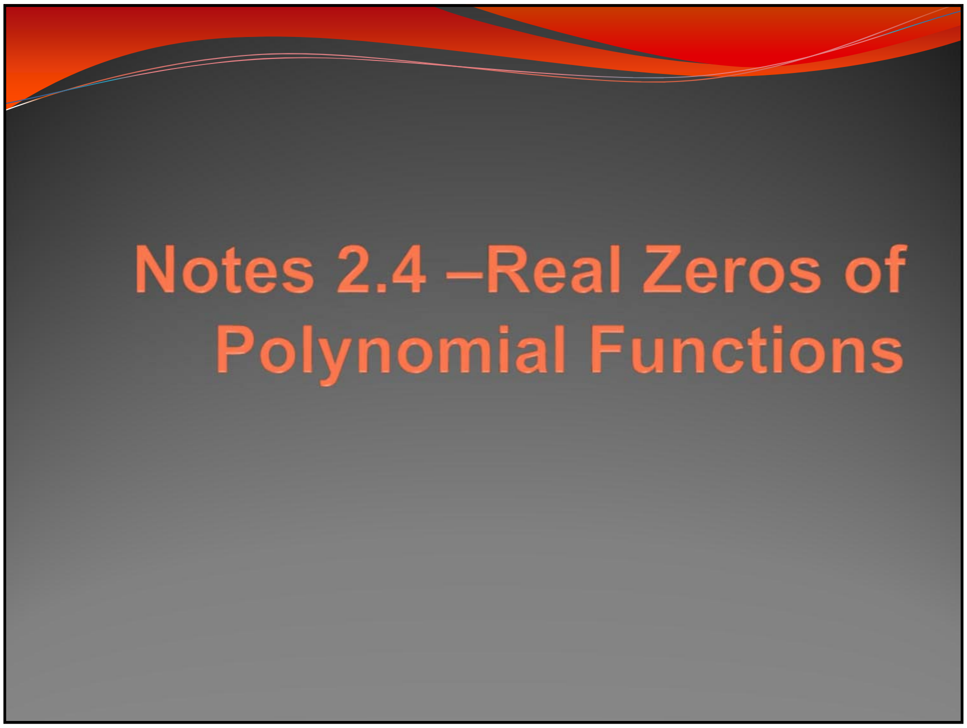
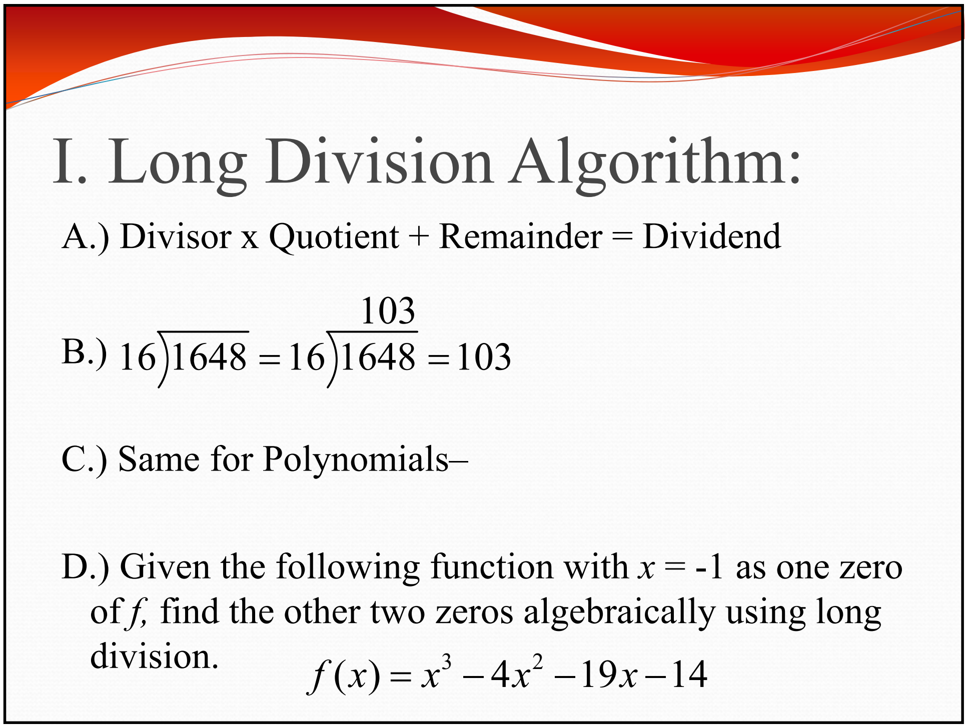


Notes 2.4 –Real Zeros of Polynomial Functions



I. Long Division Algorithm:

A.) Divisor x Quotient + Remainder = Dividend

B.) $16\overline{)1648} = 16\overline{)\overset{103}{1648}} = 103$

C.) Same for Polynomials–

D.) Given the following function with x = -1 as one zero of f, find the other two zeros algebraically using long division. $f(x) = x^3 - 4x^2 - 19x - 14$

$$\begin{array}{rl}
 & x^2 \quad -5x \quad -14 \\
x+1 \big) & \overline{x^3 - 4x^2 - 19x - 14} \\
 & \underline{x^3 + x^2} \\
 & \qquad -5x^2 - 19x - 14 \\
 & \qquad \underline{-5x^2 - 5x} \\
 & \qquad\qquad -14x - 14 \\
 & \qquad\qquad \underline{-14x - 14}
\end{array}$$

$$(x+1)(x^2 - 5x - 14) = 0$$

$$(x+1)(x-7)(x+2) = 0$$

$$x = -1, -2, 7$$

II. L. D. Alg. For Polynomials

A.) Polynomial Form: $$f(x) = \underset{\overset{\uparrow}{\text{divisor}}}{d(x)} \cdot \underset{\overset{\uparrow}{\text{quotient}}}{q(x)} + \underset{\overset{\uparrow}{\text{remainder}}}{r(x)}$$

B.) Fraction Form: $$\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$$

C.) Divide *f*(*x*) by *d*(*x*) and write the statement in both polynomial and fraction form.

$$f(x) = x^3 + 4x^2 + 7x - 7$$

$$d(x) = x + 3$$

$$\begin{array}{r}
x^2 \quad +x \quad\quad +4 \\
x+3\overline{\big)\, x^3 + 4x^2 + 7x - 7} \\
\underline{x^3 + 3x^2} \quad\quad\quad\quad \\
x^2 + 7x - 7 \\
\underline{x^2 + 3x} \quad\quad \\
4x - 7 \\
\underline{4x + 12} \\
-19
\end{array}$$

POLY. FORM:

$$f(x) = (x+3)\left(x^2 + x + 4\right) - 19$$

FRACTION FORM:

$$\frac{f(x)}{(x+3)} = \left(x^2 + x + 4\right) - \frac{19}{(x+3)}$$

III. Remainder and Factor Theorems

A.) Special Case: $d(x) = x - k$ where k is a real number- because $x - k$ is degree one, the remainder is always a real number.

B.) Remainder Thm: If $f(x)$ is divided by $x - k$, then the remainder $r = f(k)$.

1.) Ex. – Find the remainder when the following is divided by $x + 3$. $f(x) = x^3 - x^2 + 2x - 1$

$$f(-3) = (-3)^3 - (-3)^2 + 2(-3) - 1 \qquad f(-3) = -43$$

C.) Factor Thm: A poly. fn. $f(x)$ has a factor of $x - k$ if $f(k) = 0$.

1.) Ex. – Use the factor theorem to decide if $x - 2$ is a factor of $f(x) = x^3 + 3x - 4$

$$f(2) = (2)^3 + 3(2) - 4 \quad f(2) = 10$$

NO!!

2.) Rule – FACTOR FIRST!!! You may not need long div. or remainder and factor theorems.

IV. Fundamental Connections for Polys.

The following statements are equivalent:

A.) $x = k$ is a **solution** (or root) of the equation $f(x) = 0$.

B.) k is a **zero** of the function f.

C.) k is an ***x*-intercept** of the graph of $y = f(x)$.

D.) $x - k$ is a **factor** of $f(x)$.

V. Synthetic Division

Shortcut method when $x - k$ is a factor of $f(x)$.

A.) Process: Bring down the leading coefficient of the dividend, multiply it by *k*, add the 2nd coefficient to the product and repeat the process.

B.) Ex – Use synthetic division and write the answer in fraction form.

$$\frac{x^3 - 5x^2 + 3x - 2}{x+1}$$

$$\begin{array}{r|rrrr} \underline{-1} & 1 & -5 & 3 & -2 \\ & & -1 & 6 & -9 \\ \hline & 1 & -6 & 9 & -11 \end{array}$$

$$\frac{x^3 - 5x^2 + 3x - 2}{x+1} = \quad x^2 - 6x + 9 - \frac{11}{x+1}$$

VI. Rational Zeros Thm

A.) SPSE f is a poly. fn. of degree $n \geq 1$ of the form $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x^1 + a_0$ with every coefficient an integer and $a_0 \neq 0$. If $\frac{p}{q}$ is a rational zero of f where p and q have no common factors other than 1, then p is an integer factor of a_0 and q is an integer factor of a_n

B.) Ex. – Find all the rational zeros of $f(x) = 3x^3 + 4x^2 - 5x - 2$

$$\frac{a_0}{a_3} = \frac{\pm 1, \pm 2}{\pm 1, \pm 3} = \pm 1, \pm 2, \pm \frac{1}{3}, \pm \frac{2}{3}$$

$$\begin{array}{r|rrrr} \underline{1} & 3 & 4 & -5 & -2 \\ & & 3 & 7 & 2 \\ \hline & 3 & 7 & 2 & 0 \end{array}$$

$$f(1) = 3 + 4 - 5 - 2 = 0$$

$$(x-1)(3x^2 + 7x + 2) = 0$$

$$(x-1)(3x+1)(x+2) = 0$$

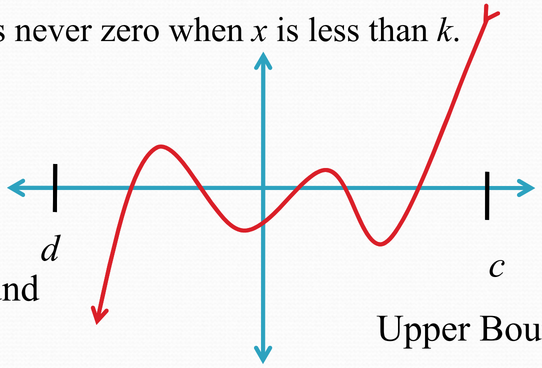
$$x = -2, -\frac{1}{3}, 1$$

VII. Upper and Lower Bounds

A.) Upper Bound - A number k is an upper bound for real zeros of f if $f(x)$ is never zero when x is greater than k.

B.) Lower Bound – A number k is an lower bound for real zeros of f if $f(x)$ is never zero when x is less than k.

C.) Graphically -



Lower Bound

Upper Bound

D.) Upper and Lower Bound Test using Synthetic Division-

SPSE $f(x)$ is divided by $x - k$ using synthetic division,

1.) If $k \geq 0$ and every number in the last line is non-negative, then k is an upper bound for the real zeros of f.

2.) If $k \leq 0$, and the numbers in the last line alternate non-negative and non-positive, then k is a lower bound for the real zeros of f.

E.) Ex.–Prove that all the real zeros of $f(x) = 2x^4 - 7x^3 + 8x^2 + 14x + 8$ must lie in the interval [-2, 5].

$$\begin{array}{r|rrrrr} \underline{5} & 2 & -7 & 8 & 14 & 8 \\ & & 10 & 15 & 115 & 645 \\ \hline & 2 & 3 & 23 & 129 & 653 \end{array}$$

All Positive – UPPER BOUND

$$\begin{array}{r|rrrrr} \underline{-2} & 2 & -7 & 8 & 14 & 8 \\ & & -4 & 22 & -60 & 92 \\ \hline & 2 & -11 & 30 & -46 & 100 \end{array}$$

Alt. Signs – LOWER BOUND

Find all the rational zeros of the following function without a calculator.

$$f(x) = 2x^4 - 7x^3 - 8x^2 + 14x + 8$$

$$x = 4, -\frac{1}{2}, \pm\sqrt{2}$$

Homework: p.205 #3-24 (multiples of 3), 29, 31