Notes 7.3 – Partial Fractions

I. Composing Fractions

A.) Ex. 1– Rewrite the following expression as a single rational expression.

$$\frac{3}{x-4} + \frac{2}{x+3}$$

$$\frac{3(x+3)+2(x-4)}{(x-4)(x+3)} = \frac{5x+1}{x^2-x-12}$$

II. Decomposing Fractions

A.) Steps:

1.) If the degree of $f \geq$ the degree of d, then

$$\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$$

2.) Factor $d(x)$ into a product of factors of the form $(mx + n)^u$ or $(ax^2 + bx + c)^v$ when $(ax^2 + bx + c)$ is irreducible.

3a.) If $d(x)$ is of the form $(mx + n)^u$ →

$$\frac{A_1}{mx+n} + \frac{A_2}{(mx+n)^2} + ... + \frac{A_u}{(mx+n)^u}$$

3b.) If $d(x)$ is of the form $(ax^2 + bx + c)^v$ →

$$\frac{B_1 x + C_1}{ax^2 + bx + c} + \frac{B_2 x + C_2}{(ax^2 + bx + c)^2} + ... + \frac{B_v x + C_v}{(ax^2 + bx + c)^v}$$

B.) For example,

$$\frac{2x+5}{x^2(x-2)(x^2+3x+1)^2} =$$

$$\frac{A_1}{x} + \frac{A_2}{x^2} + \frac{B_1}{x-2} + \frac{C_1x+D_1}{x^2+3x+1} + \frac{C_2x+D_2}{(x^2+3x+1)^2}$$

C.) Ex. 2- Find the partial fraction decomposition of $\dfrac{2x+5}{(x+3)(x-2)}$

$$\frac{2x+5}{(x+3)(x-2)} = \frac{A_1}{x+3} + \frac{A_2}{x-2}$$

Multiply both sides by the LCD

$$(x+3)(x-2)\left(\frac{2x+5}{(x+3)(x-2)} = \frac{A_1}{x+3} + \frac{A_2}{x-2}\right)$$

$$2x + 5 = A_1(x - 2) + A_2(x + 3)$$

$$2x + 5 = A_1 x - 2A_1 + A_2 x + 3A_2$$

$$2x + 5 = A_1 x + A_2 x - 2A_1 + 3A_2$$

$$2x + 5 = x(A_1 + A_2) + (-2A_1 + 3A_2)$$

Therefore

$$2 = A_1 + A_2$$
$$5 = -2A_1 + 3A_2$$

$$\begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

Solving the matrix equation gives us

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{5} \\ \dfrac{9}{5} \end{bmatrix}$$

$$\frac{2x + 5}{(x + 3)(x - 2)} = \frac{\dfrac{1}{5}}{x + 3} + \frac{\dfrac{9}{5}}{x - 2}$$

$$\frac{2x + 5}{(x + 3)(x - 2)} = \frac{1}{5(x + 3)} + \frac{9}{5(x - 2)}$$

SUPPORT GRAPHICALLY!!!

What if we were to substitute $x = 2$ into the equation below?

$$2x + 5 = A_1(x-2) + A_2(x+3)$$

$$2(2) + 5 = A_1(2-2) + A_2(2+3)$$

$$9 = 0 + 5A_2$$

$$\frac{9}{5} = A_2$$

Now, substitute $x = -3$ into the equation.

$$2(-3) + 5 = A_1(-3-2) + A_2(-3+3)$$

$$-1 = -5A_1 + A_2(0)$$

$$\frac{1}{5} = A_1$$

D.) Ex.3 - Find the partial fraction decomposition of

$$\frac{x^2 + 2x + 4}{x^3 + 6x^2 + 9x} = \frac{x^2 + 2x + 4}{x(x+3)^2}$$

$$= \frac{A_1}{x} + \frac{A_2}{x+3} + \frac{A_3}{(x+3)^2}$$

$$x^2 + 2x + 4 = A_1(x+3)^2 + A_2 x(x+3) + A_3 x$$

$$x^2 + 2x + 4 = A_1 x^2 + 6A_1 x + 9A_1 + A_2 x^2 + 3A_2 x + A_3 x$$
$$= A_1 x^2 + A_2 x^2 + 6A_1 x + 3A_2 x + A_3 x + 9A_1$$
$$= (A_1 + A_2)x^2 + (6A_1 + 3A_2 + A_3)x + 9A_1$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 6 & 3 & 1 \\ 9 & 0 & 0 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \qquad \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} \dfrac{4}{9} \\ \dfrac{5}{9} \\ -\dfrac{7}{3} \end{bmatrix}$$

Try this, substitute $x = 0$ into the equation below.

$$x^2 + 2x + 4 = A_1(x+3)^2 + A_2 x(x+3) + A_3 x$$

$$4 = A_1(3)^2 + A_2(0)(3) + A_3(0)$$

$$4 = 9A_1$$

$$\frac{4}{9} = A_1$$

Now, substitute $x = -3$ into the equation below.

$$x^2 + 2x + 4 = A_1(x+3)^2 + A_2 x(x+3) + A_3 x$$

$$7 = A_1(0)^2 + A_2(-3)(0) + A_3(-3)$$

$$7 = -3A_3$$

$$-\frac{7}{3} = A_3$$

We can now choose any value for x and substitute it into one of the equations along with A_1 and A_3 and solve for A_2.

III. Denominators of Irreducible Quadratic Factors

A.) If $d(x)$ is of the form $(ax^2 + bx + c)^v$ →

$$\frac{B_1 x + C_1}{ax^2 + bx + c} + \frac{B_2 x + C_2}{\left(ax^2 + bx + c\right)^2} + \dots + \frac{B_v x + C_v}{\left(ax^2 + bx + c\right)^v}$$

Since subscripts can get confusing, we can simply use different letters.

$$\frac{Ax + B}{ax^2 + bx + c} + \frac{Cx + D}{\left(ax^2 + bx + c\right)^2} + \dots + \frac{Yx + Z}{\left(ax^2 + bx + c\right)^v}$$

B.) Ex. 4 – Write the partial fraction decomposition of

$$\frac{x^3 + x^2}{\left(x^2 + 4\right)^2}$$

$$\frac{x^3 + x^2}{\left(x^2 + 4\right)^2} = \frac{Ax + B}{x^2 + 4} + \frac{Cx + D}{\left(x^2 + 4\right)^2}$$

$$\left(x^2 + 4\right)^2 \left(\frac{x^3 + x^2}{\left(x^2 + 4\right)^2} = \frac{Ax + B}{x^2 + 4} + \frac{Cx + D}{\left(x^2 + 4\right)^2} \right)$$

$$x^3 + x^2 = \left(x^2 + 4\right)\left(Ax + B\right) + Cx + D$$

$$x^3 + x^2 = Ax^3 + Bx^2 + 4Ax + 4B + Cx + D$$

$$x^3 + x^2 = Ax^3 + Bx^2 + (4A + C)x + 4B + D$$

$A = 1$

$B = 1$

$4A + C = 0$

$4B + D = 0$

$4(1) + C = 0$ $4(1) + D = 0$

$C = -4$ $D = -4$

$$\frac{x^3 + x^2}{(x^2 + 4)^2} = \frac{x + 1}{x^2 + 4} - \frac{4(x + 1)}{(x^2 + 4)^2}$$

Unfortunately, there is no shortcut method for denominators with irreducible quadratics.

C.) Ex. 5 – Find the partial fraction decomposition of $\dfrac{x^2 + 4x + 1}{x^3 - x^2 + x - 1}$

$$\frac{x^2 + 4x + 1}{x^3 - x^2 + x - 1} = \frac{x^2 + 4x + 1}{x^2(x - 1) + (x - 1)} = \frac{x^2 + 4x + 1}{(x^2 + 1)(x - 1)}$$

$$\frac{x^2 + 4x + 1}{(x^2 + 1)(x - 1)} = \frac{A}{x - 1} + \frac{Bx + C}{x^2 + 1}$$

$$(x^2 + 1)(x - 1)\left(\frac{x^2 + 4x + 1}{(x^2 + 1)(x - 1)} = \frac{A}{x - 1} + \frac{Bx + C}{x^2 + 1}\right)$$

$$x^2 + 4x + 1 = A(x^2 + 1) + (Bx + C)(x - 1)$$

$$x^2 + 4x + 1 = Ax^2 + A + Bx^2 + Cx - Bx - C$$

$$x^2 + 4x + 1 = x^2(A + B) + x(C - B) + A - C$$

$$
\begin{aligned}
A + B &= 1 \\
C - B &= 4 \\
A - C &= 1
\end{aligned}
\quad \xrightarrow{\ A\ } \quad
\begin{bmatrix}
1 & 1 & 0 & 1 \\
0 & -1 & 1 & 4 \\
1 & 0 & -1 & 1
\end{bmatrix}
$$

$$
\xrightarrow{\ \text{rref}(A)\ }
\begin{bmatrix}
1 & 0 & 0 & 3 \\
0 & 1 & 0 & -2 \\
0 & 0 & 1 & 2
\end{bmatrix}
\qquad
\begin{aligned}
A &= 3 \\
B &= -2 \\
C &= 2
\end{aligned}
$$

$$\frac{x^2 + 4x + 1}{x^3 - x^2 + x - 1} = \frac{3}{x - 1} + \frac{-2x + 2}{x^2 + 1}$$

IV. Applications of Partial Fractions

A.) Ex. 6 - Graph the following function on your calculator. $\dfrac{2x^2 - x - 15}{x^2 - 4}$

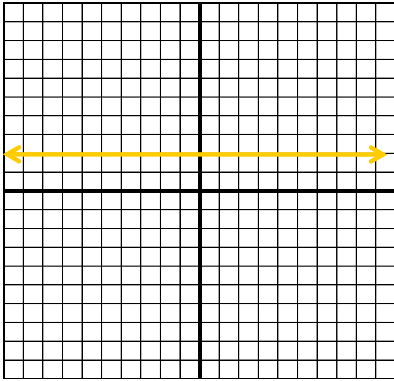Before we graph it, what can you tell me about what the graph will look like?
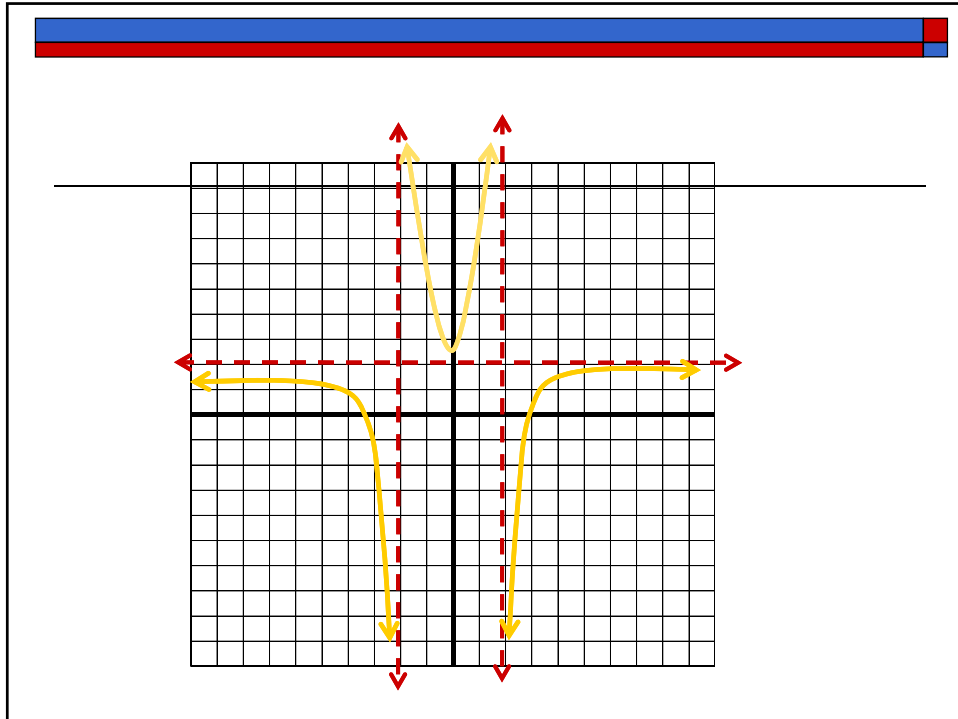
Now, lets look at the partial fraction decomposition of the function and the graphs of the individual pieces.

$$\frac{2x^2 - x - 15}{x^2 - 4} = 2 + \frac{5}{4(x+2)} - \frac{9}{4(x-2)}$$

$$\frac{2x^2 - x - 15}{x^2 - 4} = 2 + \frac{5}{4(x+2)} - \frac{9}{4(x-2)}$$
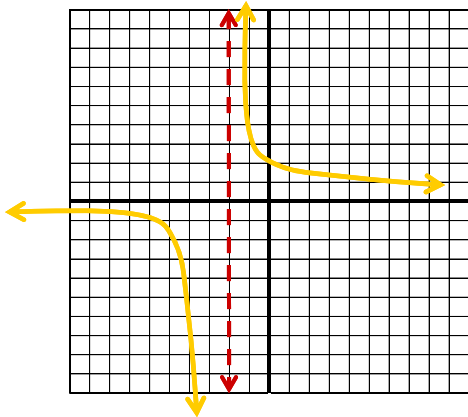
First Piece -

$$y = 2$$

Models the end behavior of the graph

$$\frac{2x^2 - x - 15}{x^2 - 4} = 2 + \frac{5}{4(x+2)} - \frac{9}{4(x-2)}$$

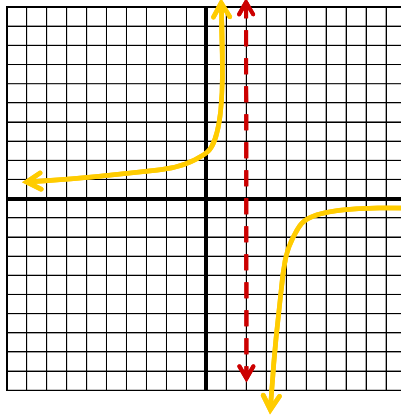2nd Piece -

$$y = \frac{5}{4(x+2)}$$

Models the behavior of the graph near the vertical asymptote $x = -2$

$$\frac{2x^2 - x - 15}{x^2 - 4} = 2 + \frac{5}{4(x+2)} - \frac{9}{4(x-2)}$$
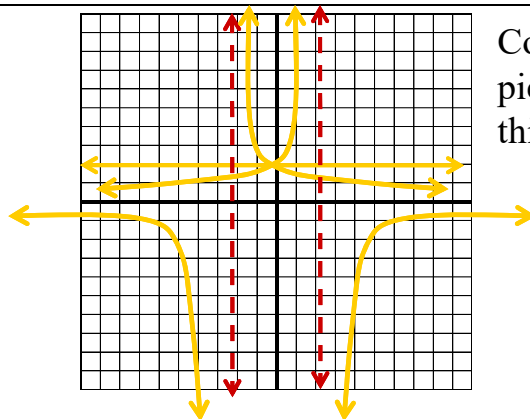
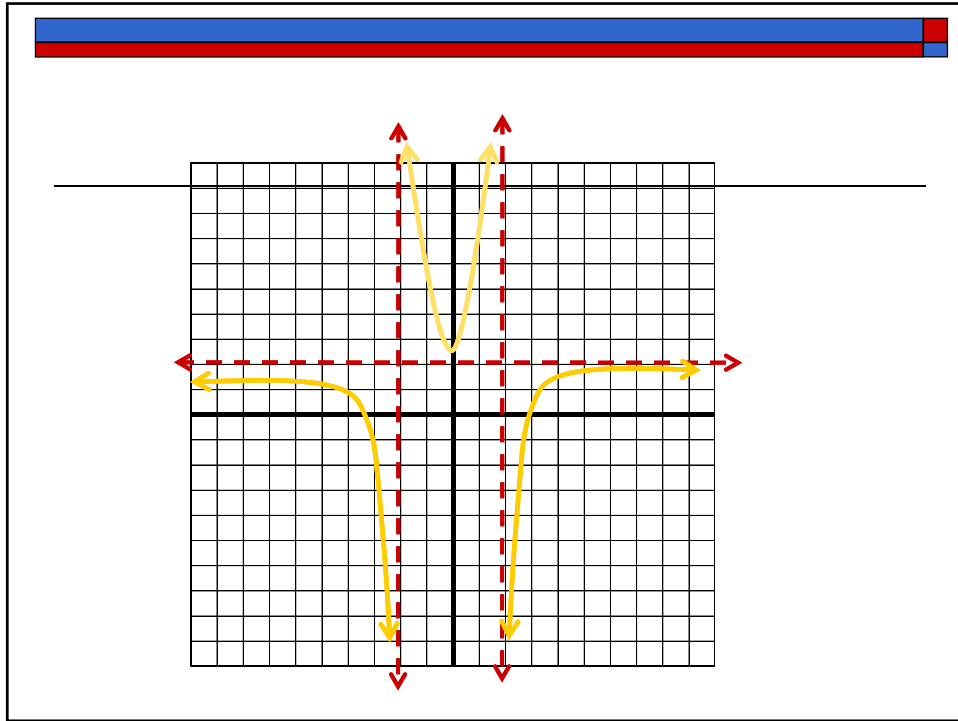3rd Piece -

$$y = -\frac{9}{4(x-2)}$$

Models the behavior of the graph near the vertical asymptote $x = 2$

$$\frac{2x^2 - x - 15}{x^2 - 4} = 2 + \frac{5}{4(x+2)} - \frac{9}{4(x-2)}$$

Combining the 3 pieces gives us this…

Or, more precisely, something like this…

B.) Ex. 7 – Explain in sentences the three different parts to the graph of the following function. Then, graph the function .

$$y = f(x) = -x + 1 + \frac{1}{x+3} - \frac{5}{x-2}$$

First, the graph's end-behavior asymptote will
be modeled by the graph $y = -x + 1$.

Second, the graph of the function at $x = -3$ will
closely resemble the behavior of the graph of
the hyperbola $y = \dfrac{1}{x+3}$ with its branches in the
upper right and lower left quadrants.

Third, the graph of the function at $x = 2$ will
closely resemble the behavior of the graph of
the hyperbola $y = -\dfrac{5}{x-2}$ with its branches in the
upper left and lower right quadrants.

Combining the three separate parts gives us a
graph that looks like