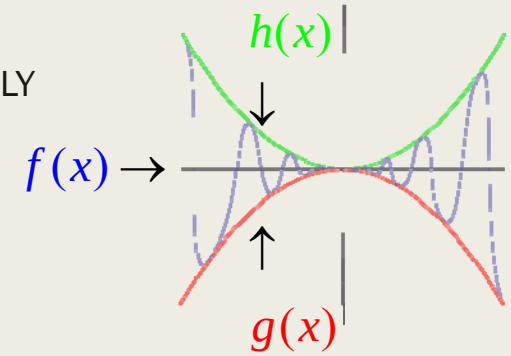
# LIMITS 5: SANDWICH THEOREM AND CHANGE OF VARIABLES

## I. Sandwich Theorem

A.) If $g(x) \le f(x) \le h(x)$ for all $x$ in an open interval containing the point $x = c$ (with the possible exception at $x = c$) and $\lim_{x \to c} g(x) = L = \lim_{x \to c} h(x)$, then $\lim_{x \to c} f(x) = L$

GRAPHICALLY



3.) Example - $\lim_{x\to 0} x^2 \sin^2\left(\frac{1}{x}\right)$

What do you know about the sin function?

$$-1 \le \sin\left(\frac{1}{x}\right) \le 1$$

$$0 \le \sin^2\left(\frac{1}{x}\right) \le 1$$

$$x^2(0) \le x^2 \sin^2\left(\frac{1}{x}\right) \le x^2(1)$$

$$0 \le x^2 \sin^2\left(\frac{1}{x}\right) \le x^2$$

$$\therefore \lim_{x\to 0} 0 \le \lim_{x\to 0} x^2 \sin^2\left(\frac{1}{x}\right) \le \lim_{x\to 0} x^2$$

$$0 \le \lim_{x\to 0} x^2 \sin^2\left(\frac{1}{x}\right) \le 0$$

$$\therefore \lim_{x\to 0} x^2 \sin^2\left(\frac{1}{x}\right) = 0$$

C.) Example - $\lim_{x\to 0} x\cos\left(\frac{1}{x^2}\right)$

$-1 \le \cos\left(\frac{1}{x^2}\right) \le 1$

$\therefore \lim_{x\to 0} -x \le \lim_{x\to 0} x\cos\left(\frac{1}{x^2}\right) \le \lim_{x\to 0} x$

$x(-1) \le x\cos\left(\frac{1}{x^2}\right) \le x(1)$

$0 \le \lim_{x\to 0} x\cos\left(\frac{1}{x^2}\right) \le 0$

$-x \le x\cos\left(\frac{1}{x^2}\right) \le x$

$\therefore \lim_{x\to 0} x\cos\left(\frac{1}{x^2}\right) = 0$

## II. Change of Variables

A.) Trig Identities – Know Sum and Difference for sin and cos!!!

B.) Sometimes it is helpful to substitute another variable when evaluating trig limits.

$\lim_{x\to -\infty}\left(1+\frac{2}{x}\right)\left(\cos\frac{1}{x}\right)$ Let $\theta = \frac{1}{x}$

$\lim_{x\to -\infty} \theta = \lim_{x\to -\infty} \frac{1}{x} = 0$

$$\lim_{x\to-\infty}\left(1+\frac{2}{x}\right)\left(\cos\frac{1}{x}\right)= \qquad \lim_{\theta\to 0}\ (1+2\theta)(\cos\theta)$$

$$=(1+2(0))(\cos(0)) \qquad =(1+0)(1)=1$$

C.) Evaluate $\lim_{x\to\frac{\pi}{2}}\left(\frac{x-\frac{\pi}{2}}{3\cos x}\right)$

$$\lim_{x\to\frac{\pi}{2}}\left(\frac{x-\frac{\pi}{2}}{3\cos x}\right) \quad =\frac{\frac{\pi}{2}-\frac{\pi}{2}}{3\cos\left(\frac{\pi}{2}\right)}=\frac{0}{0}$$

Let $\theta = x - \frac{\pi}{2}$ $\qquad \lim_{x \to \frac{\pi}{2}} \theta = 0$

$$\lim_{\theta \to 0} \left( \frac{\theta}{3\cos\left(\theta + \frac{\pi}{2}\right)} \right) = \lim_{\theta \to 0} \frac{\theta}{3\left(\cos\theta\cos\frac{\pi}{2} - \sin\theta\sin\frac{\pi}{2}\right)} =$$

$$\lim_{\theta \to 0} \frac{\theta}{3(0 - \sin\theta)} = \frac{\theta}{-3\sin\theta} = -\frac{1}{3}$$