Notes 9.4 –Series

III. Summation Notation

A. In *summation notation*, the sum of the terms of the sequence $\{a_1, a_2, \dots a_n\}$ is denoted

$$\sum_{k=1}^{n} a_k$$

Which is read "the sum of a_n from $k=1$ to n"

The variable k is called the *index of summation*.

B. Example 4:

$$\sum_{k=1}^{5} 2k = 2+4+6+8+10=30$$

C. Example 5:

$$\sum_{k=0}^{12} \cos(k\pi) = \cos(0)+\cos(\pi)+\cos(2\pi)+\ldots+\cos(12\pi)$$

$$=1-1+1-1+1+\ldots+-1+1=\boxed{1}$$

IV. Sums of Arithmetic and Geometric Sequences

A. Theorem: Sum of a finite Arithmetic Sequence.

Let $\{a_1, a_2, \ldots a_n\}$ be a finite arithmetic sequence with common difference d. Then the sum of the terms of the sequence is

$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \ldots + a_n$$

$$= n\left(\frac{a_1 + a_n}{2}\right)$$

$$= \frac{n}{2}\left[2a_1 + (n-1)d\right]$$

B. Example 6: A concert auditorium has 30 rows of seats. The first row contains 50 seats. As you move to the rear of the auditorium, each row has two more seats than the previous one. How many seats are in the auditorium?

$$a_1 = 50, d = 2$$

$$S_{30} = \frac{30}{2}\left[2(50) + (30-1)2\right] = \boxed{2370 \text{ seats}}$$

C. Theorem: Sum of a Finite Geometric Sequence.

Let $\{a_1, a_2, \ldots a_n\}$ be a finite geometric sequence with common ratio r (where r is not 0). Then the sum of the terms of the sequence is

$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \ldots + a_n$$

$$= \frac{a_1\left(1 - r^n\right)}{1 - r}$$

D. Example 7: Determine the value of the following partial sum (we say partial, because we are not adding all terms in the infinite sequence)

$$3 + \frac{9}{4} + \frac{27}{16} + \frac{81}{64}, + \ldots + 3\left(\frac{3}{4}\right)^9$$

$$a_1 = 3, r = \frac{3}{4}, n = 10$$

$$S_{10} = \frac{3\left[1 - \left(\frac{3}{4}\right)^{10}\right]}{1 - \frac{3}{4}} \approx \boxed{11.324}$$

V. Infinite Series

A. Consider the following geometric sequence:

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$$

The partial sums are below.

n	a_n	S_n
1	1/2	1/2=0.5
2	1/4	3/4=0.75
3	1/8	7/8=0.875
4	1/16	15/16=0.9375
5	1/32	31/32=0.96875
6	1/64	63/64=0.984375
7	1/128	127/128=0.9921875

Note that these numbers seem to approach 1

Taking the limit of the partial sum:

$$S = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2}\left(\frac{1}{2}\right)^{k-1}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{2}\left[1-\left(\frac{1}{2}\right)^{n}\right]}{1-\left(\frac{1}{2}\right)}$$

$$= \frac{\frac{1}{2}(1-0)}{1-\frac{1}{2}}$$

$$= 1$$

B. Definition: An *infinite series* is an expression of the form.

$$S = \sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots + a_n + \ldots$$

C. We say that the sequence of partial sums *converges* to a number if the limit of *partial sums* is a number *S*, which is called the *sum of the infinite series*.

D. If the *partial sums* do not approach a given number, then the series *diverges*.

D. Formula for an infinite series.

1. Theorem: Sum of an Infinite Geometric Series.

A geometric series converges iff |r|<1. If it does converge, then the sum is:

$$S = \frac{a}{1-r}$$

2. Example 8: Calculate the sum of the infinite geometric sequence

$$3, -3\cdot\frac{1}{5}, 3\cdot\left(\frac{1}{5}\right)^2, -3\cdot\left(\frac{1}{5}\right)^3, \ldots$$

$$S = \frac{3}{1-\left(-\frac{1}{5}\right)} = \frac{3}{\frac{6}{5}} = \boxed{\frac{5}{2}}$$

E. Example 9: Convert the following repeating decimals to fraction form.

$$0.\overline{312} = 0.312 + 0.000312 + 0.000000312 + \ldots$$

this is a geometric sequence with:

$$g_1 = 0.312, r = 0.001$$

$$S = \frac{0.312}{1-0.001} = \frac{0.312}{0.999} = \frac{312}{999} = \boxed{\frac{104}{333}}$$

$$0.\overline{9} = 0.9 + 0.09 + 0.009 + \ldots$$

this is a geometric sequence with:

$$g_1 = 0.9, r = 0.1$$

$$S = \frac{0.9}{1-0.1} = \frac{0.9}{0.9} = \boxed{1}$$