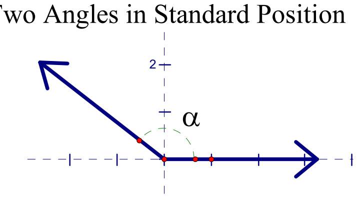
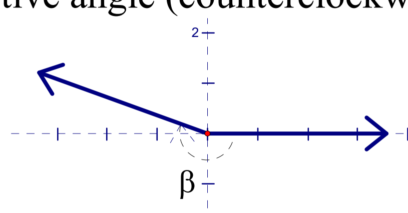
4.3: Trigonometry Extended: The Circular Functions

I. Trigonometric Functions of Any Angle

Two Angles in Standard Position

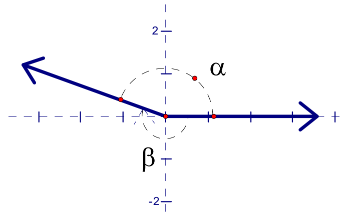


A positive angle (counterclockwise)

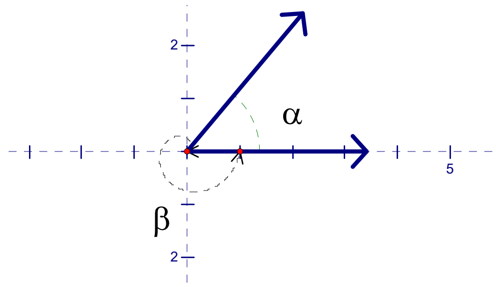


A negative angle (clockwise)

Coterminal Angles



Positive and negative coterminal angles



Two positive coterminal angles

A. Example 1: Finding coterminal angles. Find a positive and negative coterminal angle for each of the following angles.

1. 25° $\quad 25^\circ + 360^\circ = \boxed{385^\circ} \quad 25^\circ - 360^\circ = \boxed{-335^\circ}$

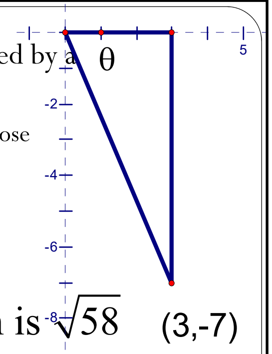
2. -400° $\quad -400^\circ + 360^\circ = \boxed{-40^\circ}$

$$-40^\circ + 360^\circ = \boxed{320^\circ}$$

3. $\frac{4\pi}{5}$ $\quad \frac{4\pi}{5} + 2\pi = \boxed{\frac{14\pi}{5}} \quad \frac{4\pi}{5} - 2\pi = \boxed{-\frac{6\pi}{5}}$

B. Example 2: Evaluating trig functions determined by a point.

Let theta be the acute angle in standard position whose terminal side contains the point *(3,-7)*. Find the six trigonometric functions of theta



the distance from the point to the origin is $\sqrt{58}$

$$\sin\theta = \frac{-7}{\sqrt{58}} = \frac{-7\sqrt{58}}{58} \approx -0.919$$

$$\csc\theta = \frac{\sqrt{58}}{-7} \approx -1.088$$

$$\cos\theta = \frac{3}{\sqrt{58}} = \frac{3\sqrt{58}}{58} \approx 0.394$$

$$\sec\theta = \frac{\sqrt{58}}{3} \approx 2.539$$

$$\tan\theta = \frac{-7}{3}$$

$$\cot\theta = \frac{3}{-7}$$

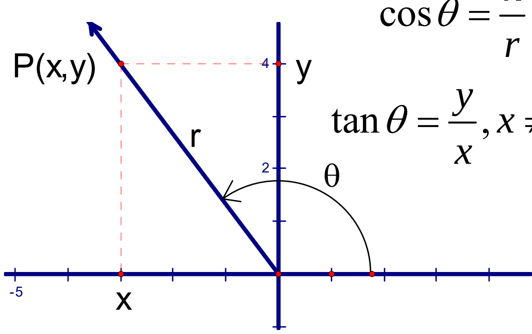
C. Definition: Trigonometric Functions of any angle.

Let theta be any angle in standard position, and let *P(x,y)* be any point on the terminal side of the angle (except for the origin). Let *r* denote the distance from *P* to the origin. Then:

$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}, y \neq 0$$

$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}, x \neq 0$$

$$\tan\theta = \frac{y}{x}, x \neq 0 \qquad \cot\theta = \frac{x}{y}, y \neq 0$$



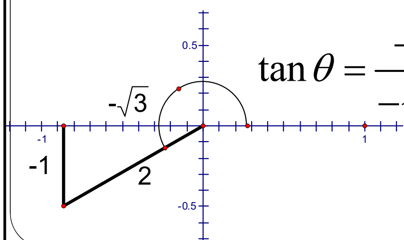
E. Example 3: Evaluating a trig function using a reference triangle. Find the six trig functions at an angle of 210 degrees.

Notice that this forms the 30-60-90 triangle below (called the reference triangle).

$$\sin\theta = \frac{-1}{2} \qquad \csc\theta = \frac{-2}{1} = -2$$

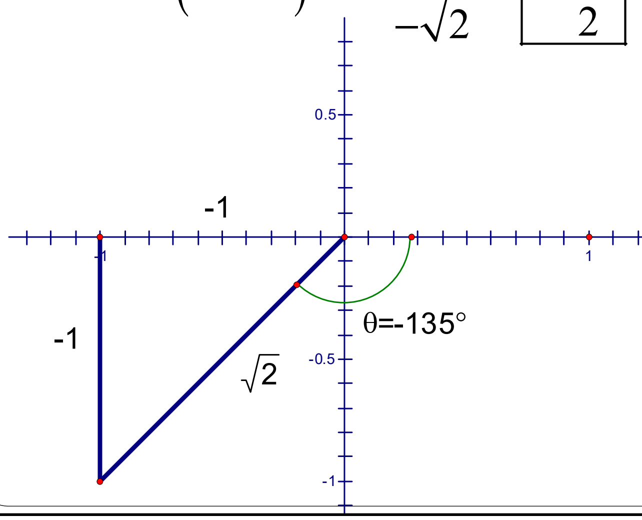
$$\cos\theta = \frac{-\sqrt{3}}{2} \qquad \sec\theta = -\frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$$

$$\tan\theta = \frac{-1}{-\sqrt{3}} = \frac{\sqrt{3}}{3} \qquad \cot\theta = \frac{-\sqrt{3}}{-1} = \sqrt{3}$$



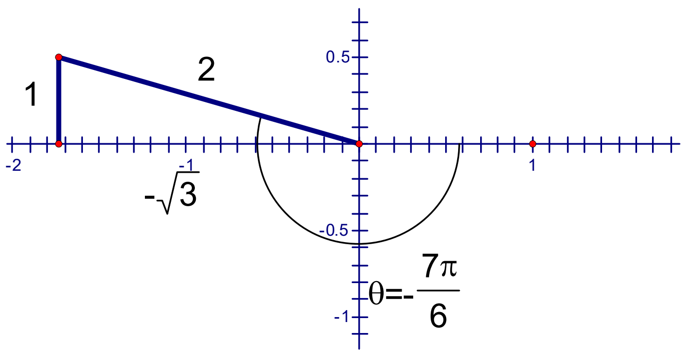
F. Example 4: Find the following without a calculator:

$$1.\ \sin\left(-135^\circ\right) = \frac{1}{-\sqrt{2}} = \boxed{-\frac{\sqrt{2}}{2}}$$



$$2.\ \csc\left(-\frac{7\pi}{6}\right) = \frac{2}{1} = \boxed{2}$$

=−



G. Example 5: Evaluating trig functions of quadrantal angles.

$$1.\ \cos\left(\frac{-3\pi}{2}\right)$$

this lies on the positive y-axis, thus the coordinates are (0,1)

$$\cos\left(-\frac{3\pi}{2}\right) = \frac{x}{r} = \frac{0}{1} = \boxed{0}$$

$$2.\ \csc\left(900^\circ\right)$$

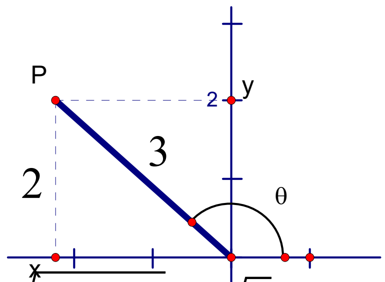
this lies on the negative x-axis, thus the coordinates are (-1,0)

$$\csc\left(900^\circ\right) = \frac{r}{y} = \frac{-1}{0}, \text{ which is undefined}$$

H. Example 4: Find the cosine and tangent of theta, given the following information.

1. $\sin\theta = \frac{2}{3}$ and $\cos\theta < 0$

second quadrant, since sin>0 and cos<0



$\sqrt{3^2 - 2^2} = \sqrt{5}$

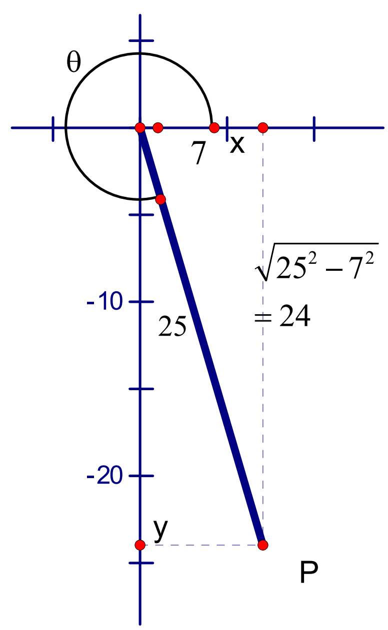
$$\cos\theta = \boxed{-\frac{\sqrt{5}}{3}}$$

$$\tan\theta = -\frac{2}{\sqrt{5}} = \boxed{-\frac{2\sqrt{5}}{5}}$$

2. $\sec\theta = \frac{25}{7}$ and $\cot\theta < 0$

$$\cos\theta = \boxed{\frac{7}{25}}$$

$$\tan\theta = \boxed{-\frac{24}{7}}$$



3. $\csc\theta$ is undefined and $\sec\theta < 0$

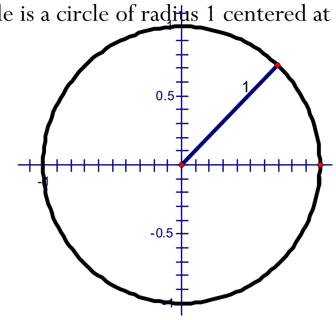
This is on the negative x-axis

$$\cos\theta = \boxed{-1} \qquad \tan\theta = \boxed{0}$$

II. Trigonometric Functions of Real Numbers

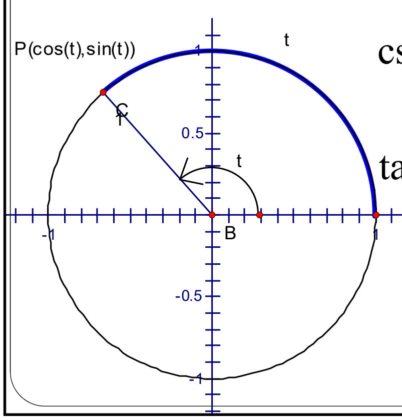
A. Definition: Unit Circle

The unit circle is a circle of radius 1 centered at the origin



C. Definition: Trigonometric Functions of Real Numbers.

Let *t* be any real number, and let *P(x,y)* be the point corresponding to *t* when the number line is wrapped onto the unit circle .



$$\sin\theta = y \qquad \cos\theta = x$$

$$\csc\theta = \frac{1}{y}, y \neq 0 \qquad \sec\theta = \frac{1}{x}, x \neq 0$$

$$\tan\theta = \frac{y}{x}, x \neq 0 \qquad \cot\theta = \frac{x}{y}, y \neq 0$$

III. Periodic Functions

A. Definition: Periodic Function

A function $y=f(x)$ is *periodic* if there is a positive number c such that $(t+c)=f(t)$ for all values of t in the domain of f.

The smallest such number c is called the *period* of the function.

B. Example 5: Using Periodicity. Find each of the following numbers without a calculator.

1. $\cos\left(\frac{645\pi}{2}\right) = \cos\left(\frac{\pi}{2}+\frac{644\pi}{2}\right) = \cos\left(\frac{\pi}{2}+322\pi\right) = \cos\left(\frac{\pi}{2}\right) = \boxed{0}$

2. $\sin(823.51\pi) - \sin(819.51\pi) = \sin(819.51\pi + 4\pi) - \sin(819.51\pi) = \boxed{0}$

since these wrap to the same point...

3. $\tan\left(231{,}457\pi - \frac{\pi}{4}\right) = \tan\left(-\frac{\pi}{4}\right) = \boxed{-1}$

IV. The 16-Point Unit Circle