Notes: DERIVATIVES

Velocity and Other Rates of Change

I. Average Rate of Change

A.) Def.- The **average rate of change** of $f(x)$ on the interval $[a, b]$ is

$$\frac{f(b)-f(a)}{b-a}$$

$$= m_{\text{secant}} \text{ through } \left(a, f(a)\right) \text{ and } \left(b, f(b)\right)$$

B.) Ex.- Find the average rate of change of $f(x)$ on [2, 5] for the following functions:

1.) $f(x) = x^2$

$$m = \frac{f(5) - f(2)}{5-2}$$

$$m = \frac{25-4}{3} = 7$$

as x changes from 2 to 5, the y-values increase at an average rate of 7 to 1.

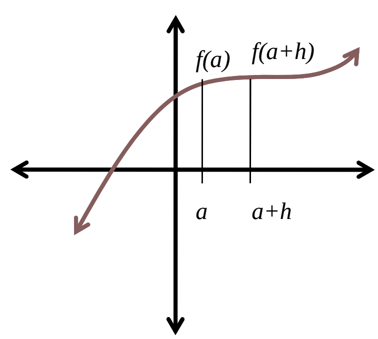
2.) $f(x) = e^{x-1}$

$$m = \frac{e^4 - e^1}{3}$$

$$m \approx 17.293$$

C.) Def.- The **average rate of change** of $f(x)$ on the interval $[a, a+h]$ is

$$\frac{f(a+h) - f(a)}{h}$$



II. Instantaneous Rate of Change

A.) Def.- The **instantaneous rate of change** of $f(x)$ at $x = a$ (if it exists) is

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

$$= m_{\text{tangent}} \text{ to } f(x) \text{ at } x = a$$

$$= \text{slope of the curve } f(x) \text{ at } x = a$$

$$= \text{the derivative of } f(x) \text{ at } x = a$$

III. Examples

A.) Find the instantaneous rate of change of the following functions at the given x values. Then, give the equation of the tangent line at that point.

1.) $f(x)=x^2,\ \text{at } x = 3$

$$\lim_{h\to 0}\frac{f(3+h)-f(3)}{h}$$

$$=\lim_{h\to 0}\frac{(3+h)^2-3^2}{h}$$

$$=\lim_{h\to 0}\frac{9+6h+h^2-9}{h}$$

$$=\lim_{h\to 0}\frac{h^2+6h}{h}=\lim_{h\to 0}\frac{h(h+6)}{h}$$

$$=\lim_{h\to 0} h+6=6$$

Tangent line – $f(3)=3^2=9$

$$y-9=6(x-3)$$

2.) $f(x)=\frac{5}{x}$, at $x=1$

$$\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

$$=\lim_{h\to 0}\frac{\left(\frac{5}{1+h}\right)-\frac{5}{1}}{h}$$

$$=\lim_{h\to 0}\frac{\frac{5-5(1+h)}{(1+h)}}{h}$$

$$=\lim_{h\to 0}\frac{-5h}{h(1+h)}$$

$$=\lim_{h\to 0}\frac{-5}{(1+0)}=-5$$

Tangent line –

$f(1)=5$

$y-5=-5(x-1)$

IV. Normal Line

A.) Def. – The **normal line** to a curve at a point is the line perpendicular to the tangent line at that point.

B.) Ex. – Find the equation of the normal line to the following graph:

$$f(x)=9-x^2,\ x=2$$

$$m=\lim_{h\to 0}\frac{9-(2+h)^2-(9-2^2)}{h}$$

$$m=\lim_{h\to 0}\frac{-4h+h^2}{h}=-4$$

$$y-5=\frac{1}{4}(x-2)$$

V. Free Fall Motion

A.) $s(t) = -16t^2 + v_0 t + s_0$ or $s(t) = -4.9t^2 + v_0 t + s_0$

B.) A rock is dropped from a cliff 1,024' high. Answer the following:

1.) Find the average velocity for the first 3 seconds.

$$\Delta s(t) = \frac{s(3) - s(0)}{3 - 0}$$

$$\Delta s(t) = \frac{-16(3^2) - 16(0^2)}{3} = -48 \text{ ft/sec}$$

2.) Find the instantaneous velocity at $t = 3$ seconds.

$$\lim_{h \to 0} = \frac{s(3+h) - s(3)}{h}$$

$$\lim_{h \to 0} = \frac{-16(3+h)^2 + 16(3^2)}{h}$$

$$\lim_{h \to 0} = \frac{-16(9 + 6h + h^2) + 16(9)}{h}$$

$$\lim_{h \to 0} = \frac{-144 - 96h - 16h^2 + 144}{h}$$

$$\lim_{h \to 0} = \frac{h(-96 - 16h)}{h}$$

$$\lim_{h \to 0} = -96 - 16h$$

$$= -96 \text{ ft/sec}$$

3.) Find the velocity at the instant the rock hits the ground.

$$-16t^2 = 1024$$
$$t = 8$$

$$\lim_{h \to 0} = \frac{-16(8+h)^2 + 16(8^2)}{h}$$

$$\lim_{h \to 0} = -256 \text{ ft/sec}$$

VI. Rectilinear Motion

A.) Def: The motion of a particle back and forth (or up and down), along an axis *s* over a time *t*.

B.) The ***displacement*** of the object over the time interval from t to Δt is

$$\Delta s = s\left(t + \Delta t\right) - s(t)$$

i.e., the ***change*** in ***position.***

C.) The ***average velocity*** of the object over the same time interval is

$$\frac{\Delta s}{\Delta t} = \frac{s\left(t+\Delta t\right)-s(t)}{\Delta t} = \frac{displacement}{\Delta t}$$

D.) The ***instantaneous velocity*** of the object at any time *t* is

$$v\left(t\right)=\frac{ds}{dt}=s'(t)=\lim_{\Delta t\to 0}\frac{s(t+\Delta t)-s(t)}{\Delta t}$$

E.) The ***speed*** of the object at any time *t* is

$$\text{speed}=\left|v\left(t\right)\right|=\left|\frac{ds}{dt}\right|$$

F.) The ***acceleration*** of the object at any time *t* is

$$a\left(t\right)=\frac{dv}{dt}=\frac{d^2s}{dt^2}=v'(t)=s''(t)$$

G.) A particle in rectilinear motion is ***speeding up*** if the signs of the velocity and the acceleration are the same.

H.) A particle in rectilinear motion is ***slowing down*** if the signs of the velocity and the acceleration are the opposite.

VII. Definitions

The **derivative** of a function *f*(*x*) at any point *x* is

A.) $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

provided the limit exists!

B.) Alternate Def. – $$f'(x) = \lim_{x\to a} \frac{f(x) - f(a)}{x-a}$$

provided the limit exists!

VIII. Calculating Derivatives

A.) Using the formal definition of derivative, calculate the derivative of $f(x) = \sqrt{x+2}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{x+h+2} - \sqrt{x+2}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{x+h+2} - \sqrt{x+2}}{h}\left(\frac{\sqrt{x+h+2} + \sqrt{x+2}}{\sqrt{x+h+2} + \sqrt{x+2}}\right)$$

$$= \lim_{h\to 0} \frac{(x+h+2) - (x+2)}{h\left(\sqrt{x+h+2} + \sqrt{x+2}\right)}$$

$$= \lim_{h\to 0} \frac{h}{h\left(\sqrt{x+h+2} + \sqrt{x+2}\right)}$$

$$= \lim_{h\to 0} \frac{1}{\left(\sqrt{x+h+2} + \sqrt{x+2}\right)}$$

$$f'(x) = \frac{1}{2\sqrt{x+2}}, x > -2$$

B.) Using the alternate definition of derivative, calculate the derivative of $f(x) = \sqrt{x+2}$

$$f'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x \to a} \frac{\sqrt{x+2} - \sqrt{a+2}}{x - a}$$

$$= \lim_{x \to a} \frac{\sqrt{x+2} - \sqrt{a+2}}{x - a} \left(\frac{\sqrt{x+2} + \sqrt{a+2}}{\sqrt{x+2} + \sqrt{a+2}} \right)$$

$$= \lim_{x \to a} \frac{(x+2) - (a+2)}{(x-a)\left(\sqrt{x+2} + \sqrt{a+2}\right)}$$

$$= \lim_{x \to a} \frac{x - a}{(x-a)\left(\sqrt{x+2} + \sqrt{a+2}\right)}$$

$$= \lim_{x \to a} \frac{1}{\left(\sqrt{x+2} + \sqrt{a+2}\right)}$$

$$f'(x) = \frac{1}{2\sqrt{a+2}}, a > -2$$

C.) Using both definitions, find the derivative of the following:

$$g(x) = 3x^2 + x - 5$$

You should get

$$g'(x) = 6x + 1$$

IX. Derivative Notation

A.) The following all represent the derivative

$$y' \qquad f'(x) \qquad \frac{df}{dx}$$

$$\frac{d}{dx} \qquad \frac{dy}{dx} \qquad \frac{d}{dx}[f(x)]$$

At a point - $y'(2) \qquad f'(2) \qquad \left.\frac{df}{dx}\right|_{x=2}$

X. Differentiability

A.) Def. – If $f'(x_0)$ exists for all points on $[a, b]$, then $f'(x)$ is **differentiable** at $x = x_0$.

B.) In order for the derivative to exist,

$$\lim_{h\to 0^-} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h}$$

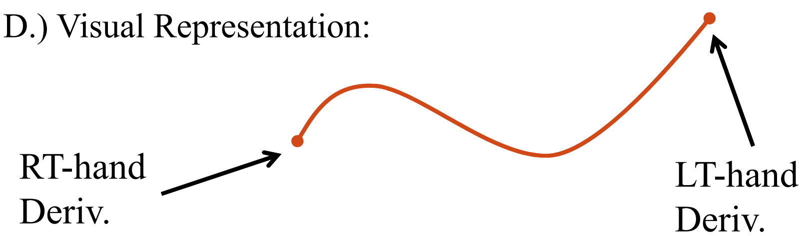
Both must exist and be equal to the same real number!

C.) One-sided derivatives- For any function on a closed interval $[a, b]$

$$\lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h} = \text{the right-hand derivative at } a.$$

$$\lim_{h\to 0^-} \frac{f(b+h)-f(b)}{h} = \text{the left-hand derivative at } b.$$

D.) Visual Representation:



XI. Examples

A.) Does $f(x) = |\, x \,|$ have a derivative at $x = 0$?

$$f(x) = \begin{cases} -x, x < 0 \\ x, x \geq 0 \end{cases}$$

Left-hand:

$$\lim_{h\to 0^-} \frac{-(0+h)-(-0)}{h} =$$

$$\lim_{h\to 0^-} \frac{-h}{h} = -1$$

Right-hand:

$$\lim_{h\to 0^+} \frac{(0+h)-(0)}{h} =$$

$$\lim_{h\to 0^+} \frac{h}{h} = 1$$

$$f'(x) = \begin{cases} -1, x < 0 \\ 1, x > 0 \end{cases}$$

Since the left-hand and right-hand derivatives equal different values, this function does not have a derivative at $x = 0$.

B.) Does the following function have a derivative at $x = 1$?

$$f(x) = \begin{cases} 3x^2, x < 1 \\ 6x - 3, x \geq 1 \end{cases}$$

Left-hand:

$$\lim_{h \to 0^-} \frac{3(1+h)^2 - 3(1)^2}{h} =$$

$$\lim_{h \to 0^-} \frac{3(1+2h+h^2) - 3}{h} =$$

$$\lim_{h \to 0^-} \frac{6h + 3h^2}{h} = 6$$

Right-hand:

$$\lim_{h \to 0^+} \frac{6(1+h) - 3 - (6(1) - 3)}{h} =$$

$$\lim_{h \to 0^+} \frac{6 + 6h - 3 - 6 + 3}{h} =$$

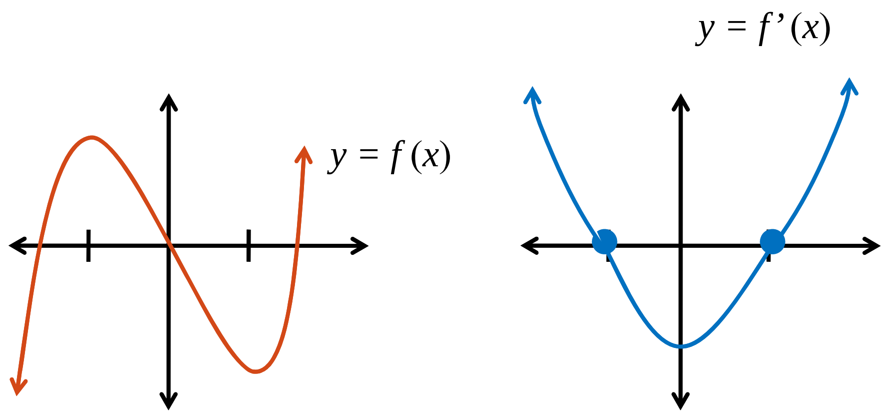
$$\lim_{h \to 0^+} \frac{6h}{h} = 6$$

Since the left-hand and right-hand derivatives both equal 6 as x approaches 1, this function does have a derivative at $x = 1$.

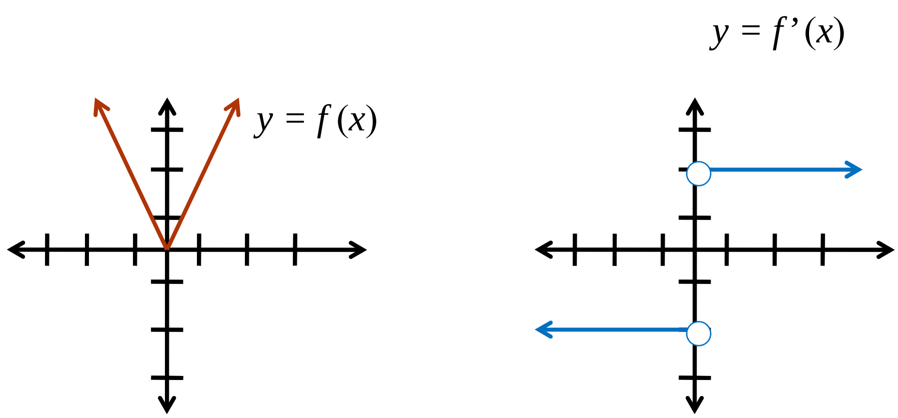
XII. Graphical Relationships

Since the derivative of f at a point a is the slope of the tangent line to f at a, we can get a good idea of what the graph of the function f' looks like by **estimating the slopes** at various points along the graph of f.

Ex. 1 – Given the graph of *f* below, estimate the graph of *f'*.



Ex. 2 – Given the graph of *f* below, estimate the graph of *f'*.



Ex. 3 – Given the graph of *f* below, estimate the graph of *f'*.