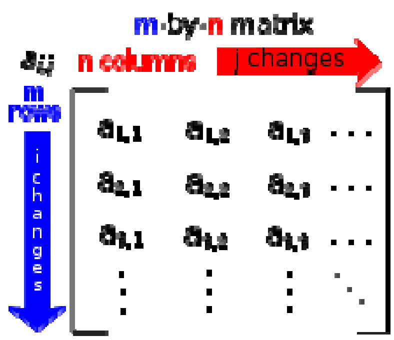
7.2: Matrix Algebra

I. Matrices

A. A matrix is a rectangular array



Where $a_{1\,1}$, $a_{2\,1}$, are algebraic expressions, called the *elements* or *entries* of the matrix.

B. Notes:

1. The *row subscript* is the first subscript i
2. The *column subscript* is the second subscript j.
3. The *order* of a matrix is its rows x columns
4. If $m=n$, it is a square matrix
5. Two matrices are equal if they have the same order and each corresponding element is equal.

II. Matrix Addition, Subtraction, and scalar multiplication.

A. In order to add or subtract matrices:

1. They must have the same order.
2. You add or subtract all corresponding entries

B. In order to multiply by a scalar, you must multiply each element of a matrix by the scalar.

C. Example 1: Perform the operations for matrices *A* and *B* below.

$$A=\begin{bmatrix}5 & 0\\ 3 & -1\end{bmatrix}, B=\begin{bmatrix}-2 & 4\\ 1 & 3\end{bmatrix}$$

1. $A+B$

$$\begin{bmatrix}5 & 0\\ 3 & -1\end{bmatrix}+\begin{bmatrix}-2 & 4\\ 1 & 3\end{bmatrix}=\begin{bmatrix}5-2 & 0+4\\ 3+1 & -1+3\end{bmatrix}=\boxed{\begin{bmatrix}3 & 4\\ 4 & 2\end{bmatrix}}$$

2. $-2A$

$$-2\begin{bmatrix}5 & 0\\ 3 & -1\end{bmatrix}=\begin{bmatrix}-2\cdot 5 & -2\cdot 0\\ -2\cdot 3 & -2(-1)\end{bmatrix}=\boxed{\begin{bmatrix}-10 & 0\\ -6 & 2\end{bmatrix}}$$

3. $3A-4B$

$$3\begin{bmatrix}5 & 0\\ 3 & -1\end{bmatrix}-4\begin{bmatrix}-2 & 4\\ 1 & 3\end{bmatrix}=\begin{bmatrix}3(5)-4(-2) & 3(0)-4(4)\\ 3(3)-4(1) & 3(-1)-4(3)\end{bmatrix}$$

$$=\boxed{\begin{bmatrix}23 & -16\\ 5 & -15\end{bmatrix}}$$

D. Properties

1. The *mxn* matrix consisting of all zeros is *O=[0]* is called the *zero matrix* and is called the *additive identity*.
2. The *mxn* matrix *B=–A* is the additive inverse of *A*.

II. Matrix Multiplication.

A. In order to multiply matrices A*B, we multiply each row from *A* by each column from *B.*

B. Note: Matrix multiplication is not commutative

$$AB \neq BA$$

(in most cases)

A. Example 2: compute the product below:

$$\begin{bmatrix} 5 & -1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 0 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 \\ 0 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 5(3)+-1(0) & 5(1)+-1(9) \\ 2(3)+0(0) & 2(1)+0(9) \end{bmatrix}$$

$$= \begin{bmatrix} 15 & -4 \\ 6 & 2 \end{bmatrix}$$

B. Note: matrix multiplication is only defined if the first matrix has as many columns as the second matrix has rows

C. Example 3: Determine which of the following products are defined and the size of those products.

1. A is 2×5, B is 5×7

$$2 \times 5 \cdot 5 \times 7$$

same, so product is defined

the solution will be 2×7

2. A is 3×4, B is 2×4

not defined

3. A is 4×4, B is 4×3

defined, 4×3

III. Identity and Inverse Matrices

A. The *nxn* matrix with 1's on the main diagonal (upper left to lower right) and 0's everywhere else is the *identity matrix of order nxn* or I_n.

$$I_1 = [1] \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

B. For any *mxn* matrix *A*:

$$I_m A = A I_n = A$$

C. Definition: Inverse of a Square Matrix

Let *A* be an *nxn* matrix. If there is a matrix *B* such that

$$AB = BA = I_n$$

Then *B* is the *inverse* of *A*. We write $B=A^{-1}$ *(read "A inverse")*

–Note that only square matrices have inverses, and that not every square matrix has an inverse

–Those with an inverse are called *nonsingular*, and those without an inverse are *singular*

Finding the Inverse Matrix A^{-1}

A.) Ex. 4 – Find the multiplicative inverse of

$$A = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \times \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$3a - 2c = 1$$
$$3b - 2d = 0$$
$$-a + c = 0$$
$$-b + d = 1$$

$$\begin{aligned} &3a - 2c = 1 \\ &\underline{3[-a + c = 0]} \\ &c = 1 \\ &a = 1 \end{aligned} \qquad \begin{aligned} &3b - 2d = 0 \\ &\underline{3[-b + d = 1]} \\ &d = 3 \\ &b = 2 \end{aligned}$$

$$\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ is the inverse of A

IV. Determinant of a Square Matrix

A. Inverse of a 2 x 2 matrix

If *ad–bc* is not equal to zero, then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

ad–bc is the *determinant* of the 2 x 2 matrix above, and is denoted:

$$\det A = \det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

If the *determinant* is 0, the matrix is *singular*

B. Example 4: Calculate the inverse of the following matrix.

$$A = \begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix}$$

$$\det A = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -d & a \end{bmatrix}$$

$$= \frac{1}{2\cdot 1 - 4(-3)}\begin{bmatrix} 1 & 3 \\ -4 & 2 \end{bmatrix}$$

$$= \frac{1}{14}\begin{bmatrix} 1 & 3 \\ -4 & 2 \end{bmatrix} = \boxed{\begin{bmatrix} 1/14 & 3/14 \\ -2/7 & 1/7 \end{bmatrix}}$$

C. Minor (minor determinant)

1. The *minor* M_{ij} corresponding to the element a_{ij} is the determinant of the *(n-1) x (n-1)* matrix obtained by deleting the row and column containing a_{ij}.
2. Example 5: calculate $M_{2\,3}$ for the matrix below:

$$B = \begin{bmatrix} -1 & 0 & 3 \\ 2 & 1 & 0 \\ 3 & 4 & -1 \end{bmatrix}$$

$$M_{2\,3} = \begin{vmatrix} -1 & 0 \\ 3 & 4 \end{vmatrix} = (-1)\cdot 4 - 3\cdot 0 = \boxed{-4}$$

D. Cofactor

The cofactor A_{ij} is:

$$A_{ij} = (-1)^{i+j} M_{ij}$$

E. Determinant of a Square matrix

The determinant of a square matrix is the sum of the entries of any row multiplied by their respective cofactors

For example, the *ith* row gives

$$\det A = |A| = a_{i1}A_{i1} + a_{i2}A_{i2} + \ldots + a_{in}A_{in}$$

F. Calculate the determinant of the matrix

$$A=\begin{bmatrix}4 & -1 & 5\\ 0 & -1 & 1\\ 1 & -1 & 1\end{bmatrix}$$

$$|A|=4(-1)^{1+1}\begin{vmatrix}-1 & 1\\ -1 & 1\end{vmatrix}+(-1)(-1)^{1+2}\begin{vmatrix}0 & 1\\ 1 & 1\end{vmatrix}+5(-1)^{1+3}\begin{vmatrix}0 & -1\\ 1 & -1\end{vmatrix}$$

$$=4\left[(-1)\cdot 1-(-1)\cdot 1\right]+1(0\cdot 1-1\cdot 1)+5\left[0\cdot(-1)-1(-1)\right]$$

$$=4\cdot 0+1(-1)+5\cdot 1$$

$$=\boxed{4}$$

note:you will get the same answer using any row

other method:

$$A=\begin{bmatrix}4 & -1 & 5\\ 0 & -1 & 1\\ 1 & -1 & 1\end{bmatrix}$$

$$A=\begin{bmatrix}4 & -1 & 5\\ 0 & -1 & 1\\ 1 & -1 & 1\end{bmatrix}\begin{matrix}4 & -1\\ 0 & -1\\ 1 & -1\end{matrix}$$

$$=\left[4\cdot(-1)\cdot 1+(-1)\cdot 1\cdot 1+5\cdot 0\cdot(-1)\right]$$

$$-\left[1\cdot(-1)\cdot 5+(-1)\cdot 1\cdot 4+1\cdot 0\cdot(-1)\right]$$

$$=\left[-4+-1+0\right]-\left[-5+-4+0\right]$$

$$=\boxed{4}$$

G. Properties of Matrix operations: see page 585 in text.

C.) The only way we have of finding the inverse of an n by n matrix for $n > 3$ is to use cofactors or our graphing calculator!

Find the inverse of A.

$$A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & 2 \\ 1 & 3 & -1 \end{bmatrix}$$

Matrix – Edit - Quit

Matrix – Names - x^{-1} - ENTER

$$A = \begin{bmatrix} \frac{3}{7} & \frac{-2}{7} & \frac{-1}{7} \\ \frac{-1}{14} & \frac{3}{14} & \frac{5}{14} \\ \frac{3}{14} & \frac{5}{14} & \frac{-1}{14} \end{bmatrix}$$

VIII. Apps: Reflection Matrices

To reflect across the… multiply $[x, y]$ by…

x-axis: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

y-axis: $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

origin: $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$