7.3: Multivariate Linear Systems and Row Operations

I. Solving Systems with Inverse Matrices

A. Systems can be written as matrices:

$$\begin{cases} x + 2y = 5 \\ 2x - 5y = -8 \end{cases} \rightarrow$$

coefficient matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & -5 \end{bmatrix}$

variable matrix $X = \begin{bmatrix} x \\ y \end{bmatrix}$

solution matrix $B = \begin{bmatrix} 5 \\ -8 \end{bmatrix}$

$$AX = B$$

Note that if *A* is nonsingular, then

$$X = A^{-1}B$$

Therefore:

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\boxed{(x, y) = (1, 2)}$$

II. Gaussian Elimination

A. Triangular (echelon) form

1. In triangular form, for a system of equations, the second equation does not involve *x*, the third equation does not involve *x* and *y*, the fourth equation does not involve *x, y*, and *z*, etc.
2. The first nonzero coefficient of each equation is 1
3. Examples:

$$\begin{cases} x+y+z=-1 \\ y-2z=4 \\ z=4 \end{cases} \qquad \begin{cases} x+5y+z=-1 \\ z=6 \end{cases}$$

4. Systems in this form are easily solved with substitution.

5. Example 1: solve the following system of equations in triangular form.

$$\begin{cases} x+2y-5z=-1 \\ y-4z=7 \\ z=-2 \end{cases}$$

$$\begin{aligned} y-4z&=7 \\ y-4(-2)&=7 \\ y&=-1 \end{aligned} \qquad \begin{aligned} x+2y-5z&=-1 \\ x+2(-1)-5(-2)&=-1 \\ x&=-9 \end{aligned}$$

$$\boxed{(x,y,z)=(-9,-1,-2)}$$

This method is called *back substitution*.

B. Elementary operations on linear systems (used to change systems into triangular form).

1. Interchange any two equations
2. Multiply an equation by a nonzero real number
3. Add to an equation a multiple of another equation.

4. Example 2: Use a sequence of elementary operations to transform the following system into triangular form. Then, solve the system

$$\begin{cases} 3y+4z=15 \\ x-y-z=-4 \\ 2x+5y+9z=39 \end{cases}$$

First, we interchange the first and second equations, because the first equation has no x-term

$$\begin{cases} x-y-z=-4 \\ 3y+4z=15 \\ 2x+5y+9z=39 \end{cases}$$

Next, add (-2) times the first equation to the third

$$\begin{cases} x-y-z=-4 \\ 3y+4z=15 \\ 7y+11z=47 \end{cases}$$

Next, multiply the second equation by 1/3 to obtain

$$\begin{cases} x - y - z = -4 \\ y + \frac{4}{3}z = 5 \\ 7y + 11z = 47 \end{cases}$$

Now, add (-7) times the second equation to the third

$$\begin{cases} x - y - z = -4 \\ y + \frac{4}{3}z = 5 \\ \frac{5}{3}z = 12 \end{cases}$$

Finally multiply the third equation by (3/5) to obtain the triangular form

$$\begin{cases} x - y - z = -4 \\ y + \frac{4}{3}z = 5 \\ z = \frac{36}{5} \end{cases}$$

Using back-substitution to solve the triangular system:

$$z = \frac{36}{5}$$

$$y = 5 - \frac{4}{3}z = 5 - \frac{4}{3}\left(\frac{36}{5}\right) = -\frac{23}{5}$$

$$x = y + z - 4 = -\frac{23}{5} + \frac{36}{5} - 4 = -\frac{7}{5}$$

III. Elementary Row Operations and Row Echelon Form

A. In order to simplify this process, we often use matrices to keep track of these steps

B. Example 3: write the following matrix in an *augmented matrix*, use elementary row operations to write in triangular form, and then solve the system.

augmented matrix

$$\begin{cases} x+3y+2z=1 \\ 2x+y-z=2 \\ x+y+z=2 \end{cases} \longrightarrow \begin{bmatrix} 1 & 3 & 2 & 1 \\ 2 & 1 & -1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 2 & 1 \\ 2 & 1 & -1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix} \xrightarrow{(-2)R_1+R_2} \begin{bmatrix} 1 & 3 & 2 & 1 \\ 0 & -5 & -5 & 0 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow{(-1)R_1+R_3} \begin{bmatrix} 1 & 3 & 2 & 1 \\ 0 & -5 & -5 & 0 \\ 0 & -2 & -1 & 1 \end{bmatrix}$$

$$\xrightarrow{\left(-\frac{1}{5}\right)R_2} \begin{bmatrix} 1 & 3 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -2 & -1 & 1 \end{bmatrix}$$

$$\xrightarrow{2R_2 + R_3} \begin{bmatrix} 1 & 3 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\begin{cases} x + 3y + 2z = 1 \\ y + z = 0 \\ z = 1 \end{cases}$$

$$z = 1$$

$$y = -z = -1$$

$$x = 1 - 3(-1) - 2(1) = 2$$

$$\boxed{(x, y, z) = (2, -1, 1)}$$

IV. Reduced Row Echelon Form

A. We can continue to apply elementary row operations until the coefficient matrix is the identity matrix

B. This is called *reduced row echelon form*.

C. In this case, we can simply read the solution from the equation.

C. Example 4: rewrite the row echelon matrix from example 3 in reduced row echelon form.

$$\begin{bmatrix} 1 & 3 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{(-3)R_2 + R_1} \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow[-R_3 + R_2]{R_3 + R_1} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\boxed{(x, y, z) = (2, -1, 1)}$$

D. Use your calculator to write the following matrices in reduced row echelon form, then solve the system.

1. $$\begin{cases} x + y - 2z = 1 \\ y - 3z = 4 \\ 2x + 3y - 7z = 5 \end{cases} \qquad A = \begin{bmatrix} 1 & 1 & -2 & 1 \\ 0 & 1 & -3 & 4 \\ 2 & 3 & -7 & 5 \end{bmatrix}$$

$$rref(A) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{since } 0 = 1, \text{ no solution}$$

$$2.\ \begin{cases} x+y+4z=1 \\ -2x-y+z=2 \\ 3x-2y+3z=1 \end{cases} \qquad A=\begin{bmatrix} 1 & 1 & 4 & 1 \\ -2 & -1 & 1 & 2 \\ 3 & -2 & 3 & 1 \end{bmatrix}$$

$$rref(A)=\begin{bmatrix} 1 & 0 & 0 & -0.5 \\ 0 & 1 & 0 & -0.5 \\ 0 & 0 & 1 & 0.5 \end{bmatrix}$$

$$\boxed{(x,y,z)=\left(-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right)}$$

$$3.\ \begin{cases} x+y+z=3 \\ 2x+y+4z=8 \\ x+2y-z=1 \end{cases} \qquad A=\begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & 1 & 4 & 8 \\ 1 & 2 & -1 & 1 \end{bmatrix}$$

$$rref(A)=\begin{bmatrix} 1 & 0 & 3 & 5 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

infinitely many solutions

$$\begin{bmatrix} 1 & 0 & 3 & 5 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{matrix} 1x+3z=5 \\ y-2z=-2 \end{matrix} \quad (-3z+5,2z-2,z)$$