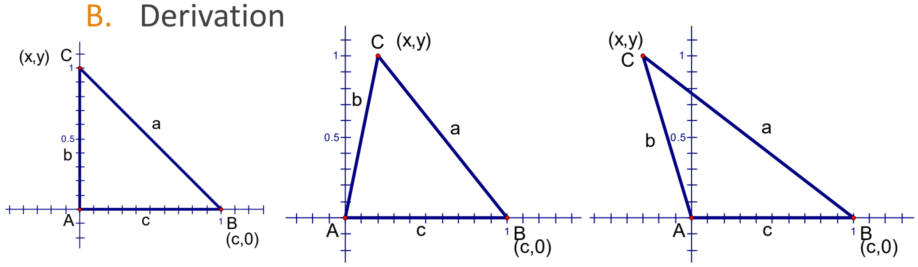
5.6: The Law of Cosines

I. Deriving the Law of Cosines
   A. Let triangle ABC be any triangle with sides and angles labeled in the usual way. Then:

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$

The Law of Cosines is used to solve any triangle where you are given, in order, SAS and SSS.

B. Derivation



In each of the three cases above,

$$\frac{x}{b} = \cos A \qquad \frac{y}{b} = \sin A$$

$$x = b\cos A \qquad y = b\sin A$$

Using the distance formula:

$$a = \sqrt{(x-c)^2 + (y-0)^2}$$

$$a^2 = (x-c)^2 + y^2$$

$$a^2 = (b\cos a - c)^2 + (b\sin a)^2$$

$$a^2 = b^2\cos^2 a - 2bc\cos A + c^2 + b^2\sin^2 a$$

$$a^2 = b^2(\cos^2 a + \sin^2 a) - 2bc\cos A + c^2$$

$$a^2 = b^2 + c^2 - 2bc\cos A$$

Since we chose the sides generally,
it is true for any set of sides and angles

C. Example 1: Solve the triangle with the following sides and angles.

$$a = 300, b = 225, C = 51^\circ$$

$$c^2 = 300^2 + 225^2 - 2(300)(225)\cos 51$$

$$c^2 \approx 55666.75 \quad \boxed{c \approx 235.94}$$

$$300^2 \approx 225^2 + 235.94^2 - 2(225)(235.94)\cos A$$

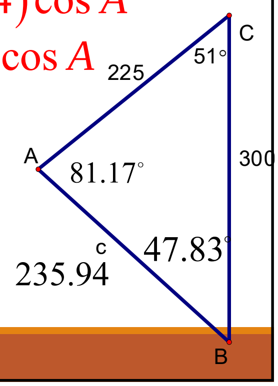
$$90000 \approx 50625 + 55666.75 - 106172.11\cos A$$

$$-16291.75 \approx -106172.11\cos A$$

$$0.153 \approx \cos A$$

$$\boxed{A \approx 81.17^\circ}$$

$$B \approx 180 - 51 - 81.17 = \boxed{47.83^\circ}$$



II. Examples

Solve the following triangle:

A.) $\angle A = 49°,\ b = 10,\ c = 18$ $\quad h = 10\sin 49 \approx 7.547$

$$a^2 = 10^2 + 18^2 - 2(10)(18)\cos 49$$

$$a \approx 13.705$$

$$18^2 = 13.705^2 + 10^2 - 2(13.075)(10)\cos C$$

$$C \approx 97.583°$$

$$B \approx 33.417°$$

Note: General rule is to use the Law of Cosines for finding the first, (or first and second) missing angle of a triangle because the Law of Cosines will return an obtuse angle. For example, if we use the Law of Sines to solve the last example for angle *C*, we get

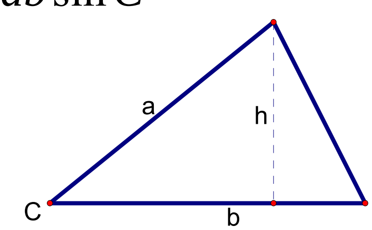
$\angle C = 82.405^\circ$ and $\angle B = 48.5$

This value of angle *B* contradicts the Law of Sines for *b*=10.

II. Triangle Area

A. Suppose that you know two sides of a triangle and the included angle. The area of the triangle is given by the formula

$$A = \frac{1}{2} ab \sin C$$



$$\sin C = \frac{h}{a}$$

$$a \sin C = h$$

$$Area = \frac{1}{2} bh = \frac{1}{2} ab \sin C$$

C. Example 2: Find the area of a triangle with sides of length 22 feet and 31 feet, with an included angle of 37 degrees.

$$A = \frac{ab}{2}\sin C$$

$$A = \frac{(22)(31)}{2}\sin 37$$

$$\boxed{A \approx 205.22\ ft^2}$$

D. Heron's Formula: Suppose that a triangle has side lengths *a*, *b*, and *c*. Then the area of the triangle is given by the formula below.

$$\boxed{A = \sqrt{s(s-a)(s-b)(s-c)}}$$

$$\text{where} \quad s = \frac{1}{2}(a+b+c)$$

E. Example 3: Suppose that a triangle has sides of length 12 cm, 15 cm, and 11 cm. Use Heron's formula to estimate the area of the triangle.

$$s = \frac{1}{2}(12+15+11) = 19$$

$$Area = \sqrt{19(19-12)(19-15)(19-11)}$$

$$Area = \sqrt{19(7)(4)(8)}$$

$$Area = \sqrt{4256}$$

$$\boxed{Area \approx 65.24cm^2}$$