6.3 PARAMETRIC EQUATIONS AND MOTION

I. Definition

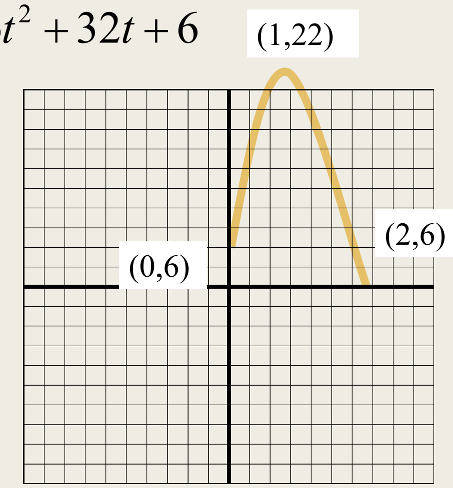
A.) Parametric Curve – Graph of ordered pairs (x, y) where $x = f(t)$ and $y = g(t)$.

B.) Parametric Equations– Any equation in the form of $x = f(t)$ and $y = g(t)$. $\boldsymbol{t}$ is called the parameter on an interval I.

Ex. 1– "Old School" – A ball is thrown in the air with an initial velocity of 32 ft./sec from a height of 6 ft. Write an equation to model the height of the ball as a function of time.

$$h(t) = -16t^2 + 32t + 6$$

Let's look at the graph



If the ball is thrown directly upward, we can model the position of the ball using two different functions of *t*.

$$y = -16t^2 + 32t + 6 \qquad x = 2$$

The horizontal position of the ball is now on a vertical axis, as it is in "real-life". While the graph models its vertical position with respect to time, we can also see the velocity and the acceleration by changing our mode to parametric and our line to -0 (Tracer Mode in Y=)

II. Graphing by Hand

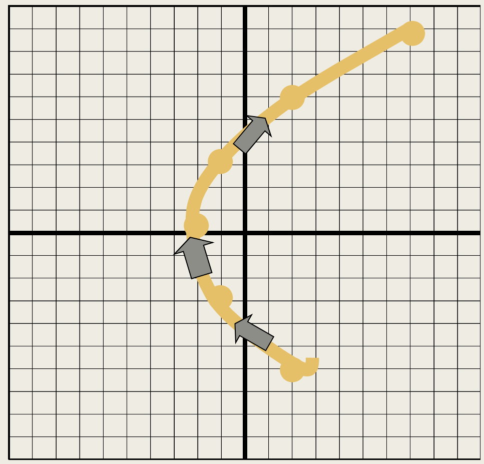
Ex. 2- Graph the following parametric equation by hand.

$$x = t^2 - 2$$

$$y = 3t \quad \text{for } -2 \le t \le 3$$

Make a table:

t	x	y
-2	2	-6
-1	-1	-3
0	-2	0
1	-1	3
2	2	6
3	7	9



III. Eliminating the Parameter

A.) Ex. 3- Determine the graph to the parametric equation by eliminating the parameter.

$$x = 2t + 3$$

$$y = t^2 \quad \text{for } -3 \le t \le 3$$

$$x = 2t + 3 \qquad y = \left(\frac{x-3}{2}\right)^2$$

$$t = \frac{x-3}{2} \qquad y = \frac{1}{4}x^2 - \frac{3}{2}x + \frac{9}{4} \qquad -3 \le x \le 9$$

B.) Ex. 4- Determine the graph to the parametric equation by eliminating the parameter.

$$x = \cos t \quad y = \sin t$$

$$t = \cos^{-1} x$$

$$y = \sin\left(\cos^{-1} x\right)$$ ← Very unfamiliar and hard to work with!

What if we squared both sides of both equations?

$$x^2 = \cos^2 t$$

$$y^2 = \sin^2 t$$

Now, add the equations!

$$x^2 + y^2 = \cos^2 t + \sin^2 t$$

$$x^2 + y^2 = 1$$ ← The circle centered at the origin with a radius of 1

IV. Lines and Line Segments

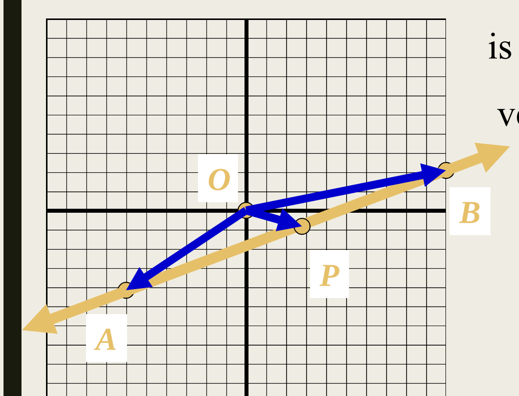
A.) Ex. 5- Find the parameterization for the line through the points (-3, -3) and (5, 1).

Let $P(x, y)$ be a point on the line through A and B and let O be the origin.

The vector $\overrightarrow{OP} - \overrightarrow{OA} = \overrightarrow{AP}$

is a scalar multiple of the

vector $\overrightarrow{OB} - \overrightarrow{OA} = \overrightarrow{AB}$



$$\overrightarrow{OP} = \langle x, y \rangle \quad \overrightarrow{OA} = \langle -3, -3 \rangle \quad \overrightarrow{OB} = \langle 5, 1 \rangle$$

$$\overrightarrow{OP} - \overrightarrow{OA} = t\left(\overrightarrow{OB} - \overrightarrow{OA}\right)$$

$$\langle x+3, y+3 \rangle = t\langle 8, 4 \rangle$$

$$x + 3 = 8t \quad y + 3 = 4t$$

$$x = 8t - 3 \quad y = 4t - 3$$

$$-\infty < t < \infty$$

Ex. 6 - Find the parameterization for the line through the points *A*=(3, 4) and B=(6, -3).

$$\overrightarrow{OP} = \langle x, y \rangle \quad \overrightarrow{OA} = \langle 3, 4 \rangle \quad \overrightarrow{OB} = \langle 6, -3 \rangle$$

$$\overrightarrow{OP} - \overrightarrow{OA} = t\left(\overrightarrow{OB} - \overrightarrow{OA}\right)$$

$$\langle x-3, y-4 \rangle = t\langle 3, -7 \rangle$$

$$x - 3 = 3t \quad y - 4 = -7t$$

$$x = 3t + 3 \quad y = -7t + 4 \qquad -\infty < t < \infty$$

What if we wanted the line segment with endpoints *A* and *B*?

Which values of *t* produce these endpoints?

$$3t + 3 = 3 \quad 3t + 3 = 6$$

$$t = 0 \quad t = 1 \qquad 0 \le t \le 1$$

V. Simulating Motion

Ex. 7-A particle moves along a horizontal line so that the position s (in feet) at any time $t \geq 0$ seconds is given by the function $s(t) = t^3 - 8t + 1$. Estimate the values at which the particle changes direction given $-3 \leq t \leq 3$.

1. Graph it. Place the equation in X_{1T} and choose an arbitrary Y_{1T} value.

$$X_{1T} = T^3 - 8T + 1$$
$$Y_{1T} = 4$$

2. What kind of window do we need?

$$T:[-3,3] \quad X:[-10,50] \quad Y:[0,6]$$

3. Trace it.

$$t = \pm 1.637$$

Another way: Let $Y_{1T} = T$.

$$X_{1T} = T^3 - 8T + 1$$
$$Y_{1T} = T$$

Window:

$$T:[-3,3] \quad X:[-10,50] \quad Y:[-10,10]$$

Now we can visually see when the particle changes direction!

VI. Projectile Motion

A.) Not everything is straight vertically or horizontally. Take for example, hitting a baseball. There is both a horizontal and vertical component. The **velocity vector** is

$$\boldsymbol{v} = \langle v_0 \cos\theta, v_0 \sin\theta \rangle$$

The path that the object takes can be modeled by the parametric equations

$$x = (v_o \cos\theta)t$$ ← Horizontal Distance

$$y = -16t^2 + (v_o \sin\theta)t + y_0$$ ← Vertical Position

B.) Ex. 8- Sally hits a softball 2.5 feet above the plate with an initial speed of 110 feet per second at an angle of 22º with the horizontal. Will the ball clear the 5 ft. fence 250 feet away?

First, determine the equations for the ball:

$$X_{T1} = (110\cos 22°)t$$

$$Y_{T1} = -16t^2 + (110\sin 22°)t + 2.5$$

Now, determine the window

$$T:[0,5] - T\text{step}:.1 \quad X:[0,280] \quad Y:[0,30]$$

Now, graph and trace to see if it clears the wall

C.) Determine the equations for the wall:

$$X_{T2} = 250$$
$$Y_{T2} = 5$$

Change your MODE to Simultaneous

Now, just graph to see if it clears the wall

D.) General the equation for the wall:

$$X_{T2} = \text{Distance}$$
$$Y_{T2} = \frac{(\text{Wall's height}) \times T}{T\text{max}}$$

VII. Modeling Ferris Wheels

Ex. 9 - A Ferris wheel with a radius of 20 feet makes 1 complete revolution every 10 seconds. The lowest point of the Ferris wheel is 5 feet above the ground. Determine the parametric equations which will model the height of a rider starting in the 3 o'clock position at $t = 0$.

- Center the Ferris wheel on the vertical axis such that the center will be at the point (0, 25).
- We know, from Chapter 5 that

$$x = 20\cos\theta$$
$$y = 25 + 20\sin\theta$$

- But, θ must be in terms of t. Since it takes 10 sec. to complete 1 revolution,

$$\frac{360°}{10 \text{ sec}} = \frac{36°}{1 \text{ sec}}$$

- Therefore,

$$\theta = 36T \quad \text{and} \quad \begin{pmatrix} X_{T1} = 20\cos(36t) \\ Y_{T1} = 25 + 20\sin(36t) \end{pmatrix}$$