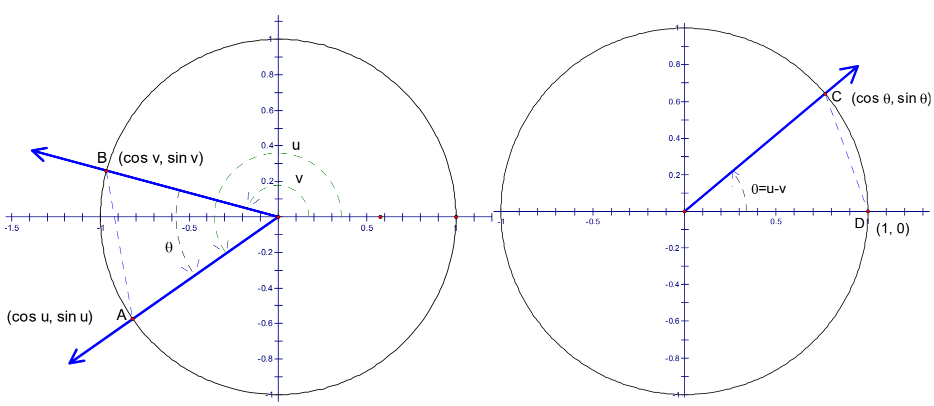
5.3: Sum and Difference Identities

I. Cosine of a Difference:

A. Derivation: Consider the two diagrams, (Note that for the chords AB=CD).



$$AB = CD$$

$$\sqrt{(\cos v - \cos u)^2 + (\sin v - \sin u)^2} = \sqrt{(\cos\theta - 1)^2 + (\sin\theta - 0)^2}$$

$$\cos^2 u - 2\cos u\cos v + \cos^2 v + \sin^2 u - 2\sin u\sin v + \sin^2 v$$
$$= \cos^2\theta - 2\cos\theta + 1 + \sin^2\theta$$

$$\left(\cos^2 u + \sin^2 u\right) + \left(\cos^2 v + \sin^2 v\right) - 2\cos u\cos v - 2\sin u\sin v$$
$$= \left(\cos^2\theta + \sin^2\theta\right) + 1 - 2\cos\theta$$

$$2 - 2\cos u\cos v - 2\sin u\sin v = 2 - 2\cos\theta$$

$$\cos u\cos v + \sin u\sin v = \cos\theta$$

$$\boxed{\cos u\cos v + \sin u\sin v = \cos(u - v)}$$

B. Find the formula for the cosine of a sum.

$$\cos(u + v) = \cos\left[u - (-v)\right]$$
$$= \cos u\cos(-v) + \sin u\sin(-v)$$
$$= \cos u\cos v + \sin u(-\sin v)$$
$$= \boxed{\cos u\cos v - \sin u\sin v}$$

C. Cosine sum and difference formulas:

$$\boxed{\cos(u \pm v) = \cos u\cos v \mp \sin u\sin v}$$

D. Example 1: Determine the value of: $\cos\frac{5\pi}{12}$

First, note that $\frac{5\pi}{12}=\frac{3\pi}{4}-\frac{\pi}{3}$

$$\cos\frac{5\pi}{12}=\cos\left(\frac{3\pi}{4}-\frac{\pi}{3}\right)=\cos\left(\frac{3\pi}{4}\right)\cos\left(\frac{\pi}{3}\right)+\sin\left(\frac{3\pi}{4}\right)\sin\left(\frac{\pi}{3}\right)$$

$$=\left(-\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)+\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$$

$$=-\frac{\sqrt{2}}{4}+\frac{\sqrt{6}}{4}$$

$$=\boxed{\frac{\sqrt{6}-\sqrt{2}}{4}}$$

E. Example 2: Verify the following identities:

$$1.\ \cos(\pi+\theta)=-\cos\theta$$

$$\boxed{\cos(\pi+\theta)}=\cos\pi\cos\theta-\sin\pi\sin\theta$$

$$=(-1)\cos\theta-(0)\sin\theta$$

$$=\boxed{-\cos\theta}$$

$$2.\ \cos\left(\theta - \frac{3\pi}{2}\right) = -\sin\theta$$

$$\boxed{\cos\left(\theta - \frac{3\pi}{2}\right)} = \cos\theta\cos\left(\frac{3\pi}{2}\right) + \sin\theta\sin\left(\frac{3\pi}{2}\right)$$

$$= \cos\theta(0) + \sin\theta(-1)$$

$$= \boxed{-\sin}$$

Note: these are called reduction formulas,
since they reduce the complexity of the expression

II. Sine of a difference or sum

$$\text{A. } \sin(u+v) = \cos\left[\frac{\pi}{2} - (u+v)\right]$$

$$= \cos\left[\left(\frac{\pi}{2} - u\right) - v\right]$$

$$= \cos\left(\frac{\pi}{2} - u\right)\cos v + \sin\left(\frac{\pi}{2} - u\right)\sin v$$

$$= \sin u\cos v + \cos u\sin v$$

B. $\sin(u-v) = \cos\left[\frac{\pi}{2} - (u-v)\right]$

$$= \cos\left[\left(\frac{\pi}{2} - u\right) + v\right]$$

$$= \cos\left(\frac{\pi}{2} - u\right)\cos v - \sin\left(\frac{\pi}{2} - u\right)\sin v$$

$$= \sin u \cos v - \cos u \sin v$$

$$\boxed{\sin(u \pm v) = \sin u \cos v \pm \cos u \sin v}$$

C. Example 3: Find the value of each of the following.

1. $\sin(27.35^\circ)\cos(2.65^\circ) + \cos(27.35^\circ)\sin(2.65^\circ)$

$$= \sin(27.35^\circ + 2.65^\circ)$$

$$= \sin(30^\circ) \quad = \boxed{\frac{1}{2}}$$

2. $\cos\left(\frac{4\pi}{11}\right)\cos\left(\frac{7\pi}{11}\right) - \sin\left(\frac{4\pi}{11}\right)\sin\left(\frac{7\pi}{11}\right)$

$$= \cos\left(\frac{4\pi}{11} + \frac{7\pi}{11}\right)$$

$$= \cos(\pi) \quad = \boxed{-1}$$

III. Tangent of a difference or sum

A. $$\boxed{\tan(u \pm v) = \frac{\sin(u \pm v)}{\cos(u \pm v)} = \frac{\sin u \cos v \pm \cos u \sin v}{\cos u \cos v \mp \sin u \sin v}}$$

or

$$\boxed{\tan(u \pm v) = \frac{\tan u \pm \tan v}{1 \mp \tan u \tan v}}$$

B. Prove the reduction formula below:

$$\tan\left(\frac{\pi}{2} + \theta\right) = -\cot\theta$$

$$\boxed{\tan\left(\frac{\pi}{2} + \theta\right)} = \frac{\sin\left(\frac{\pi}{2} + \theta\right)}{\cos\left(\frac{\pi}{2} + \theta\right)} = \frac{\sin\left(\frac{\pi}{2}\right)\cos\theta + \cos\left(\frac{\pi}{2}\right)\sin\theta}{\cos\left(\frac{\pi}{2}\right)\cos\theta - \sin\left(\frac{\pi}{2}\right)\sin\theta}$$

$$= \frac{(1)\cos\theta + (0)\sin\theta}{(0)\cos\theta - (1)\sin\theta}$$

$$= \frac{\cos\theta}{-\sin\theta}$$

$$= \boxed{-\cot\theta}$$

IV. Verifying a Sinusoid Algebraically

Express the function below as a single sinusoid

$$f(x) = 2\sin 2x + 3\cos 2x$$

$$a\sin(bx+c) = a\left(\sin bx\cos c + \cos bx\sin c\right)$$

$$= a\sin bx\cos c + a\cos bx\sin c$$

$$2\sin 2x + 3\cos 2x = \left(a\cos c\right)\sin bx + \left(a\sin c\right)\cos bx$$

$$\boxed{b=2}$$

$$2 = a\cos c \qquad 3 = a\sin c$$

$$\left(a\cos c\right)^2 + \left(a\sin c\right)^2 = (2)^2 + (3)^2$$

$$a^2\cos^2 c + a^2\sin^2 c = 13$$

$$a^2\left(\cos^2 c + \sin^2 c\right) = 13$$

$$a^2 = 13$$

$$a = \pm\sqrt{13}$$

$$\text{let's use } a = \sqrt{13}$$

$$\sqrt{13}\cos c = 2 \qquad \sqrt{13}\sin c = 3$$

$$\cos c = \frac{2}{\sqrt{13}} \qquad \sin c = \frac{3}{\sqrt{13}}$$

$$c = \cos^{-1}\left(\frac{2\sqrt{13}}{13}\right) \text{ or } c = \sin^{-1}\left(\frac{3\sqrt{13}}{13}\right)$$

Exact answer:

$$\boxed{f(x) = \sqrt{13}\sin\left[2x + \cos^{-1}\left(\frac{2\sqrt{13}}{13}\right)\right]}$$

or

$$\boxed{f(x) = \sqrt{13}\sin\left[2x + \sin^{-1}\left(\frac{3\sqrt{13}}{13}\right)\right]}$$

Approximate:

$$\boxed{3.606\sin\left[2x + 0.983\right]}$$