Notes 9.5 – Mathematical Induction

I. Recursion

A.) Question: Consider the recursive sequence

$$a_1 = 2$$

$$a_n = \left(a_{n-1}\right)^2 + 3a_{n-1} \text{ for all } n \geq 2$$

we know a_{364} is defined because we defined it to be for all $n \geq 2$.

II. Towers of Hanoi

A.) Demo

B.) Minimum # of Moves to move the tower containing:

WASHERS	MOVES
1	1
2	3
3	7
4	15
5	31
n	$2^n - 1$

C.) **Thm:** The minimum number of moves required to move a stack of n washers in a Tower of Hanoi game is

$$2^n - 1$$

D.) **Proof:**

1.) Step 1: **The Anchor:**

(Prove the formula is true for the first step)

$$2^1 - 1 = 1 \quad \checkmark\checkmark$$

2.) Step 2A: **The Inductive Hypothesis:**

(Assume the assertion holds true for $n = k$)

i.e., the minimum number of moves for k washers is

$$2^k - 1$$

Step 2B: **The Inductive Step:**

(If the assertion is true for $n = k$, then prove the assertion is true for $k + 1$)

$$k+1+k$$

$$\left(2^k-1\right)+\left(1\right)+\left(2^k-1\right)=2^{k+1}-1$$

$$2\left(2^k\right)-1=2^{k+1}-1$$

$$2^{k+1}-1=2^{k+1}-1 \quad \checkmark$$

III. PRINCIPLE OF MATHEMATICAL INDUCTION:

A.) Let P_n be a statement about integer n. Then P_n is true for all positive integers n provided the following conditions are satisfied:

1.) (The anchor) P_1 is true.

2.) (The inductive step) if P_k is true, then P_{k+1} is true.

Think of dominoes!

Examples:

A. Use mathematical induction to prove the following is true for all positive integers *n*.

$$1+3+5+...+(2n-1)=n^2$$

Step 1:

$$P_1=1=1^2$$

Step 2: Assume that

$$P_k=1+3+5+...+(2k-1)=k^2$$

Step 3:

$$P_{k+1}=(k+1)^2$$

$$P_{k+1}=k^2+2k+1$$

$$P_{k+1}=1+3+5+...+(2k-1)+2k+1$$

$$P_{k+1}=1+3+5+...+(2k+1)$$

$$P_{k+1}=1+3+5+...+(2k+2-1)$$

$$P_{k+1}=1+3+5+...+(2(k+1)-1)$$

B. Use mathematical induction to prove that the following is true for all positive integers *n*.

$$8+10+12+\ldots+(2n+6)=n^2+7n$$

Step 1:

$$P_1=8=(1)^2+7(1)$$

Step 2: Assume that

$$P_k=8+10+12+\ldots+(2k+6)=k^2+7k$$

$$8+10+12+\ldots+(2k+6)+2(k+1)+6=k^2+7k+2(k+1)+6$$

What do we need to get on the right side?

$$(k+1)^2+7(k+1)$$

$$=k^2+7k+2(k+1)+6$$

$$=k^2+7k+2k+2+6$$

$$=k^2+9k+8$$

$$=(k^2+2k+1)+(7k+7)$$

$$(k+1)^2+7(k+1) \quad \checkmark$$

C. Use mathematical induction to prove the following is true for all integers *n*.

$$1^2+2^2+3^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}$$

Step 1:

$$P_1=1^2=\frac{1(1+1)\left[2(1)+1\right]}{6}=\frac{1(2)(3)}{6}=1$$

Step 2: Assume that

$$P_k=1^2+2^2+3^2+\ldots+k^2=\frac{k(k+1)(2k+1)}{6}$$

Step 3:

$$P_{k+1}=\frac{(k+1)\left[(k+1)+1\right]\left[2(k+1)+1\right]}{6}$$

$$P_{k+1}=\frac{2k^3+9k^2+13k+6}{6}$$

$$P_{k+1}=\frac{\left(2k^3+3k^2+k\right)+\left(6k^2+12k+6\right)}{6}$$

$$P_{k+1}=\frac{\left(2k^3+3k^2+k\right)}{6}+\frac{\left(6k^2+12k+6\right)}{6}$$

$$P_{k+1}=\frac{k(k+1)(2k+1)}{6}+\left(k^2+2k+1\right)$$

$$P_{k+1}=\frac{k(k+1)(2k+1)}{6}+(k+1)^2$$

$$P_{k+1}=1^2+2^2+3^2+\ldots+(k+1)^2$$

Example 4: Prove that the following is true for all positive inters n.

$$1+2+2^2+...+2^{n-1}=2^n-1$$

1. $P_1 = 1 = 2^1 - 1$

2. $P_k = 1+2+2^2+...+2^{k-1} = 2^k - 1$

3.
$$P_{k+1} = 2^{k+1} - 1 = 2\left(2^k\right) - 1$$
$$= 2^k + 2^k - 1$$
$$= \left(2^k - 1\right) + 2^k$$
$$= 1+2+2^2+...+2^{k-1}+2^k$$