrt{5}}{5} \approx 0.447}$$

Example 6: Show that an ellipse of eccentricity 0 is a circle

If e=0, then c=0, therefore

$$0 = a^2 - b^2$$

$$a^2 = b^2$$

$$a = b$$

Thus, the semimajor and semiminor axes have the same length, and the ellipse is a circle.

Example 7: According to Kepler's first law, every planet orbits the sun in an elliptical orbit, with the sun at one focus. The eccentricity of the earth's orbit about the sun is approximately 0.0167. The closest distance between the earth and sun is approximately 93 million miles. What is the furthest distance between the earth and the sun?

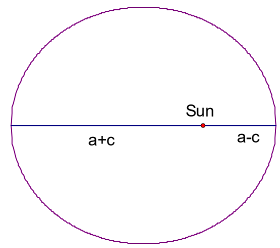
Let *a* be the semimajor axis of the orbit. Assume that the center of the orbit is *(0,0)*, and the sun is at the focus *(c,0)*.

$$a = c + 93$$

$$e = \frac{c}{a} = \frac{c}{c+93} \approx 0.0167$$

$$c \approx 0.0167c + 1.5531$$

$$c \approx 1.5795$$



$$\text{distance: } a + c = 2c + 93$$

$$= \boxed{96.159 \text{ million miles}}$$

ELLIPSOIDS OF REVOLUTION

Rotate ellipse about its focal axis to get an ellipsoid of revolution.

Examples of these include whispering galleries and a lithotripter, a device which uses shockwaves to destroy kidney stones.