Notes 2.5 –Complex Zeros and the Fundamental Theorem of Algebra

I. The Fundamental Theorem of Algebra-

A.) Thm: A polynomial function of degree $n > 0$ has n complex zeros. Some of these zeros may be repeated.

B.) Thm: The Linear Factorization Theorem–

If f is a polynomial function of degree $n > 0$, then f has precisely n linear factors and

$$f(x) = a(x - z_1)(x - z_2)...(x - z_n)$$

where a is the leading coefficient of $f(x)$. The z_i are not necessarily distinct numbers, some may be repeated.

C.) Fundamental Polynomial Connections to the Complex Case

1.) $x = k$ is a **solution** (or root) of the equation $f(x) = 0$.

2.) k is a **zero** of the function f.

3.) k is an ***x*-intercept** of the graph of $y = f(x)$.

NOTE – $(k, 0)$ is not an x-intercept if k is complex.

D.) Ex.- Write the polynomial in standard form and identify the zeros of the function and the x- intercepts.

$$f(x) = (x+2)(x-3)(x-2i)(x+2i)$$

$$f(x) = (x^2 - x - 6)(x^2 + 4) =$$

$$f(x) = x^4 - x^3 - 2x^2 - 4x - 24$$

Real zeros of -2, 3

Complex zeros of $\pm 2i$

$x-$intercepts of $(-2,0)$ & $(3,0)$

$y-$intercept of $(0,-24)$

III. Complex Conjugate Zeros

A.) For any poly. fn. $f(x)$ with REAL COEFFICIENTS, if a and b are real numbers with $b \neq 0$ and $a + bi$ is a zero of $f(x)$, then the complex conjugate $a - bi$ is also a zero of $f(x)$.

B.) Ex. – Write a poly. fn. of minimum degree in standard form with real coefficients whose zeros include -3, 1, 1 – 2i.

$$f(x)=(x+3)(x-1)(x-(1-2i))(x-(1+2i))$$

$$f(x)=(x^2+2x-3)(x-1+2i)(x-1-2i)$$

$$f(x)=(x^2+2x-3)(x^2-2x+5)$$

$$f(x)=x^4-2x^2+16x-15$$

IV. Factoring a Poly. with Complex Zeros

Ex. – Use your calculator to help find all the zeros of the following function and write in a linear factorization form.

$$f(x)=x^5-3x^4-5x^3+5x^2-6x+8$$

$$f(x)=a(x-z_1)(x-z_2)\ldots(x-z_n)$$

- Real Zeros – $x=-2,\ 1,\ 4$
- Use synthetic division to find the "irreducible quadratic"

$$
\begin{array}{r|rrrrrr}
-2 & 1 & -3 & -5 & 5 & -6 & 8 \\
 & & -2 & 10 & -10 & 10 & -8 \\
\hline
 & 1 & -5 & 5 & -5 & 4 & 0
\end{array}
$$

$$
\begin{array}{r|rrrrr}
1 & 1 & -5 & 5 & -5 & 4 \\
 & & 1 & -4 & -5 & 4 \\
\hline
 & 1 & -4 & 1 & -4 & 0
\end{array}
$$

$$
\begin{array}{r|rrrr}
4 & 1 & -4 & 1 & -4 \\
 & & 4 & 0 & 4 \\
\hline
 & 1 & 0 & 1 & 0
\end{array}
$$

$$x^2 + 1 = 0$$

$$x = \pm i$$

$$f(x) = (x+2)(x-1)(x-4)(x-i)(x+i)$$

Ex.- Find all the complex zeros of $f(x) = 4x^4 + 17x^2 + 14x + 65$
If one zero is $1-2i$.

$$\left(x-(a-bi)\right)\left(x-(a+bi)\right) = \left(x^2 - 2ax + \left(a^2+b^2\right)\right)$$

$$\left(x-(1-2i)\right)\left(x-(1+2i)\right) = \left(x^2 - 2x + 5\right)$$

Now use Long Division or Synthetic Division to find the other factor(s)

$$
\begin{array}{r}
4x^2 + 8x + 13 \\
x^2 - 2x + 5 \overline{\big) 4x^4 + 17x^2 + 14x + 65}
\end{array}
$$

Now use quadratic formula to linearize the other factor

$$x = \frac{-8 \pm \sqrt{8^2 - 4(4)(13}}{2(4)} \qquad x = \frac{-8 \pm \sqrt{64 - 208}}{8}$$

$$x = \frac{-8 \pm \sqrt{-144}}{8} = \frac{-8 \pm 12i}{8} = -1 \pm \frac{3}{2}i$$

$$x = 1 \pm 2i, -1 \pm \frac{3}{2}i$$

$$f(x) = 4(x - 1 + 2i)(x - 1 - 2i)\left(x + 1 + \frac{3}{2}i\right)\left(x + 1 - \frac{3}{2}i\right)$$

$$f(x) = (x - 1 + 2i)(x - 1 - 2i)(2x + 2 + 3i)(2x + 2 - 3i)$$

Homework: p.215 #3-42 (mult. of 3)