3-2: Exponential and Logistic Modeling

I. Constant Percentage Rate and Exponential Functions.

A. Exponential Population Model:

1. If a population P is changing at a constant percentage rate *r* each year, then $P(t) = P_0\left(1+r\right)^t$

where P_0 is the initial population, r is the growth rate (decimal) , and t is time.

2. r > 0→ exponential growth
3. r < 0 → exponential decay

B. Example 1: finding growth and decay rates

a. $P(t) = 32,459 \cdot 0.9845^t$

decay, since $1 + r < 1$

$r = -0.0155$

decay rate is 1.55%

b. $P(t) = 890,234 \cdot 1.0239^t$

growth, since $1 + r > 1$

$r = 0.0239$

growth rate is 2.39%

C. Example 2: Find the exponential function with the initial value of 28, and decaying at a rate of 4.5%.

$$f(x) = 28(1 - 0.045)^x = 28(0.955)^x$$

II. Exponential Growth and Decay Models

A. Doubling Time

$$f(t) = a \cdot 2^{\frac{t}{d}} = a\left(\sqrt[d]{2}\right)^t$$

Where *f(t)* is the population/amount, a is the initial amount, t is the time, and d is the doubling time.

B. Half-Life

$$f(t) = a \cdot \left(\frac{1}{2}\right)^{\frac{t}{h}} = a\left(\sqrt[h]{\frac{1}{2}}\right)^t$$

Where *f(t)* is the population/amount, a is the initial amount, t is the time, and h is the half-life.

C. Example 3: Radioactive iodine is used in various medical diagnostic tests. Its half-life is 8 days.

1. Write a formula for *f(t)*—the amount of iodine—in terms of the number of days.

$$f(t) = a\left(\frac{1}{2}\right)^{\frac{t}{8}}$$

2. What proportion of the original radioactive iodine is present after 48 days?

$$f(48) = a \cdot \left(\frac{1}{2}\right)^{\frac{48}{8}} = a \cdot \left(\frac{1}{2}\right)^{6} = \boxed{\frac{1}{64}a}$$

3. If there is initially 12mL of iodine present, after how many days will there be less than 1 mL? (solve graphically)

$$t \approx 28.68 days$$

D. Example 4: suppose that a colony of bacteria exhibits exponential growth. The colony initially contains 5000 bacteria, and takes 4 hours to double.

1. Write a function for *P(t)*—the population of bacteria—as a function of time.

$$P(t) = 5000(2)^{\frac{t}{4}}$$

2. How many bacteria are there after 10 hours?

$$P(10) = 5000(2)^{\frac{10}{4}} \approx 28,284$$

3. How long will it take until the colony contains 160,000 bacteria?

Algebraically:

$$5000 \cdot 2^{\frac{t}{4}} = 160,000$$

$$2^{\frac{t}{4}} = 32$$

$$2^{\frac{t}{4}} = 2^5$$

$$\therefore \frac{t}{4} = 5$$

$$\boxed{t = 20 hours}$$

III. Using Regression to Model Population

A. Example 5: The world population is given in the table to the right.

1. Find an exponential model for the data (in years after 1900)

$$P(t) \approx 1300.7007(1.0149)^t$$

2. Is this a reasonable model?

$$yes, r^2 \approx 0.9804 \approx 1$$

3. Predict the population in 2010

$$P(110) \approx 6,629,000,000 \text{ people}$$

Year	Population (millions)
1950	2,555
1960	3,039
1970	3,707
1980	4,456
1990	5,283
2000	6,080
2010	6,823*
2020	7,518*
2030	8,140*
	*projections

B. Example 6: The world population is given in the table to the right.

1. Find a logistic model for this population data. Use years after 1900 for t.

$$P(t)=\frac{10929.493}{1+14.364e^{-0.029t}}$$

2. Use your model to predict the population in the year 2050

$$P(150)\approx 9175\text{million people}$$

3. Use this model to predict the maximum sustainable population for the world.

$$\lim_{t\to\infty}P(t)=10{,}929{,}493{,}000$$

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C. Example 8: Find the logistic model with the following characteristics:

1. Initial value of 16 $f(0)=16$
2. Maximum capacity of 80 $c=80$
3. Passing through the point (2,40) $f(2)=40$

$$f(t)=\frac{c}{1+ab^{x}}$$

$$f(t)=\frac{80}{1+ab^{x}}$$

$$f(0) = 16 = \frac{80}{1 + ab^0}$$

$$16 = \frac{80}{1 + a}$$

$$1 + a = \frac{80}{16}$$

$$a = 4$$

$$f(2) = 40 = \frac{80}{1 + 4b^2}$$

$$1 + 4b^2 = \frac{80}{40}$$

$$4b^2 = 1$$

$$b = \frac{1}{2}$$

$$f(t) = \frac{80}{1 + 4(0.5)^t}$$