6.6 DeMoivre's Theorem and n^{th} Roots

Day 2

I. n^{th} Roots of Complex Numbers

$v = a + b\mathbf{i}$ is an n^{th} root of z iff $v^n = z$.

If $z = 1$, then v is an n^{th} ROOT OF UNITY.

Suppose $v = s(cos\alpha + isin\alpha)$ is an nth root of $z = r(cos\theta + \theta sin\theta)$, then:

$$v^n = z$$

$$\left(s(\cos\alpha + i\sin\alpha)\right)^n = r\left(\cos\theta + i\sin\theta\right)$$

$$\left(s^n(\cos n\alpha + i\sin n\alpha)\right) = r\left(\cos\theta + i\sin\theta\right)$$

$$\left|\left(s^n(\cos n\alpha + i\sin n\alpha)\right)\right| = \left|r\left(\cos\theta + i\sin\theta\right)\right|$$

$$\sqrt{\left(s^{2n}(\cos^2 n\alpha + \sin^2 n\alpha)\right)} = \sqrt{r^2\left(\cos^2\theta + \sin^2\theta\right)}$$

$$\sqrt{s^{2n}} = \sqrt{r^2}$$

$$s^n = r$$

$$s = \sqrt[n]{r}$$

Furthermore

$$\left(\sqrt[n]{r}\right)^n(\cos n\alpha + i\sin n\alpha) = r\left(\cos\theta + i\sin\theta\right)$$

$$(\cos n\alpha + i\sin n\alpha) = \left(\cos\theta + i\sin\theta\right)$$

Therefore, $n\alpha$ is any coterminal angle with θ.
For any integer k, v is an nth root of z if

$$s = \sqrt[n]{r},\ n\alpha = \theta + 2k\pi,\ \text{and } n = \frac{\theta + 2k\pi}{\alpha}$$

If $z = r(\cos\theta + i\sin\theta)$, then the n distinct complex numbers

$$\sqrt[n]{r}\left(\cos\frac{\theta+2k\pi}{n} + \boldsymbol{i}\sin\frac{\theta+2k\pi}{n}\right)$$

Where k = 0, 1, 2, …, n-1 are the nth roots of the complex number z.

Find the 4th roots of $z = 5cis\left(\frac{\pi}{3}\right)$

$$z_1 = \sqrt[4]{5}cis\left(\frac{\frac{\pi}{3}}{4}\right) = \sqrt[4]{5}cis\left(\frac{\pi}{12}\right) \quad z_2 = \sqrt[4]{5}cis\left(\frac{\frac{\pi}{3}+2\pi}{4}\right) = \sqrt[4]{5}cis\left(\frac{7\pi}{12}\right)$$

$$z_3 = \sqrt[4]{5}cis\left(\frac{\frac{\pi}{3}+4\pi}{4}\right) = \sqrt[4]{5}cis\left(\frac{13\pi}{12}\right)$$

$$z_4 = \sqrt[4]{5}cis\left(\frac{\frac{\pi}{3}+6\pi}{4}\right) = \sqrt[4]{5}cis\left(\frac{19\pi}{12}\right)$$

II. Finding Cube Roots

Find the cube roots of -1.

$$z^3 = -1$$

$$z^3 + 1 = 0$$

$$(z+1)\left(z^2 - z + 1\right) = 0$$

$$z = -1 \text{ or } z = \frac{1 \pm \sqrt{1-4(1)(1)}}{2(1)}$$

$$z = -1,\ \frac{1}{2} + \frac{\sqrt{3}}{2}\boldsymbol{i},\ \frac{1}{2} - \frac{\sqrt{3}}{2}\boldsymbol{i}$$

Now.... $z = -1 + 0i$

$$z = 1\left(\cos\pi + i\sin\pi\right)$$

$$z_1 = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + \frac{\sqrt{3}}{2}\boldsymbol{i}$$

$$z_2 = \cos\frac{3\pi}{3} + i\sin\frac{3\pi}{3} = -1 + 0i$$

$$z_3 = \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3} = \frac{1}{2} - \frac{\sqrt{3}}{2}\boldsymbol{i}$$

Plot these points on the complex plane.

What do you notice about them?

III. Roots of Unity

Any complex root of the number 1 is also known as a ***Root of Unity***.

Find the 6 roots of unity. $z = 1 + 0i = cis0$

$$z_1 = cis0 = 1$$

$$z_2 = cis\frac{0+2\pi}{6} = cis\frac{\pi}{3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$$

$$z_3 = cis\frac{0+4\pi}{6} = cis\frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

$$z_4 = cis\frac{0+6\pi}{6} = cis\pi = -1$$

$$z_5 = cis\frac{0+8\pi}{6} = cis\frac{4\pi}{3} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

$$z_6 = cis\frac{0+10\pi}{6} = cis\frac{5\pi}{3} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$$