6.6 DeMoivre's Theorem and n^{th} Roots

Day 1

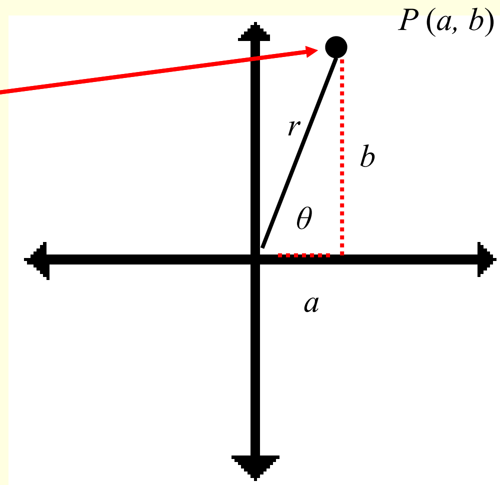
I. Trigonometric Form of Complex Numbers

A.) The standard form of the complex number $z = a + b\mathbf{i}$ is very similar to the component form of a vector $\boldsymbol{v} = ai + bj$. If we look at the trigonometric form of ***v***, we can see

$$\boldsymbol{v} = |\boldsymbol{v}|(\cos\theta i + \sin\theta j)$$

B.) If we graph the complex $z = a + b\boldsymbol{i}$ on the complex plane, we can see the similarities with the polar plane.

$z = a + b\boldsymbol{i}$



C.) If we let $a = r\cos\theta$ and $b = r\sin\theta$ then,

$$z = a + b\boldsymbol{i} = r\cos\theta + \left(r\sin\theta\right)\boldsymbol{i}$$

where $r = \left|z\right| = \sqrt{a^2 + b^2}$,

$\tan\theta = \dfrac{b}{a}$ and $\boldsymbol{i} = \sqrt{-1}$

D.) Def. – The trigonometric form of a complex number z is given by

$$z = r\left(\cos\theta + \boldsymbol{i}\sin\theta\right)$$

or

$$z = r cis\theta$$

Where r is the ***Absolute Value*** or ***Modulus*** of z and θ is the ***Argument*** of z.

E.) Ex.1 - Find the trig form of the following:

1.) $2\boldsymbol{i}$

$$r = \sqrt{0^2 + 2^2} = 2$$

$$\theta = \tan^{-1}\left(\frac{2}{0}\right) = \frac{\pi}{2}$$

$$z = 0 + 2i$$

$$= r(\cos\theta + \boldsymbol{i}\sin\theta)$$

$$= 2\left(\cos\frac{\pi}{2} + \boldsymbol{i}\sin\frac{\pi}{2}\right)$$

$$= 2cis\frac{\pi}{2}$$

2.) $\frac{1}{2} - \frac{\sqrt{3}}{2}\boldsymbol{i}$

$$r = \sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2} = 1$$

$$\theta = \tan^{-1}\left(-\sqrt{3}\right) = \frac{5\pi}{3}$$

$$z = \frac{1}{2} - \frac{\sqrt{3}}{2}i$$

$$= r(\cos\theta + \boldsymbol{i}\sin\theta)$$

$$= 1\left(\cos\frac{5\pi}{3} + \boldsymbol{i}\sin\frac{5\pi}{3}\right)$$

$$= cis\frac{5\pi}{3}$$

II. Products and Quotients

A.) Let $z_1 = r_1\left(\cos\theta_1 + i\sin\theta_1\right)$ and $z_2 = r_2\left(\cos\theta_2 + i\sin\theta_2\right)$

Mult.- $z_1 \cdot z_2 = r_1 \cdot r_2\left(\cos\left(\theta_1+\theta_2\right) + i\sin\left(\theta_1+\theta_2\right)\right)$

Div. - $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\left(\cos\left(\theta_1-\theta_2\right) + i\sin\left(\theta_1-\theta_2\right)\right)$

DERIVE THESE!!!!

Given $z_1 = 5\left(\cos 15° + i\sin 15°\right)$ & find $z_1 \cdot z_2$ and $\dfrac{z_1}{z_2}$

$z_2 = 2\left(\cos 105° + i\sin 105°\right)$

$z_1 \cdot z_2 = (5\cdot 2)\left(\cos(120°) + i\sin(120°)\right)$

$= 10\left(-\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i\right)$

$= -5 + 5\sqrt{3}i$

$\dfrac{z_1}{z_2} = \dfrac{5}{2}\left(\cos -90° + i\sin -90°\right)$

$= \dfrac{5}{2}(0 - i)$

$= -\dfrac{5}{2}i$

III. Powers of Complex Numbers

DeMoivre's (di-'mȯi-vərz) Theorem –

If $z = r(\cos\theta + \boldsymbol{i}\,\sin\theta)$ and n is a positive integer, then,

$$z^n = \left[r(\cos\theta + i\sin\theta)\right]^n = r^n\left(\cos n\theta + i\sin n\theta\right)$$

Why??? – Let's look at z^2-

$$z^2 = r\left(\cos\theta + i\sin\theta\right)\cdot r\left(\cos\theta + i\sin\theta\right)$$

$$z^2 = r^2\left(\cos\theta + i\sin\theta\right)\left(\cos\theta + i\sin\theta\right)$$

$$z^2 = r^2\left(\cos^2\theta + 2i\cos\theta\sin\theta + i^2\sin^2\theta\right)$$

$$z^2 = r^2\left(\cos^2\theta + (-1)\sin^2\theta + (i)2\cos\theta\sin\theta\right)$$

$$z^2 = r^2\left(\cos 2\theta + i\sin 2\theta\right)$$

Find $\left(1 + i\sqrt{3}\right)^3$ by "Foiling"

$$= \left(1 + i\sqrt{3}\right)\left(1 + i\sqrt{3}\right)\left(1 + i\sqrt{3}\right)$$

$$= \left(1 + 2i\sqrt{3} - 3\right)\left(1 + i\sqrt{3}\right)$$

$$= \left(-2 + 2i\sqrt{3}\right)\left(1 + i\sqrt{3}\right)$$

$$= -2 - 6$$

$$= -8$$

Now find $\left(1+i\sqrt{3}\right)^3$ using DeMoivre's Theorem

$$r=2 \quad \theta=\frac{\pi}{3}$$

$$z^3=\left(2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)\right)^3$$

$$z^3=2^3\left(\cos 3\left(\frac{\pi}{3}\right)+i\sin 3\left(\frac{\pi}{3}\right)\right)$$

$$z^3=8\left(\cos\pi+i\sin\pi\right)$$

$$z^3=8(-1+0)=-8$$

Use DeMoivre's Theorem to simplify $\left(\frac{\sqrt{3}}{2}+i\frac{\sqrt{3}}{2}\right)^{10}$

$$r=\sqrt{2\left(\frac{\sqrt{3}}{2}\right)^2}=\frac{\sqrt{6}}{2} \quad \theta=\frac{\pi}{4}$$

$$z^{10}=\left(\frac{\sqrt{6}}{2}\right)^{10}\left(\cos\frac{10\pi}{4}+i\sin\frac{10\pi}{4}\right)$$

$$z^{10}=\frac{6^5}{2^{10}}\left(\cos\left(\frac{5\pi}{2}\right)+i\sin\left(\frac{5\pi}{2}\right)\right)$$

$$z^{10}=\frac{6^5}{2^{10}}\left(i\right)$$