5-4: SUM AND DIFFERENCE IDENTITIES

CP Precalculus
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I. COSINE OF A SUM OR DIFFERENCE

Cosine of a Sum Formula

$$cos(\alpha + \beta) = cos\alpha \cdot cos\beta - sin\alpha \cdot sin\beta$$

Cosine of a Difference Formula

$$cos(\alpha - \beta) = cos\alpha \cdot cos\beta + sin\alpha \cdot sin\beta$$

EXAMPLE 1

Evaluate $cos\frac{\pi}{12}$

$$\begin{aligned}\cos(\alpha-\beta) &= \cos\left(\frac{\pi}{3}-\frac{\pi}{4}\right)\\ &= \cos\alpha\cos\beta+\sin\alpha\sin\beta\\ &= \cos\frac{\pi}{3}\cos\frac{\pi}{4}+\sin\frac{\pi}{3}\sin\frac{\pi}{4}\\ &= \frac{1}{2}\left(\frac{\sqrt{2}}{2}\right)+\frac{\sqrt{3}}{2}\left(\frac{\sqrt{2}}{2}\right)\\ &= \frac{\sqrt{2}+\sqrt{6}}{4}\end{aligned}$$

EXAMPLE 2:

Verify the following identity:

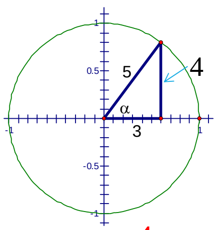
$$cos\left(\frac{\pi}{2}-x\right) = sinx$$

$$\begin{aligned}\cos\left(\frac{\pi}{2}-x\right) &= \cos(\alpha-\beta)\\ &= \cos\alpha\cos\beta+\sin\alpha\sin\beta\\ &= \cos\frac{\pi}{2}\cos x+\sin\frac{\pi}{2}\sin x\\ &= (0)\cos x+(1)\sin x\\ &= \sin x\end{aligned}$$

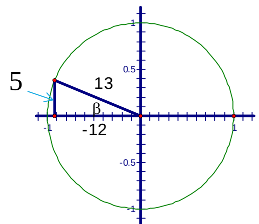
EXAMPLE 3:

Evaluate $cos(\alpha + \beta)$ given that $cos\alpha = \frac{3}{5}\left(0 < \alpha < \frac{\pi}{2}\right)$, and $cos\beta = -\frac{12}{13}\left(\frac{\pi}{2} < \beta < \pi\right)$.

Find $\sin\alpha$ and $\sin\beta$:



$\sin\alpha = \frac{4}{5}$



$\sin\beta = \frac{5}{13}$

$$= \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$= \frac{3}{5}\left(-\frac{12}{13}\right) - \frac{4}{5}\left(\frac{5}{13}\right)$$
$$= \frac{-36-20}{65}$$
$$= -\frac{56}{65}$$

II. SINE OF A SUM OR DIFFERENCE

Formulas:

$$sin(\alpha + \beta) = sin\alpha \cdot cos\beta + cos\alpha \cdot sin\beta$$
$$sin(\alpha - \beta) = sin\alpha \cdot cos\beta - cos\alpha \cdot sin\beta$$

Example 4: evaluate $sin75°$

$$\sin 75° = \sin(30+45)$$
$$= \sin(\alpha+\beta)$$
$$= \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$= \sin 30\cos 45 + \cos 30\sin 45$$
$$= \frac{1}{2}\left(\frac{\sqrt{2}}{2}\right) + \frac{\sqrt{3}}{2}\left(\frac{\sqrt{2}}{2}\right)$$
$$= \frac{\sqrt{2}+\sqrt{6}}{4}$$

EXAMPLE 5

C. Evaluate

$$sin(5°)cos(-50°) + cos(5°)sin(-50°)$$

$$\begin{aligned}\sin(5)\cos(-50)+\cos(5)\sin(-50) &= \sin\alpha\cos\beta+\cos\alpha\sin\beta \\ &= \sin(\alpha+\beta) \\ &= \sin(5+(-50)) \\ &= \sin(-45) \\ &= -\frac{\sqrt{2}}{2}\end{aligned}$$

EXAMPLE 6

Verify the identity

$$\sin\left(\frac{\pi}{2}-x\right)=\cos x$$

$$\begin{aligned}\sin\left(\frac{\pi}{2}-x\right) &= \sin(\alpha-\beta) \\ &= \sin\alpha\cos\beta-\cos\alpha\sin\beta \\ &= \sin\frac{\pi}{2}\cos x-\cos\frac{\pi}{2}\sin x \\ &= (1)\cos x-(0)\sin x \\ &= \cos x\end{aligned}$$

Homework: p.341 #1, 3, 13, 15, 19, 21, 35, 37, 39

III. TANGENT OF A SUM OR DIFFERENCE

Formulas:

$$tan(\alpha + \beta) = \frac{tan\alpha + tan\beta}{1 - tan\alpha \cdot tan\beta} = \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)}$$

OR

$$tan(\alpha - \beta) = \frac{tan\alpha - tan\beta}{1 + tan\alpha \cdot tan\beta} = \frac{\sin(\alpha - \beta)}{\cos(\alpha - \beta)}$$

EXAMPLE 5

Simplify $tan(\pi - \theta)$

$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$
$$=\frac{\tan\pi-\tan\theta}{1+\tan\pi\tan\theta}$$
$$=\frac{0-\tan\theta}{1+(0)\tan\theta}$$
$$=\frac{-\tan\theta}{1}$$
$$=-\tan\theta$$

EXAMPLE 8:

Evaluate

$$\frac{\tan\frac{4\pi}{3}-\tan\frac{\pi}{2}}{1+\tan\frac{4\pi}{3}\tan\frac{\pi}{2}}$$

$$=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$
$$=\tan(\alpha-\beta)$$
$$=\tan\left(\frac{4\pi}{3}-\frac{\pi}{2}\right)$$
$$=\tan\left(\frac{5\pi}{6}\right)$$
$$=-\frac{\sqrt{3}}{3}$$

Verify the identity $\tan\left(x-\frac{\pi}{4}\right)=\frac{\tan x-1}{\tan x+1}$

$$\tan\left(x-\frac{\pi}{4}\right)=\tan(\alpha-\beta)$$

$$=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

$$=\frac{\tan x-\tan\frac{\pi}{4}}{1+\tan x\tan\frac{\pi}{4}}$$

$$=\frac{\tan x-1}{1+\tan x(1)}$$

$$=\frac{\tan x-1}{\tan x+1}$$

Homework: p.341 #5, 6, 11, 16, 17, 22